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On the Approximation Through Polyharmonic Operators

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1 INTRODUCTION

Let us first introduce the necessary notions and notations.

We shall consider a bounded domain D in R^n. By $C(D)$ we will denote the space of functions continuous in D supplied with the norm:

$$\|f\| = \sup_{x \in \bar{D}} |f(x)| .$$

By $HC^r(D)$ we shall denote the space of functions $f(x)$ in $C(D)$ such that the p^{th} degree of the Laplacian, $p \geq 0$, is continuous in $\bar{D}$ and satisfies the inequality:

$$|\Delta^p f(x)| \leq 1 , \quad x \in \bar{D} . \tag{1}$$

The Laplacian is the operator given by

$$\Delta f(x) = \sum_{k=1}^{n} (\partial^2/\partial x_k^2) f(x) ,$$

and the powers of the Laplacian are defined inductively by the equalities: $\Delta^0 = id$, and for every $k \geq 0$, $\Delta^{k+1} = \Delta\Delta^k$.

By $B(x;t)$ we shall denote the open ball in R^n given by

$$B(x;t) = \{ y \in R^n : |x - y| < t \} .$$

We will assume that the domain D is regular in the sense of solubility of the Dirichlet problem, Helms (1963). Let the function $f(x)$ be continuous in the closure $\bar{D}$. Then there exists a harmonic function $h_f(x)$ solving the Dirichlet problem in D (Helms (1963)), i.e.

$$\Delta h_f(x) = 0 , \quad \text{for} \quad x \in D;$$

$$h_f(x) = f(x) , \quad \text{for} \quad x \in \partial D.$$

For every function continuous in $\bar{D}$ we shall define the function $Hf(x)$ by the following conditions:

$$Hf(x) = f(x) - h_f(x) , \quad x \in \bar{D}$$
$$Hf(x) = 0 , \quad x \bar{\in} \bar{D} . \tag{2}$$

Notice that the function $Hf(x)$ is continuous in the whole of R^n, and $HHf = Hf$.

2 POLYHARMONIC OPERATORS

Let for some $p \geq 1$, $K(x)$ be a polyharmonic function of order p, i.e. satisfies the following equation

$$\Delta^p K(x) = 0 ,$$

and is continuous for x in the ball

$$B(0;1) = \{ x \in R^n : 0 \leq |x| < 1 \}$$

We shall further assume that the domain $\bar{D}$ lies inside the ball $B(0;1/2)$. This will not be an essential restriction on the generality of our main Theorem since we may use similarity transforms preserving the result.

For every integrable in D function $f(x)$ we define the integral operator T with kernel K through the equality:

$$T[f;x] = \int_D K(x - y) f(y)\, dy =$$
$$= \int_{R^n} K(x - y) f(y)\, dy = \int_{R^n} K(z) f(x - z)\, dz . \tag{3}$$

Obviously, $T[f;x]$ is a function polyharmonic of order p. This fact follows from the inclusion $supp(f) \subseteq B(0;1/2)$.

Thanks to the second equality in (3) we obtain the following one:

$$\Delta T[f;x] = T[\Delta f;x] \tag{4}$$

which is true for every function $f(x) \in HC^1(D)$.

We put

$$E = \sup_{f \in HC^1} \| f(x) - T[f;x] \| . \tag{5}$$

Further, for every integrable in D function $f(x)$, and every integer $r \geq 0$ we define the operator A^r

by putting

$$A^{o}[f;x] = T\,[f;x]$$

and inductively, for every $r \ge 1$ we put

$$A^{r+1}[f;x] = HA^{r}[f;x] + T\,[H(f(\xi) - A^{r}[f;\xi])\ ;\ x\ ] \qquad (6)$$

for every $x \in \bar{D}$.

It is evident that the opeartors A^{r} are linear and $A^{r}[f;x]$ is a polyharmonic function of order p. This is easy to see inductively. Indeed , we have the equality

$$\Delta^{p}A^{r+1}[f;x] = \Delta^{p}HA^{r}[f;x] +$$

$$+ \Delta^{p}T\,[H(f(\xi) - A^{r}[f;\xi])\ ;\ x\ ] \ .$$

Since $\operatorname{supp} H(f-A^{r}f) \subseteq D(0;1/2)$, we obtain the equality

$$\Delta^{p}A^{r+1}[f;x] = \Delta^{p}HA^{r}[f;x] \ .$$

Let us also remark that due to (4) we have the equality

$$\Delta A^{r+1}[f;x] = \Delta HA^{r}[f;x] + \Delta T\,[H(f(\xi) - A^{r}[f;\xi])\ ;\ x\ ] =$$

$$= \Delta A^{r}[f;x] + T\,[\Delta f(\xi) - \Delta A^{r}[f;\xi]\ ;\ x\ ] \ .$$

The last proves by induction in r that

$$\Delta A^{r}[f;x] = A^{r}[\Delta f;x] \ . \qquad (7)$$

The operators A^{r} are also well defined for the functions

$f(x)$ integrable in D since the kernels K are defined and continuous in the ball $B(0;1)$.

