Chapter 6

Application of polysplines to magnetism and CAGD

One may say that all arguments provided so far in the present book, which prove the advantageous properties of the polysplines (and thus of the *polyharmonic paradigm*) by comparing them with other smoothing methods, are speculative in character since they appeal mainly to purely mathematical criteria for beauty and naturalness.

This would be the normal reaction for example of the people working with practical data. For them we have prepared two case studies where the superiority of the polysplines over *Kriging*, *Minimum curvature*, and *Radial Basis Functions* (RBFs) (not to talk about polynomial splines) is irrefutable.

6.1 Smoothing airborne magnetic field data

First we consider an interesting application to magnetic data. The case concerns the airborne data (collected through airplanes) over the Cobb Offset (in the ocean near California) where the cooled magma creates a natural magnetic anomaly. Due to the reversals of the magnetic field the neighboring layers of the magma (going somewhat North–South) have opposite signs and thus the data *oscillate wildly*.[1] The 13 tracks of the airplanes are approximately horizontal (East–West), i.e. transversal to the magma layers with nearly 200 data points on each. At these points the magnetic field (the so-called total value) has been measured and they are seen in Figure 6.1. According to the usual terminology in approximation theory these data are “scattered”.

Figure 6.2 provides a sample of data on a vertical straight line (going North–South), where we see how strong the oscillation in this direction is. (So far this is not the worst oscillation in the North–South direction!)

[1] I have been provided with this data set by Dr. Richard O. Hansen, Pearson&de Ridder&Johnson at Denver, Colorado.

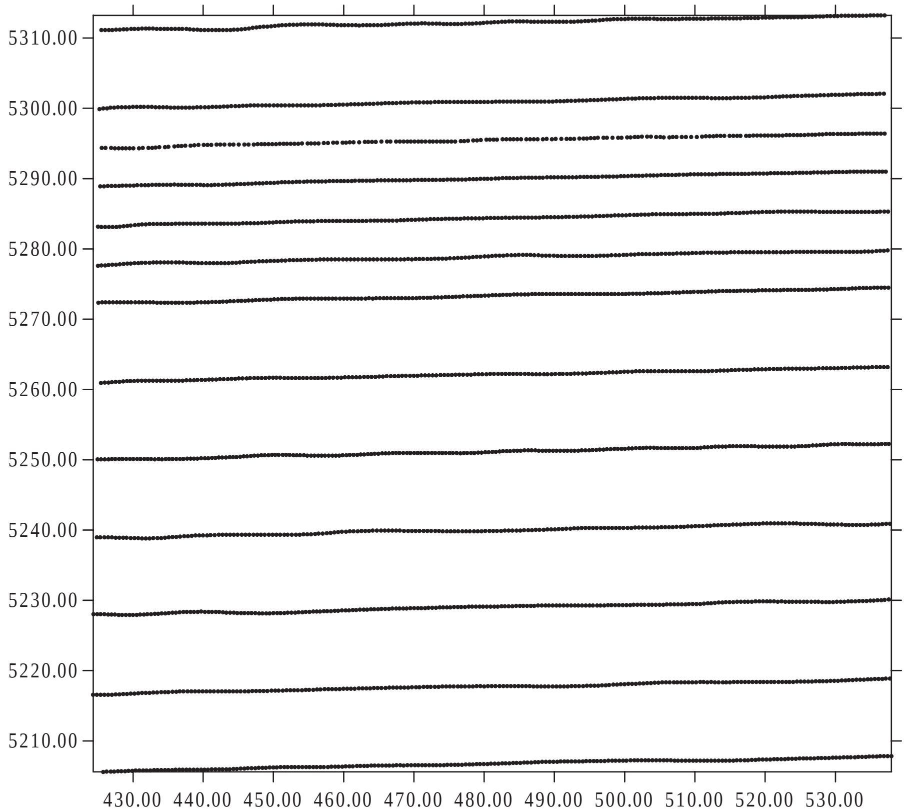

Figure 6.1. The 13 tracks of the airplanes with the data points.

