Limit theorems for subcritical age-dependent branching processes with two types of immigration

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We prove a strong law of large numbers and a central limit theorem for such processes.

Similar conclusions are obtained for their discrete-time counterparts (lifetime per individual equals one), called Galton- Watson processes with immigration at zero and immigration of renewal type (GWPIOR).

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Our approach is based on the theory of regenerative processes, renewal theory and occupation measures and is quite different from those in earlier related work using analytic tools.

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Galton- Watson process with immigration at 0 (Foster-Pakes model)- Foster [4](1971) and Pakes [10–12] (1971, 1972, 1978)

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BHPIOR - Slavtchova-Bojkova (2002) [15] - LLN under stronger conditions and by analytic means

3. Model description

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$\{Z(t)\}_{t\geq 0}$ be a Bellman-Harris branching process with immigration only in the state zero (BHBPIO) with model parameters an individual lifetime distribution G with $G(0) = 0$, an offspring distribution $(p_j)_{j\geq 0}$ with p.g.f. $f(s)$, a number of immigrants distribution $(g_j)_{j\geq 0}$ with p.g.f. $g(s)$, a distribution D of the delay times elapsing after extinction epochs before new immigrants enter the population.

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The discrete-time variant $(Z(n))_{n\geq 0}$, where $t \in [0, \infty]$ is replaced with $n \in \mathbb{N}_0$, and where $G = \delta_1$ (Dirac measure at 1) and D is a distribution on $\mathbb{N}_0$, will be called a Galton-Watson process with immigration at 0 (GWPIO).

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$(\sigma_n)_{n\geq 0}$ - zero-delayed renewal process with increment distribution F

The numbers of immigrants $(Y_n)_{n\geq 1}$ are assumed to be iid r.v.'s with probability generating function (pgf) $h(s)$. The Y_n are supposed to be the numbers of individuals entering the population at times σ_n. A further integer-valued random variable Y_0 gives the number of ancestors of the considered population. It is assumed that σ_n, $(Y_n)_{n\geq 1}$, Y_0 and all Z_{ij} are mutually independent.

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$$N(t) \stackrel{def}{=} \sup\{n : \sigma_n \leq t\}$$

the number of renewal events in the sequence $\sigma_n, n = 1, 2, \ldots$ during the time interval $[0, t]$.

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and

$$Z_i(t) = \sum_{i=0}^{Y_i} Z_{ij}(t), \quad t \geq 0$$

is a BHPIO with Y_i ancestors.

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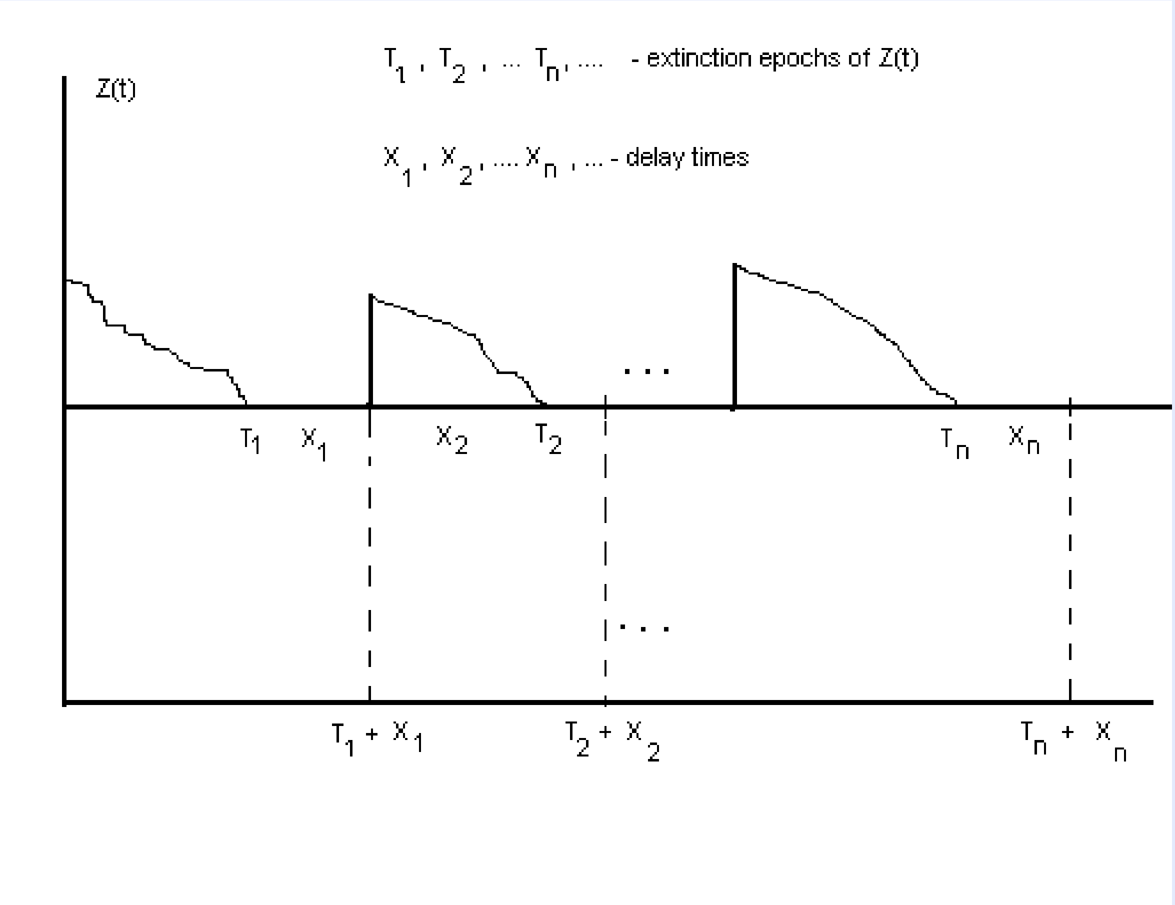
Its discrete time variant, where the Z_i are GWPIO and σ_n forms a discrete renewal process, is called a Galton-Watson process with immigration at zero and immigration of renewal type (GWPIOR).

