Polyharmonically exact formula of Euler-Maclaurin, multivariate Bernoulli functions, and Poisson type formula

Dimiter Dryanov, Ognyan Kounchev

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Authors: Dimiter Dryanov, Ognyan Kounchev

ABSTRACT. In the present paper we find multivariate Bernoulli functions and two formulas of Euler-Maclaurin type.

RÉSUMÉ. Dans cet article nous introduisons fonctions multivariables de Bernoulli et deux formules de type d'Euler-Maclaurin.

Version française abrégée:

Nous donnons ici une généralisation multidimensionnelle de la formule d'Euler-Maclaurin qui est basée sur une généralisation en plusieurs variables des fonctions de Bernoulli en une variable.

L'idée centrale de cette Note est la généralisation de la propriété de Réproduction par Rapport aux Domaines Voisins (RRDV) qui est une propriété remarquable et caractéristique de la formule d'Euler-Maclaurin unidimensionnelle (1).

Dans la Note présente on approche les intégrales $d-$dimensionelles sur le cube unitaire $D_{uc}$ par les combinaisons linéaires d'intégrales sur des surfaces de dimensions plus petites, respectivement 0 (des valeurs fonctionnelles), 1, ..., $d-1$.

La propriété RRDV est conservée. Les formules généralisées d'Euler-Maclaurin sont exactes pour les fonctions polyharmoniques d'un degré donné (c'est un espace de dimension infinie), ou pour les solutions d'une autre équation différentielle partielle.

C'est une manifestation du paradigme polyharmonique qui trouve beaucoup d'applications dans les théories d'approximation et splines multivariables, voir [7], [8], [9].
Nous avons deux formules d’intégration approximative de type d’Euler-Maclaurin. La première, (11),(12),(13),(14), se base sur les fonctions pairs de Bernoulli, \( P_{2k,d}(x) \), \( k = 1, \ldots, m \). Cette formule est exacte pour les solutions de l’équation polyharmonique \( \Delta^{m} f(x) = 0 \), \( x \in D_{uc} \).

La seconde formule, (15),(16),(17),(18), se base sur les fonctions de Bernoulli d’ordre impair, \( P_{2k+1,d}(x) \), \( k = 1, \ldots, m \). Cette formule d’intégration approximative est exacte pour les solutions de l’équation \( \Delta^{m} \partial_{x_1}^d \cdots \partial_{x_d}^d f(x) = 0 \), \( x \in D_{uc} \).

Remarquons que les deux formules, (11) et (15), possèdent la propriété RRDV pareillement au cas unidimensionnel. Cette propriété assure une complexité minimale calculatoire par analogie à la formule d’Euler-Maclaurin unidimensionnelle.

Les deux formules d’Euler-Maclaurin engendrent des formules du type de Poisson (d’ordre pair et impair). Nous donnons ici la formule qui correspond au cas impair, voir (19). La formule de Poisson (19) est exacte pour les polynômes trigonométriques de degré \( n-1 \), où \( n \) est l’ordre de la règle composée trapézoïdale.

On peux trouver les résultats complets dans [2].

**Main text (English).**

One of the most beautiful devices of classical analysis which has found numerous applications in number theory and divergent series [1, p. 7 in Vol. 1, Chapters 7,8], [6, Chapter 13], and which plays profound role in numerical integration [3, Chapter 2.9], is the Euler-Maclaurin formula. In the present paper we provide a multivariate generalization of the Euler-Maclaurin formula which is based on an appropriate generalization of the Bernoulli functions.

1. The celebrated quadrature formula of Euler-Maclaurin is the following (cf. [3, p. 109], [6, p. 323]):

\[
\int_{a}^{b} f(t) \, dt = TR(f) - BT(f) + R_{n,k}(f) \tag{1}
\]

with remainder of even order or of odd order, for \( f \in C^{2k+1}[a,b] \),

\[
R_{n,k}(f) = h^{2k} \int_{a}^{b} P_{2k} \left( n \frac{t-a}{b-a} \right) f^{(2k)}(t) \, dt = h^{2k+1} \int_{a}^{b} P_{2k+1} \left( n \frac{t-a}{b-a} \right) f^{(2k+1)}(t) \, dt, \tag{2}
\]

compound trapezoidal rule \( TR(f) \) and boundary terms \( BT(f) \) given by

\[
TR(f) = h \left[ \frac{1}{2} f(a) + \sum_{i=1}^{n-1} f(a + ih) + \frac{1}{2} f(b) \right], \quad BT(f) = \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!} h^{2j} \left[ f^{(2j-1)}(b) - f^{(2j-1)}(a) \right].
\]

Here the functions \( P_k(t) = B_k(t) / k! \) are the normalized Bernoulli functions (see [3, p. 109], [6, p. 323], [10, Chapter 1]) and the Bernoulli numbers are related to them by \( B_k / k! = P_k(0) = P_k(1) \). Note that \( B_{2k+1} = 0 \), \( k \geq 1 \).
Now let us explain the RRND property. Imagine that we write formula (1) (putting \( n = 1 \)) for each subinterval \([a + jh, a + (j + 1)h]\), \( j = 0, ..., n - 1 \). After summing up over \( j \) we see that the terms with interior derivatives cancel each other and we obtain formula (1) itself for the interval \([a, b]\), i.e. it has reproduced itself!