Let us note that due to the mentioned scattered character of the data and their high oscillation *the test with magnetic data is one of the most difficult tests for every smoothing method!*

Next we provide the figures which show that the polysplines perform distinctly better than well established methods in the area of *magnetic explorations* such as *kriging, minimum curvature, thin plate splines,* and last but not least *RBFs* (the polynomial splines fail completely in such tests!).[2]

Figure 6.3 shows the result of the application of the *polysplines* to the Cobb Offset data. We provide the "level curves" of the graph of the interpolation polyspline. We have "posted" also the 13 data tracks.

When we apply *kriging*, *minimum curvature*, or *RBFs* to the same data, the result does not differ essentially for the three methods (see Figure 6.4).

On both pictures we have "posted" the locations of the original data by small points. Comparing both pictures we see that on the second one, Figure 6.4, one may easily recognize the thirteen parallel lines where the data points lie since the data points are

[2] These methods are incorporated in professional software for geodesists, geophysicists, geographers, etc., e.g. the program "SURFER" [19].

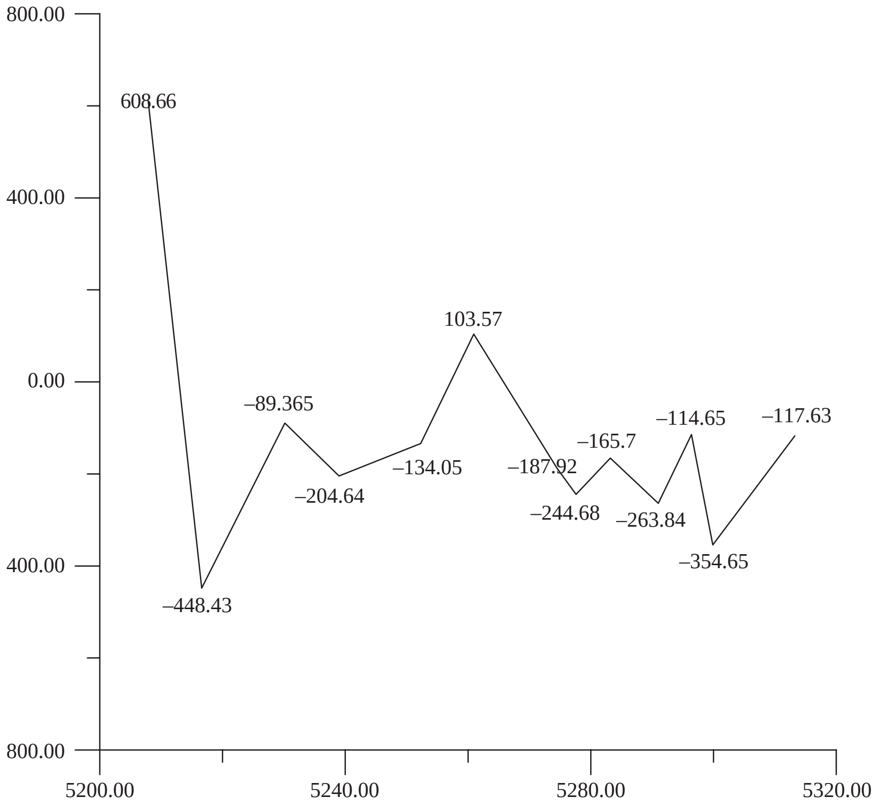

Figure 6.2. Easternmost vertical line of the data, connected by a linear function. This corresponds to $x \approx 425$ on Figure 6.1.

now local extrema (maxima or minima) of the smoothing function. Also the typical "pock marks" located at the extremal points are seen, which is a typical vermin effect of the smoothing with *kriging, minimum curvature* and *RBFs*. We have encircled with two ellipses two such locations with pock marks. However, we see that in Figure 6.4, created with polysplines, these effects are almost invisible! The polysplines minimize the artifacts while the kriging, minimum curvature and RBFs cannot get rid of them.

An important advantage of the polysplines in the above applications is that the result is an **interpolation** polyspline! All the above methods give only approximations at the data points.[3] This *interpolation property* of the polysplines is much more important for computer-aided geometric design (CAGD) which will be considered in Section 6.2.

The conclusion may be drawn that, at least for magnetic data, the polysplines show definite superiority.