4. Sample Path of BHIO



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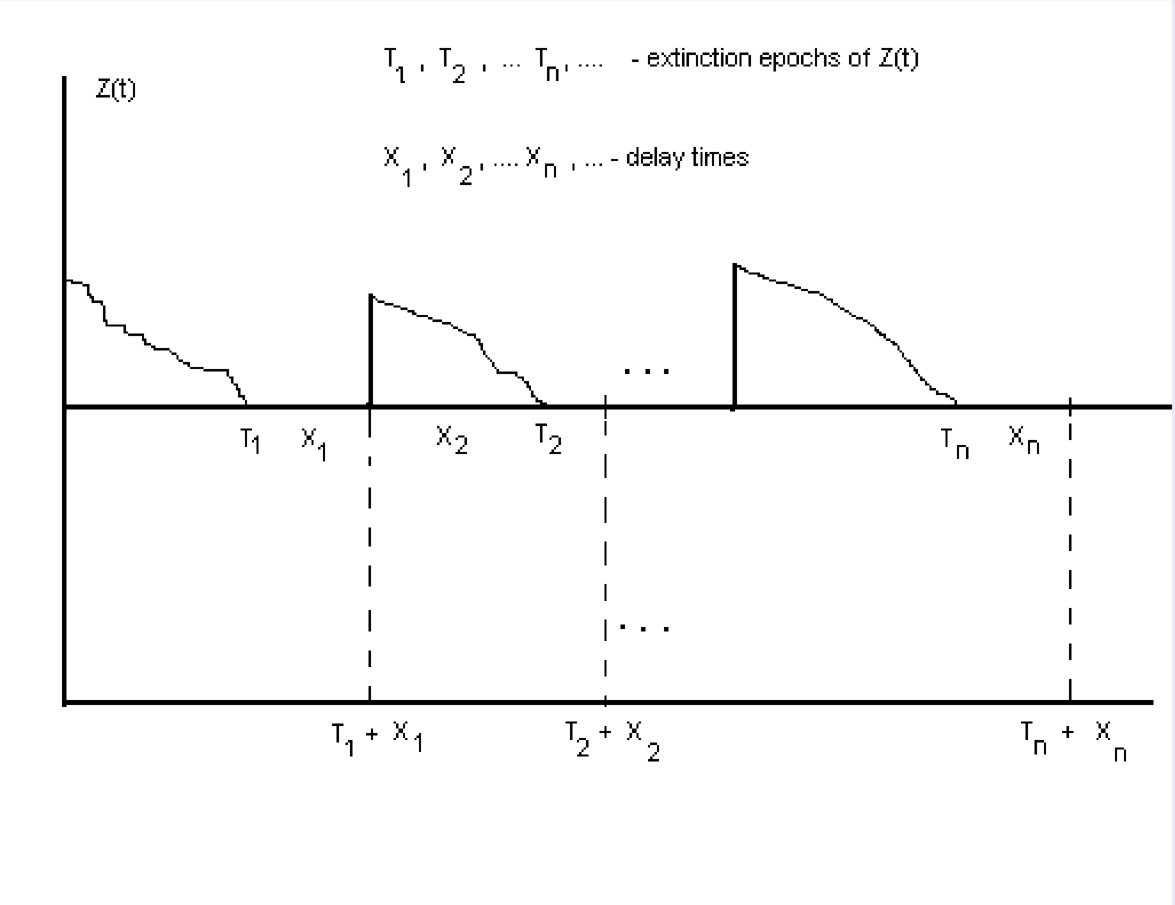
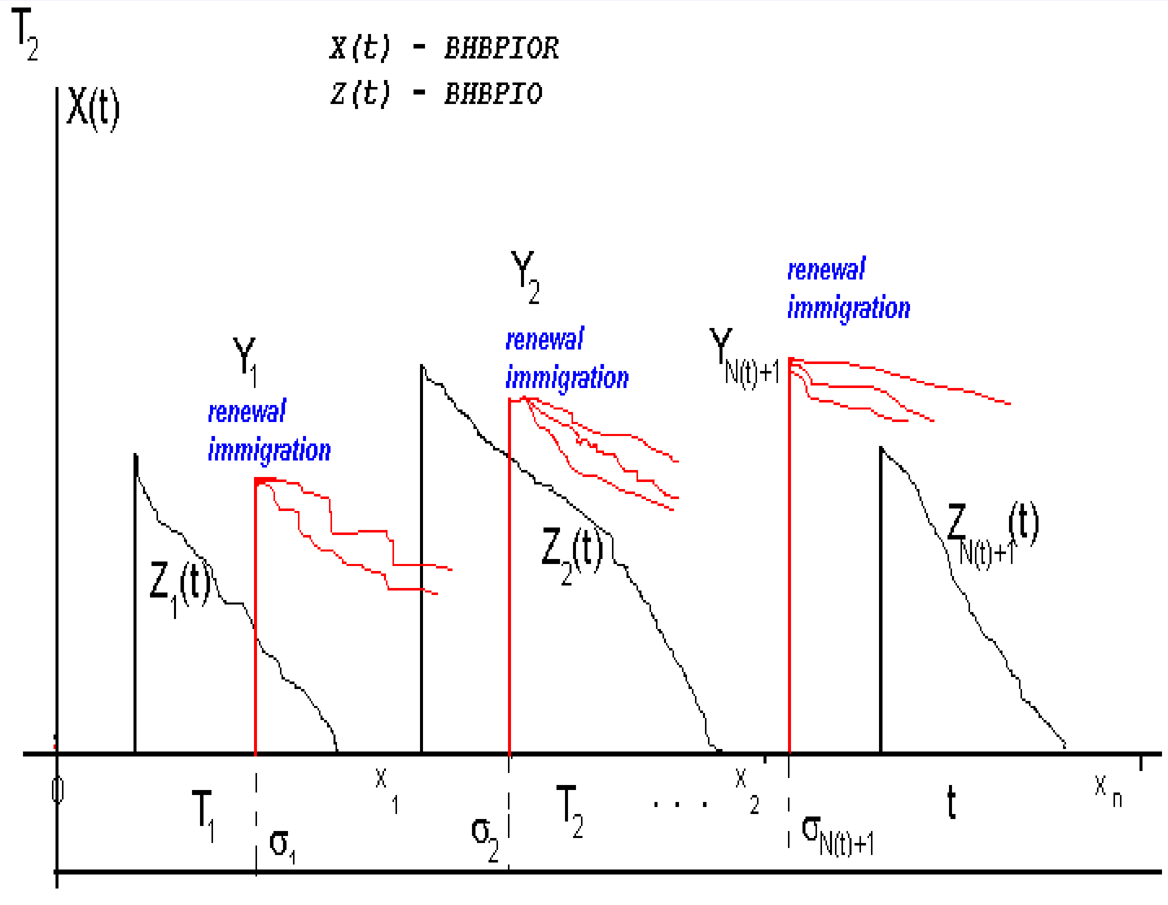