3 THE THEOREM

The following statement, which is a generalization of a theorem of Jackson (for the one-dimensional result see Dzyadyk (1977), Ch. IV or Meinardus (1967), holds true:

THEOREM Let the domain $\bar{D}$, lying in the ball $B(0;1/2)$, be regular in the sense of the solubility of the Dirichlet problem. Then for every integer $r \ge 0$ and every function $f(x) \in HC^{r}(D)$, the following inequality holds:

$$|f(x) - A^{r}[f;x]| \le \| \Delta^{r}f(x) - T\,[\Delta^{r}f;x]\ \|.E^{r} \ , \qquad (8)$$

where A^{r} is the operator defined by (6).

PROOF. We will proceed by induction in r.

For $r = 0$ the inequality is evident since $A^{o} = T$ by definition.

Let us suppose that (8) is true for some $r \ge 0$. We will prove it for $r + 1$.

Suppose now that we have a function $f(x) \in HC^{r+1}(D)$, i.e. it satisfies

$$\Delta^{r+1}f(x) \in C(\bar{D}). \qquad (9)$$

Clearly, we also have that $\Delta f(x) \in HC^{r}(D)$.

Let us put

$$\phi(x) = Hf(x) - HA^r[f;x]. \qquad (10)$$

According to (7) we have the equality:

$$\Delta A^r[f;x] = A^r[\Delta f;x]. \qquad (11)$$

Let us apply the inductive hypothesis to the function $\Delta f(x)$, which gives the following inequality for $\Delta\phi(x)$:

$$|\Delta\phi(x)| = |\Delta Hf(x) - \Delta HA^r[f;x]| = \qquad (12)$$

$$= |\Delta f(x) - A^r[\Delta f;x]| \le$$

$$\le \| \Delta^{r+1}f(x) - A^o[\Delta^{r+1}f;x] \|.E^r =$$

$$= \| \Delta^{r+1}f(x) - T[\Delta^{r+1}f;x] \|.E^r ,$$

for every $x \in D$.

The last inequality gives that

$$M^{-1}\phi(x) \in HC^1(D)$$

where we have defined the constant M through the equality:

$$M = \| \Delta^{r+1}f(x) - T[\Delta^{r+1}f;x] \|.E^r .$$

Equality (5) for the constant E gives the following estimate:

$$\| M^{-1}\phi(x) - A^o[M^{-1}\phi;x] \| \le E ,$$

for every $x \in D$.

Multiplying the last inequality by M, and in view of the definition of the operator A^r , we obtain the following:

$$\| \phi(x) - A^o[\phi;x] \| =$$

$$= \| Hf(x) - HA^r[f;x] - T[H(f(\xi) - A^r[f;\xi]) ; x] \| =$$

$$= \| f(x) - A^{r+1}[f;x] \| \le ME =$$

$$= \| \Delta^{r+1}f(x) - T[\Delta^{r+1}f;x] \|.E^{r+1} .$$

This is in fact the statement of the Theorem for $r + 1$.

The Theorem is proved. Q.D.E.

Let us finally remark that the construction used in the Theorem proved above is basic for the proof of a Jackson type result about the approximation through polyharmonic functions in the domain D , Kounchev (1991).

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Inequalities for Some Special Functions

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1. INTRODUCTION.

The purpose of the present paper is to describe some elementary methods which can be used to establish inequalities for some special functions. In particular we derive inequalities for the modified Bessel functions $I_\nu(t)$ and $K_\nu(t)$. The bounds are obtained directly from the recurrence relations satisfied by these functions.

Furthermore we describe a method for getting a simple inequality for elliptic integrals K and E of the first and the second kind, respectively.

2. BOUNDS FOR MODIFIED BESSEL FUNCTIONS.

Let $I_\nu(t)$ and $K_\nu(t)$ be the modified Bessel functions of the first and the third kind, respectively. Several authors studied inequalities for these functions. For example Bordelon [2] and Ross [8] proved the following result

$$e^{x-y}\left(\frac{x}{y}\right)^\nu < \frac{I_\nu(x)}{I_\nu(y)} < e^{y-x}\left(\frac{x}{y}\right)^\nu, \quad \nu > 0,\ 0 < x < y$$

and Ifantis and Siafarikas [3] established the upper bound

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