[3] At least only these implementations are available. There are implementations using (polynomial) splines which interpolate the data but they fail to produce a nice result in the magnetic data case due to the high oscillation of the data. The reader may check e.g. that the polynomial spline algorithms available in the NAG-algorithms in Matlab in www.nag.com/advisory/gamswww/k1a1b.html, are approximation but not interpolation.

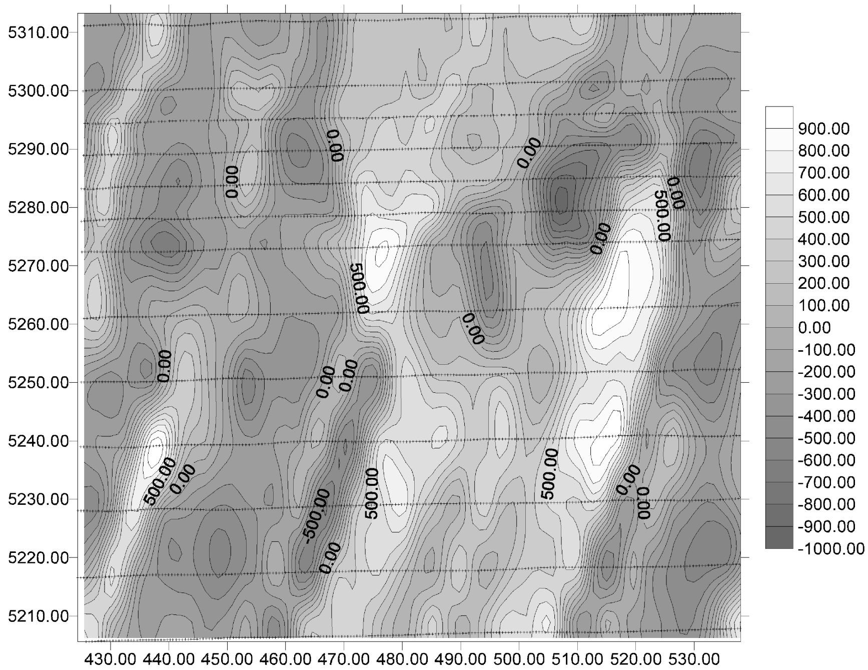

Figure 6.3. Result of applying the polysplines to the Cobb Offset data-level curves of the interpolation polyspline.

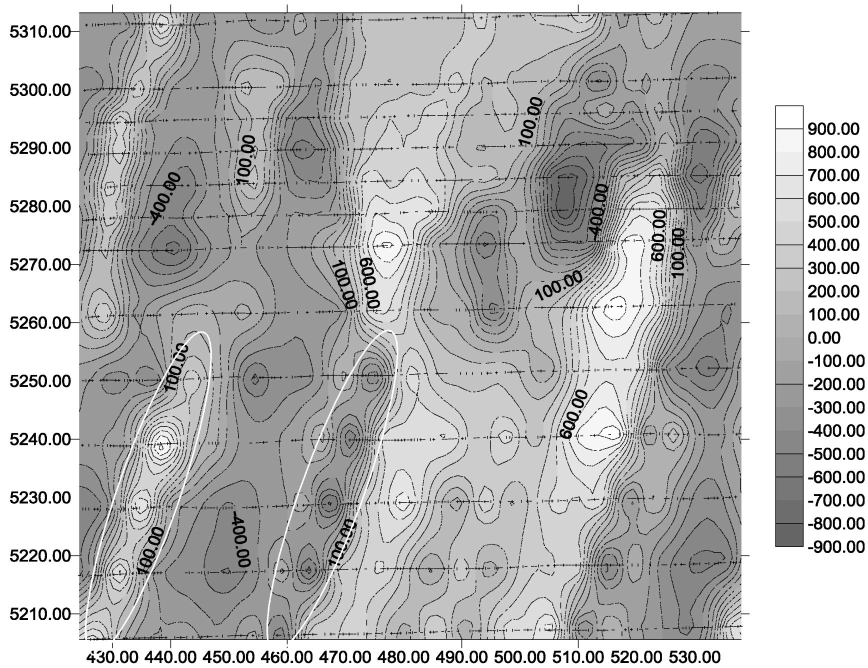

Figure 6.4. Result of applying radial basis functions (level curves). The two ellipses surround areas with strong “pock marks”.