5. Sample Path of BHIOR



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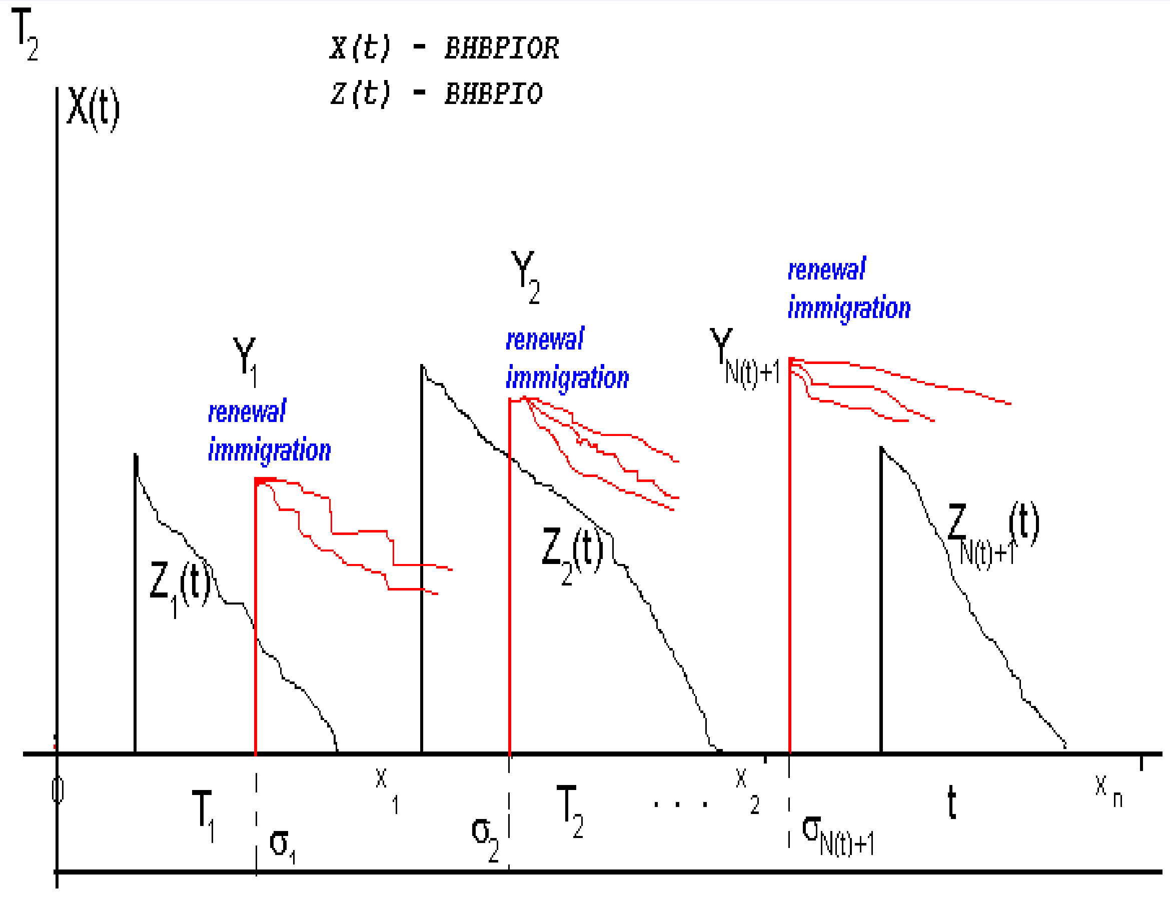

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Let the pth moments of $(p_k)_{k\geq 0}$, G, F, D be denoted as m_p , $m_{G,p}$, $m_{F,p}$ and $m_{D,p}$, respectively.

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Put $\mathbb{P}_k = \mathbb{P}(.|Z(0) = k)$ for $k \geq 0$ and $\mathbb{P}^* \stackrel{def}{=} \sum_{k\geq 0} g_k \mathbb{P}_k$, so that the initial distribution of $(Z(t))_{t\geq 0}$ under $\mathbb{P}^*$ is $(g_k)_{k\geq 0}$.

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is an ordinary BHP with lifetime distribution G (or GWP with $G = \delta_1$), offspring distribution $(p_j)_{j\geq 0}$ and extinction time T_1, which has finite mean under every $\mathbb{P}_k$.

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When moving to the process $(X(t))_{t\geq 0}$ we put $Z(t) \stackrel{def}{=} Z_0(t)$ for $t \geq 0$ and retain the previous notation.