Why is the RRND property interesting and important from computational point of view? To answer this question, let us make further subdivision into \( 2n \) equal parts in the interval \([a, b]\) and write (1) for \( h = (b - a) / 2n \). We see that we do not have to compute anything else except the functional values by the trapezoidal rule, and put \( h/2 \) instead of \( h \) in (1). This shows the minimal computational complexity of formula (1) with respect to subdivisions which makes it very efficient from algorithmic point of view.

2. What concerns the interest in the RRND property in the multivariate case let us say that it has been alluded in [3, Section 5.8, p. 286, top of page] as a compound rule. An analog to the RRND property is considered in [15, Chapter 4] where it is understood as "extension of formulas" from a lower–dimensional to higher–dimensional domains.

In the known generalizations of the Euler-Maclaurin formula (see [4], [11], [12], [14], [15], and references there) the integral is approximated in terms of values of the function at the integer points (or the points of some lattice), and the formulas are exact for polynomials up to a certain degree. So far (to the best of our knowledge) none of the results in these references generalizes the RRND property.

On the other hand, the main idea of the present work is to represent the integral of the function \( f(x) \) over a \( d \)-dimensional domain like an integral of the same function \( f(x) \) over surfaces of lower dimension, \( d - 1, d - 2, \) etc., in such a way that the formula obtained would possess the property of RRND (at least for some special domains). An important feature of this approach is that we obtain formulas which are exact for functions which are polyharmonic of a certain degree or which are solutions to similar higher order partial differential equations. The last fact is a manifestation of what we call polyharmonicity paradigm which has proved to be very successful in multivariate approximation and spline theory (see [7], [8], [9]).

3. One of the interesting discoveries of the present research was the observation that the theory of the one-dimensional Bernoulli polynomials may be viewed in the framework of the Hodge theory [5]. Namely, the sequence \( P_k \) splits into odd and even order parts as solutions to the Poisson problems \( u''(t) = P_k(t), k \geq 0 \), in 1–periodic functions \( u \), starting with the 1–periodic functions \( P_0(t) = 1 - \sum_{j=-\infty}^{\infty} \delta(t - j) \) and \( P_1(t) = t - \frac{1}{2}, 0 < t < 1; P_1(j) = 0, j \in \mathbb{Z}. \) The existence and uniqueness follow from the orthogonality condition \( \int_0^1 u(t) \, dt = 0. \)

Now, for the multivariate generalization of the Bernoulli functions, it is of basic importance to choose proper initial functions \( P_{0,d}(x) \) and \( P_{1,d}(x) \), which are multidimensional analogs to \( P_0 \) and \( P_1 \). For the proper choice of such we are lead by the idea explained above to represent the integral \( \int_D f dx \) like a sum of
integrals over manifolds of lower dimension, not only zero-dimensional.

We choose as initial functions the following

\[ P_{0,d}(x) = d - \sum_{k=1}^{d} \sum_{j=-\infty}^{\infty} \delta(x_k - j) \quad x = (x_1, ..., x_d) \]  

(3)

and

\[ P_{1,d}(x) = P_1(x_1) P_1(x_2) ... P_1(x_d) , \]  

(4)

and construct two series of Bernoulli functions. The first one \( P_{2k,d}(x) \) is the multidimensional generalization of the even order Bernoulli functions \( P_{2k} \) and the second series \( P_{2k+1,d}(x) \) corresponds to the odd order Bernoulli functions \( P_{2k+1} \). They are \( 1 \)-periodic in every variable and satisfy

\[ \Delta P_{k,d} = P_{k-2,d} \quad k \geq 2 \]  

(5)

as well as the orthogonality condition

\[ \int_{D_{uc}} P_{k,d}(x) \, dx = 0 \quad k \geq 0, \]  

(6)

where \( D_{uc} \) denotes the \( d \)--dimensional unit cube. The last condition is necessary for the solubility of the Poisson problem

\[ \Delta u = P_{k,d}(x) \]  