6.2 Applications to computer-aided geometric design

One might speculate that the polysplines are extremely successful in the applications to magnetic field data owing to the similarity in the physical nature of the functions. Indeed, the magnetic field is related to harmonic functions and the polysplines which we have used are composed of biharmonic functions. The experiments below will put an end to such doubts.

6.2.1 Parallel data lines Γ_j

The first experiment we consider is the simplest one where all data curves Γ_j are parallel lines. There are 200 sample points (x_i, y_i) on every line Γ_j. So we have 7×200 sample points (see Figure 6.5). The surface is prescribed at the points of these curves, i.e. the value z_i is given at (x_i, y_i). They are uniform. This is a situation typical for the CAGD

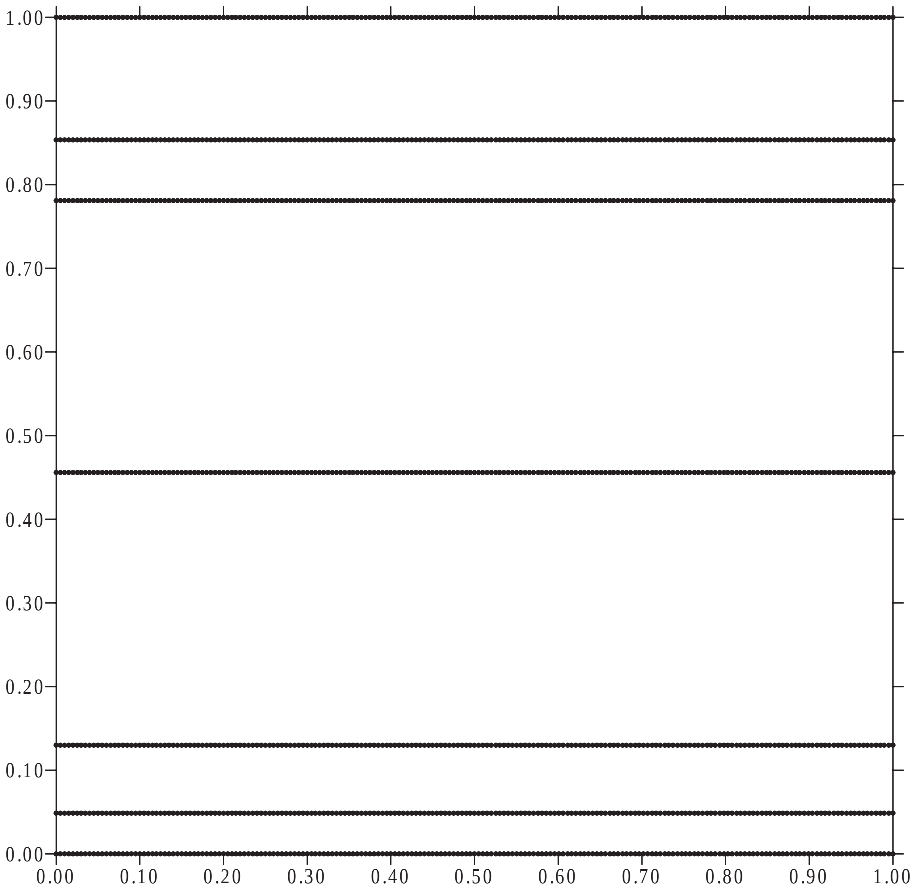

Figure 6.5. Data lines Γ_j are seven parallel straight lines.