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Let $(Z(t))_{t \geq 0}$ be a subcritical BHPIO with arbitrary ancestor distribution, $g'(1) < \infty$, and $m_G < \infty$. Suppose also $m_D < \infty$, and that the convolution $G * D$ is nonarithmetic.

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$$\Lambda_2(\infty) = \frac{g'(1)m_G}{(1-m)\beta} + \frac{1}{\beta}(\frac{g'(1)f''(1)}{1-m} + g''(1)) \int_0^\infty m^2(t)dt < \infty$$

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Then

$$\frac{X(t)}{t} \xrightarrow{\mathbb{P}} \frac{g'(1)h'(1)m_G}{(1-m)m_F\beta}, \quad t \to \infty$$

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Then

$$\frac{X(t) - (N(t)+1)h'(1)\Lambda(\infty)}{t^{1/2}} \xrightarrow{d} N(0, m_F \Xi(\infty)^2)$$

Theorem 7.3.

Let $(X(t))_{t\geq 0}$ be a subcritical BHPIOR with arbitrary ancestor distribution, $g'(1) < \infty$, $f''(1) < \infty$, $h''(1) < \infty$ and $m_{G,2} < \infty$.

Suppose also $m_F < \infty$, $m_{D,2} < \infty$, and that at least one of G or D is spread out.

Then

$$\frac{X(t) - (N(t)+1)h'(1)\Lambda(\infty)}{t^{1/2}} \xrightarrow{d} N(0, m_F \Xi(\infty)^2)$$

where

$$\Xi(\infty)^2 \stackrel{def}{=} (h''(1) - h'(1)^2)\Lambda(\infty)^2 + h'(1)(\Lambda_2(\infty) - \Lambda(\infty)^2)$$

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denotes the variance of $Z^*(\infty)$, the limiting variable of $Z_i(t)$.

Proposition 7.4.

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Let $(Z(k))_{k\geq 0}$ be a subcritical GWPIO with arbitrary ancestor distribution, $g'(1) < \infty$. Suppose also $m_D < \infty$, and that the convolution $G * D = \delta_1 * D$ is 1-arithmetic.

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$$\mathbb{P}(Z(\infty) = n) = \begin{cases} \frac{m_D}{\beta}, & n = 0 \\ \frac{1}{\beta} \sum_{k\geq 0} \mathbb{P}^*(\widehat{Z}(k) = n), & n \geq 1 \end{cases}$$

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Then $Z(k)\xrightarrow{d}Z(\infty)$, $t\to\infty$, for an integer-valued random variable $Z(\infty)$ satisfying

$$\mathbb{P}(Z(\infty)=n)=\begin{cases}\frac{m_D}{\beta}, & n=0\\ \frac{1}{\beta}\sum_{k\geq 0}\mathbb{P}^*(\widehat{Z}(k)=n), & n\geq 1\end{cases}$$

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$$\Phi(s,\infty)=\frac{m_D}{\beta}+\frac{1}{\beta}\sum_{k\geq 0}(g(\Phi(s,k))-g(\Phi(0,k)))$$

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$$\Phi(s,\infty) = \frac{m_D}{\beta} + \frac{1}{\beta}\sum_{k\geq 0}(g(\Phi(s,k)) - g(\Phi(0,k)))$$

and mean

$$\Lambda(\infty) = \frac{g'(1)}{(1-m)\beta}.$$

Moreover,

$$\lim_{k\to\infty} \mathbb{E}_j Z(k) = \lim_{k\to\infty} \Lambda(k) = \Lambda(\infty), \quad j \geq 0$$

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$$\lim_{k\to\infty} \mathbb{E}_j Z^2(k) = \lim_{k\to\infty} \Lambda_2(k) = \Lambda_2(\infty)$$

$$\Lambda_2(\infty) = \frac{g'(1)}{(1-m)\beta} + \frac{1}{\beta(1-m^2)}\left(\frac{g'(1)f''(1)}{1-m} + g''(1)\right).$$

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$$\frac{X(t)}{t} \xrightarrow{\mathbb{P}} \frac{g'(1)h'(1)}{(1-m)m_F\beta}, \quad t \to \infty$$

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$$\frac{X(t)}{t} \xrightarrow{\mathbb{P}} \frac{g'(1)h'(1)}{(1-m)m_F\beta}, \quad t \to \infty$$

as $t \to \infty$ through the integers.

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