# Fast Domain Decomposition Solver for Internal Problems of 3D hierarchical hp-FEM

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# Plan

1. Fast DD solvers for 3D *hp*-discretizations
2. Terminology. What we mean under "fast"
3. Hierarchical *hp*–version
4. Decomposition
5. Local Dirichlet problems

## Deriving fast DD solvers for *hp* discretizations of 2nd order 3D elliptic PDEs

1. V. Korneev, U. Langer, L. S. Xanthis. `On fast domain decomposition solving procedures for hp-discretizations of 3-d elliptic problems`. Computational Methods in Applied Mathematics, 2003, V.3, N4, p.536–559.
2. Korneev V, Langer U and Xanthis L. `Fast adaptive domain decomposition algorithms for` *hp*`-discretizations of 2-`$d$` and 3-`$d$` elliptic equations: recent advances`. Hermis J. Comput. Math. Appl., 2003, 4, p.27–44.

✓ DD algorithms are `almost optimal`

## Terminology: optimality, almost optimality

$$Ax = y, \quad A - n \times n$$

An `algorithm` for solving system is termed

`almost optimal (fast) in computational cost`

if it requires $O(n(\log n)^k)$ arithmetic operations, $k \geq 0$ not large and fixed.
If $k = 0$, it is called `optimal`.

A `preconditioner` $\boldsymbol{\mathcal{A}}$ for spd matrix $A$ is termed almost optimal in condition, if $\mathrm{cond}(\boldsymbol{\mathcal{A}}^{-1}A) = O(\log^k n)$.

A `preconditioner-solver` is called `almost optimal (fast)`, if PCG with this preconditioner requires $O(n \log^k n)$ arithmetic operations

- ✓ We do not take into account matrix-vector multiplications

## Hierarchical *hp*–version

**Dirichlet problem** $\Omega \subset R^3$, find $u \in H_0^1(\Omega)$

$$a_\Omega(u,v) = (f,v)_\Omega, \qquad \forall v \in H_0^1(\Omega)\,,$$

$$a_\Omega(u,v) = \int_\Omega \varrho(x)\nabla u \cdot \nabla v\, dx\,, \quad (f,v)_\Omega = \int_\Omega uv\, dx\,, \quad \varrho \geq \mu_1 > 0.$$

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**Computational domain** $\overline{\Omega} = \bigcup_{r=1}^{\mathcal{R}} \overline{\tau}_r\,, \qquad x = X^{(r)}(y) : \overline{\tau}_0 \to \overline{\tau}_r$

$\tau_0 = (-1,1) \times (-1,1) \times (-1,1)$ – reference element.

$X^{(r)}$ are nondegenerate mappings and satisfy the conditions of the angular quasiuniformity.

**Reference element** $\tau_0 = (-1,1) \times (-1,1) \times (-1,1)$

$Q_p$ – polynomials of the order not greater $p \geq 1$, in each variable.

Hierarchical basis

$$\mathcal{M}_p = \{\mathcal{L}_\alpha(x) = L_{\alpha_1}(x_1)L_{\alpha_2}(x_2)L_{\alpha_3}(x_3)\,, \quad 0 \leq \alpha_1, \alpha_2, \alpha_3 \leq p\}$$

$$L_0(s) = \frac{1}{2}(1+s)\,, \quad L_1(s) = \frac{1}{2}(1-s)\,,$$

$$L_i(s) := \beta_i \int_{-1}^{s} P_{i-1}(t)\,dt\,, \quad i \geq 2\,,$$

$P_i$ are Legendre's polynomials,

$\beta_i \Rightarrow \quad \|L_i(s)\|_{0,(-1,1)} = 1$ for $i \geq 2$.