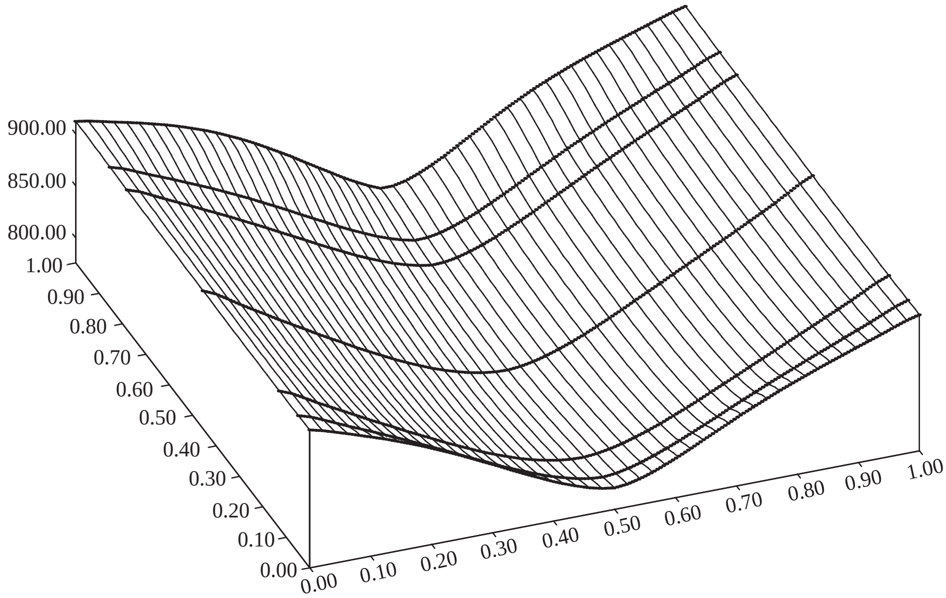

Figure 6.6. Biharmonic polyspline (surface) defined on the rectangle $[0, 1] \times [0, 1]$. Sample data points are shown on the surface u.

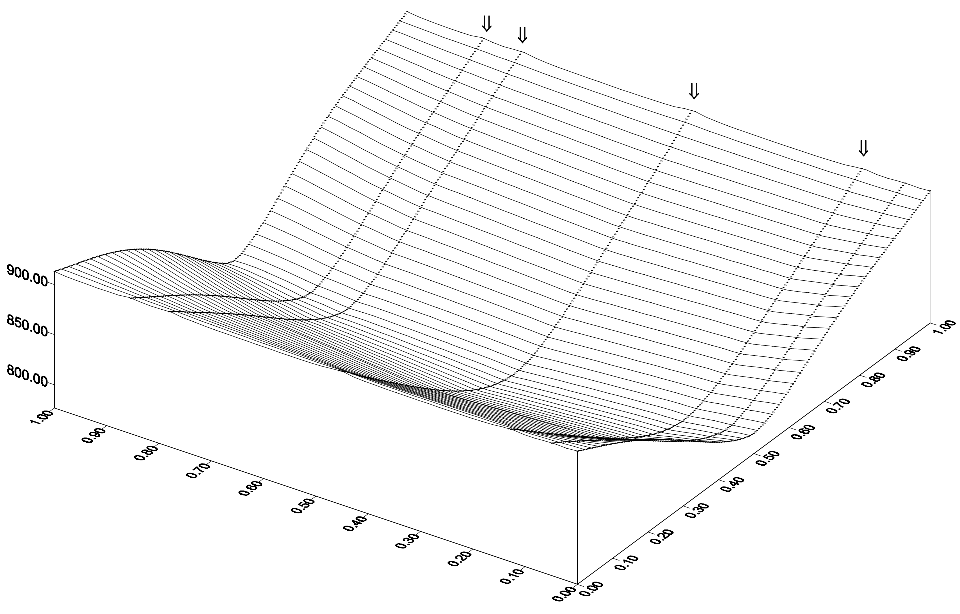

Figure 6.7. Surfaces created by radial basis functions. Arrows indicate places where there is non smoothness locations (non C^2-smoothness). Obviously these places are near the data lines Γ_j. They are very well visible on the border of Γ_j.