(7)

on the torus \( T^d = (S^1)^d \) and represents the orthogonality to the harmonic space consisting of the constants only [5, p. 89]. Now by the same arguments the solution \( u \) is determined on the torus up to a constant. Hence, \( u \) is completely determined by putting

\[ \int_{D_{uc}} u(x) \, dx = 0. \]  

(8)

One may prove that

\[ P_{2k,d}(x) = \sum_{j=1}^{d} P_{2k}(x_j) , \]  

(9)

where \( P_{2k} \) are the one-dimensional normalized Bernoulli functions. For the odd order we have only multiple Fourier series expression

\[ P_{2k+1,d}(x) = (-1)^{d+k} \frac{1}{(2\pi)^{2k}} \sum_{l_1=1}^{\infty} ... \sum_{l_d=1}^{\infty} \frac{1}{|l|^2} \left\{ \prod_{j=1}^{d} \frac{2 \sin(2\pi l_j x_j)}{2\pi l_j} \right\} , \]  

(10)

where \( |l|^2 = l_1^2 + ... + l_d^2 \).

Let us remark that these functions are multivariate polysplines in the sense of [8].

4. Thus we have generated two series of multivariate Bernoulli functions. Respectively, we obtain two different Euler-Maclaurin type cubature formulas. The following one is based on the even order multivariate Bernoulli functions.
Theorem 1  For every function \( f \in C^{2m} (\overline{D}_{uc}) \) the following formula holds

\[
I (f) = \int_{D_{uc}} f (y) \, dy = TR_{n,d} (f) - BT_{n,m,d} (f) + R_{n,m,d} (f) \tag{11}
\]

where by using the notation \( \widehat{dy}_j = dy_1...dy_{j-1}dy_{j+1}...dy_d \), \( j = 1, ..., d \), the compound trapezoidal sum is given by

\[
TR (f) = (nd)^{-1} \sum_{k=1}^{n-1} \sum_{j=1}^{d} \int_0^1 ... \int_0^1 f (y)|_{y_j=k/n} \, \widehat{dy}_j + (2nd)^{-1} \sum_{k=0}^{d} \sum_{j=1}^{d} \int_0^1 ... \int_0^1 f (y)|_{y_j=k/n} \, \widehat{dy}_j, \tag{12}
\]

the boundary terms by

\[
BT_{n,m,d} (f) = d^{-1} \sum_{j=1}^{d} \sum_{k=1}^{m} n^{-2k} \int_0^1 ... \int_0^1 \left\{ \frac{\partial}{\partial y_j} \Delta^{k-1}_y f (y)|_{y_j=1} - \frac{\partial}{\partial y_j} \Delta^{k-1}_y f (y)|_{y_j=0} \right\}, \tag{13}
\]

and the remainder by

\[
R_{n,m,d} (f) = d^{-1} n^{-2m} \int_{D_{uc}} \Delta^m f (y) P_{2m,d} (ny) \, dy. \tag{14}
\]

Let us remark that the above cubature formula (11) is exact for polyharmonic functions of order \( m \).

In order to formulate the second multivariate Euler-Maclaurin type formula we introduce the following subdivision of \( D_{uc} \), into smaller cubes \( D_{\alpha,n} = \prod_{i=1}^{d} [\alpha_i/n, \alpha_{i+1}/n] \), \( \alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{Z}^d \). Evidently, we have \( D_{uc} = \bigcup \{ D_{\alpha,n} : \alpha \in \mathbb{Z}^d, 0 \leq \alpha_i \leq n-1, i = 1, ..., d \} \).

Let \( \Omega_j \) denote the set of all \( j \)-dimensional edges of the cubes \( D_{\alpha,n} \) for \( 0 \leq \alpha_i \leq n-1, i = 1, ..., d \). For any \( K \in \Omega_j \) let the number of the cubes \( D_{\alpha,n} \) for which \( K \subset D_{\alpha,n} \) be denoted by \( ind_{D_{uc}} (K) \). It is easy to see that \( ind_{D_{uc}} (K) = 2^{r-j} \) where \( K \) is contained in an \( r \)-dimensional edge of \( D_{uc} \), \( j \leq r \leq d \).

The odd order Bernoulli functions generate the following cubature formula of Euler-Maclaurin type.