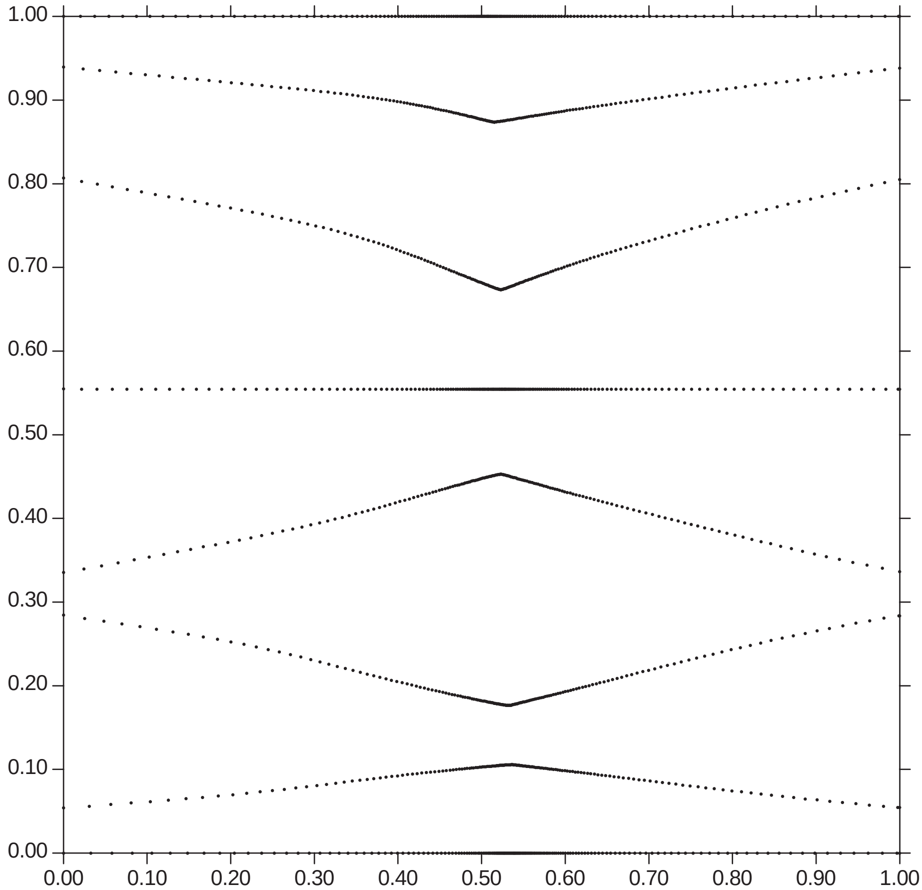

Figure 6.8. The data lines Γ_j are 8 curves with 200 sample points on each which are non–uniformly distributed on the curves.

where the "control curves" are prescribed (perhaps through control points) and we may supply the above points in a sufficient quantity. Thus the situation differs from the one we had in airborne magnetic data.

Figure 6.6 provides the result of the application of the polysplines, and Figure 6.7 provides the result of smoothing the same data set with RBFs. (The result is almost the same if we use kriging or minimum curvature.) We see that the result of the smoothing is reasonable and does not differ essentially from the result obtained using polysplines if one does not deepen. But a closer scrutiny reveals some non-smoothness locations which are essential for CAGD and make the result unsuitable for computerized design. They are indicated by arrows. Since the data do not oscillate as in the case of magnetic data, we provide the graphs of the functions in the usual "surface map" form.

Let us note that such a result would meet the standards in magnetism research but CAGD is much more sensitive to the small details of the form.

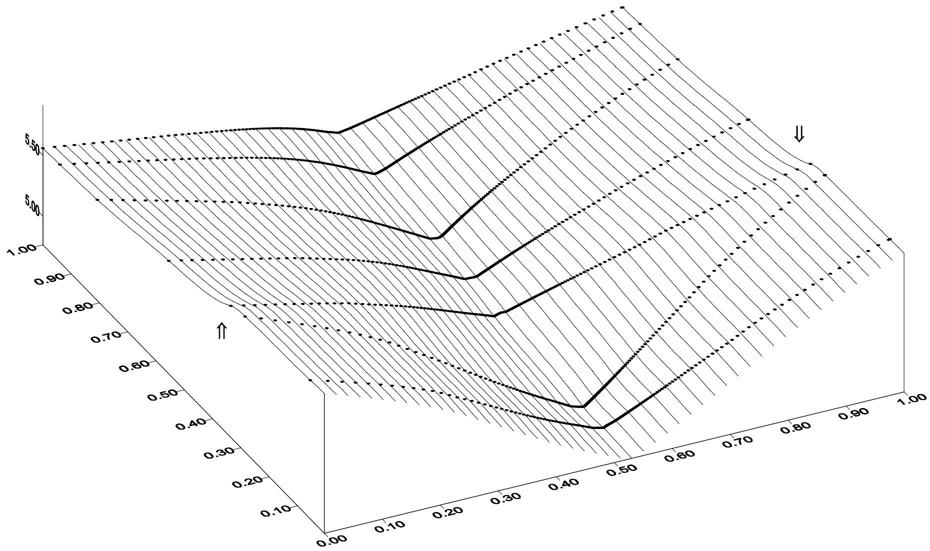

Figure 6.9. This is the surface of the interpolation polyspline. The arrows indicate locations of big tension – hence of big curvature.

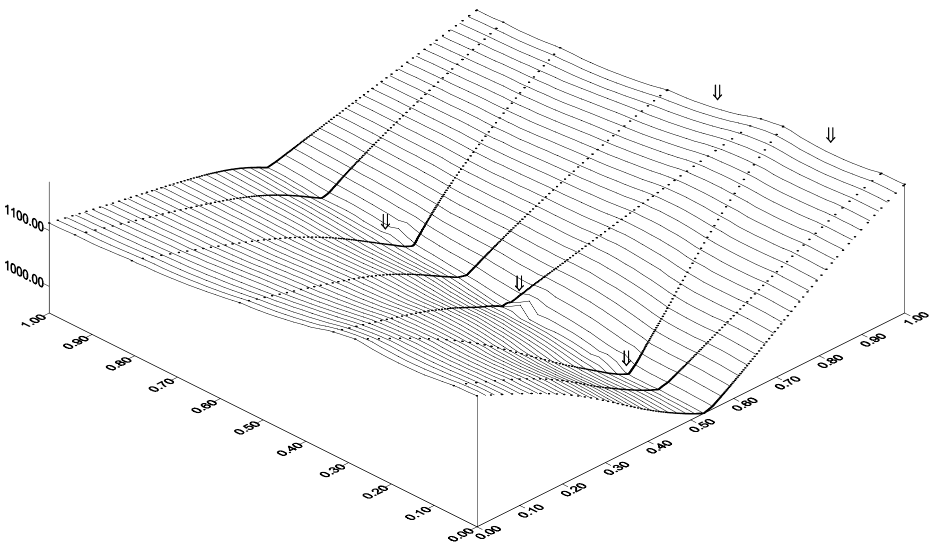

Figure 6.10. This is the surface created by kriging. The arrows show locations with obvious non-smoothness.

6.2.2 Nonparallel data curves Γ_j

The major strength of the theory of polysplines as presented in Part IV is that it allows extending data which are measured on curvilinear boundaries.

The next experiment is with data samples on eight curves Γ_j. Figure 6.8 provides the location of the sample points (x_i, y_i).

The result of the application of the interpolation biharmonic *polysplines* to the data is provided by Figure 6.9. We have indicated with arrows two locations where, due to the data, the curvature is greater. The result of applying kriging to the same data (neither RBFs nor minimum curvature gives a better result), shown in Figure 6.10, indicates that this problem is really difficult.

We have indicated with arrows the rather unpleasant for the eye roughnesses.

6.3 Conclusions

The above experiments and also other experiments carried out for similar data bring us to the following conclusions (which of course do not have the meaning of rigorous mathematical statements and extend only to the classes of data which have been considered):

- The polysplines definitely perform *better* than kriging, RBFs, minimum curvature etc., in magnetic field data problems. Unlike these methods they do not expose any "pock marks" or "line effects".
- The polysplines perform *better* than Kriging, RBFs and minimum curvature in CAGD problems where the controlling curves have parallel projections on the coordinate plane (x, y).[4]
- The polysplines perform *much better* than the above methods if the controlling curves do not have parallel projections on the coordinate plane (x, y).
- The above experiments and many others show that the biharmonic polysplines apparently have some shape-preserving properties. They do not oscillate more than the data in the magnetic field data case. The same is also true in the CAGD case where shape-preserving property is extremely important.

Remark 6.1 *The theoretical comparison between the polysplines and RBFs will be considered in the next volume to this book. Let us note only that the polysplines and the polyharmonic splines (of Madych) of the same order are close relatives due to the second Green formula (20.9), p. 422, by which the polysplines may be expressed in every subdomain where they are polyharmonic functions.*

[4] The comparison with the polynomial spline methods also favours the polysplines. The results are not given here.