Theorem 2  For every function \( f \in C^{2k+d} (\overline{D}_{uc}) \) the following formula holds

\[
I (f) = \int_{D_{uc}} f (y) \, dy = (-1)^d \left[ TR_{n,d} (f) + BT_{n,m,d} (f) + R_{n,m,d} (f) \right], \tag{15}
\]

where the compound trapezoidal sum is

\[
TR_{n,d} (f) = \sum_{k=0}^{d-1} (-1)^{k+1} \frac{1}{2d-k} \frac{1}{n^{d-k}} \sum_{\dim(K_1)=k} ind_{D_{uc}} (K_1) \int_{K_1} f (y) \, d\sigma_{k,y}, \tag{16}
\]
the boundary terms are
\[ BT_{n,m,d}(f) = \sum_{i=1}^{m} n^{-(2i+d)} \int_{\partial \overline{D_{uc}}} \Delta_{y}^{i-1} \frac{\partial^{d}}{\partial y_{1}\ldots \partial y_{d}} f(y) \frac{\partial}{\partial \nu_{y}} P_{2i+1,d}(ny) \, dy, \]
\[ (17) \]
the remainder is
\[ R_{n,m,d}(f) = n^{-(2m+d)} \int_{D_{uc}} \Delta_{y}^{m} \frac{\partial^{d}}{\partial y_{1}\ldots \partial y_{d}} f(y) P_{2m+1,d}(ny) \, dy, \]
\[ (18) \]
where \( d\sigma_{k,y} \) is the \( k \)-dimensional measure on \( K_{1} \) and \( \frac{\partial}{\partial \nu_{y}} \) denotes the exterior normal derivative to \( \partial D_{uc} \).

Let us remark that formulas (11) and (15) have the RRND property, namely, if we sum the formulas which are written for two neighboring cubes the interior terms containing derivatives cancel. Let us remark that, as in the one-dimensional case (see [3, p. 109, f. (2.9.16), [10, p. 216]), the boundary terms \( BT_{n,m,d} \) in both Theorem 1 and 2 may be used for convergence acceleration of the compound trapezoidal rules \( TR_{n,d} \).

The following estimates of the error hold

**Theorem 3**

1. Let the function \( f \in C^{2m}(D_{uc}) \). Let \( |\Delta_{y}^{m} f| \leq M \). Then in the notations of Theorem 1 we have
\[ |I(f) - TR_{n,d}(f) + BT_{n,m,d}(f)| \leq MC_{1} \zeta(2m)/n^{2m}, \]
2. Let the function \( f \in C^{2m+d}(D_{uc}) \). Let \( |\Delta_{x}^{m} \frac{\partial^{d}}{\partial x_{1}\ldots \partial x_{d}} f(x)| \leq M \). Then in the notations of Theorem 2 we have
\[ |I(f) - (-1)^{d} TR_{n,d}(f) - (-1)^{d} BT_{n,m,d}(f)| \leq MC_{2} \left[ \zeta \left( 1 + \frac{2m}{d} \right) \right]^{d}/n^{2m+d}, \]
Here \( \zeta \) denotes the Riemann zeta function, and \( C_{1} = 2/(2\pi)^{2m} \), \( C_{2} = 2^{d}/\left[ d^{m}(2\pi)^{2m+d} \right] \).

Both above Euler-Maclaurin formulas are related to Poisson type formulas. The one corresponding to Theorem 2 is:
\[ \int_{D_{uc}} f(x) \, dx = (-1)^{d} \left[ TR_{n,d}(f) + \sum_{k=1}^{d} \left\{ \hat{f}(nl) : l \in Z^{d}, l_{k} \neq 0 \right\} \right], \]
\[ (19) \]
where \( f \in C(D_{uc}), \hat{f}(\xi) = \int_{D_{uc}} f(x) e^{-2\pi i x.\xi} \, dx \), and satisfies the asymptotic condition \( |\hat{f}(\xi)| \leq C (1 + |\xi|)^{-d-\delta} \) for \( \xi \in Z^{d} \).

The cubature formula (19) is exact for all \( d \)-dimensional trigonometric polynomials of degree \( \leq n - 1 \) with respect to every coordinate variable. It is convenient for approximate integration of smooth periodic functions, that is for functions with fast decreasing Fourier coefficients.
The complete proofs are available in [2].

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References


D. DRYANOV
Permanent address:
Faculty of Mathematics, University of Sofia
J. Boucher Str. 5, Sofia, Bulgaria
ddryan@bgearn.acad.bg

O. KOUNCHEV
Permanent address:
Institute of Mathematics, Bulgarian Academy of Sciences
Acad. G. Bonchev Str. 8, 1113 Sofia, Bulgaria
e-mail: kounchev@math.uni-duisburg.de, kounchev@bgearn.acad.bg

Address for correspondence:
O. KOUNCHEV
Department of Mathematics,
University of Duisburg, Lotharstr. 65, 47048 Duisburg, GERMANY;
telephone: ++49-203-379 2671
fax: ++49-203-379 3139
e-mail: kounchev@math.uni-duisburg.de, kounchev@bgearn.acad.bg