`System of linear equations` $\mathbf{K}\mathbf{u} = \mathbf{f}$

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`Decomposition` $\tau_r$ – subdomain of decomposition

$$V = V_I \oplus V_F \oplus V_W$$

$V_I$ – internal unknowns
$V_F$ – face unknowns
$V_W$ – wire-basket unknowns

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IF} & \mathbf{K}_{IW} \\ \mathbf{K}_{FI} & \mathbf{K}_F & \mathbf{K}_{FW} \\ \mathbf{K}_{EI} & \mathbf{K}_{EF} & \mathbf{K}_{EW} \end{pmatrix}$$

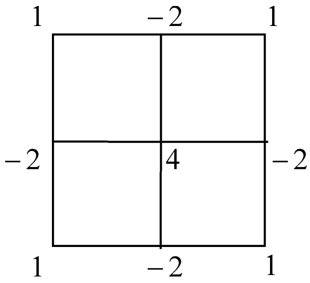
$$\mathbf{K}_I = diag\left(\mathbf{K}_I^{(1)}, \mathbf{K}_I^{(2)}, \ldots, \mathbf{K}_I^{(\mathcal{R})}\right) , \qquad \mathbf{K}_I^{(r)} \leftrightarrow \tau_r$$

## Finite-Difference preconditioner for $\mathbf{K}_I^{(r)}$

$$z^2\frac{\partial^4 u}{\partial^2 x \partial^2 y} + y^2\frac{\partial^4 u}{\partial^2 x \partial^2 z} + x^2\frac{\partial^4 u}{\partial^2 y \partial^2 z} = f(x,y,z), \quad (x,y,z) \in \delta,$$

$$u|_{\partial\delta} = 0, \qquad \delta = (0,1)\times(0,1)\times(0,1)$$

Uniform grid in $\delta$
with $h = 1/(N+1)$
$(p = 2N+1)$



$\Rightarrow \mathbf{\Lambda u} = \mathbf{f}$

$$\mathbf{\Lambda}_I^{(r)} = diag\,\underbrace{(\mathbf{\Lambda}, \mathbf{\Lambda}, \ldots, \mathbf{\Lambda})}_{8\ \text{times}}, \qquad \mathbf{\Lambda}_I^{(r)} \prec \mathbf{K}_I^{(r)} \prec (1+\log p)^2\mathbf{\Lambda}_I^{(r)}$$

How to obtain fast solver for $\mathbf{\Lambda}$ ?

## Local Dirichlet problems with $\mathbf{K}_I^{(r)}$

Approaches

1. S. Beuchler. `Multigrid solver for the inner problem in domain decomposition methods for` $p$`-FEM`. SIAM J. Num. Anal., 40 (2002), No. 4, 928-944 p.p.

2. S. Beuchler, R. Schneider and C. Schwab. `Multiresolution weighted norm equivalence and applications`. Preprint SFB393/02-09, Technische Universitat Chemnitz, Chemnitz, 2002.

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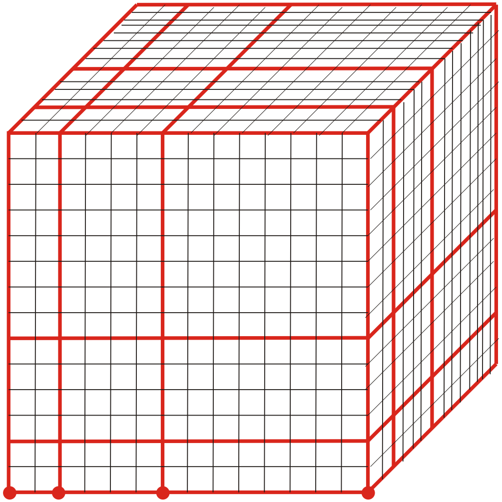
3. V. G. Korneev. `Local Dirichlet problems on subdomains of decomposition in` $hp$ `discretizations, and optimal algorithms for their solution`. Mathematical Modelling. 2002, V.14, N5, p.51–74. ← DD for 2D case

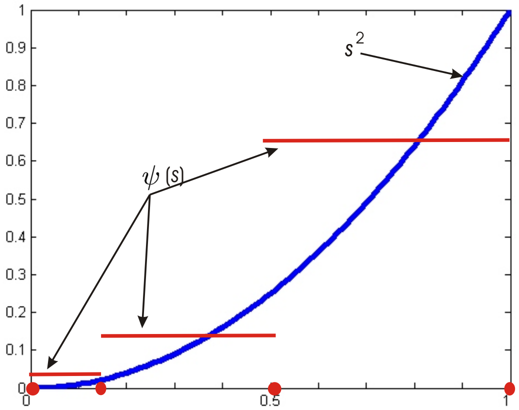
# DD approach to 3D case

$$z^2 \frac{\partial^4 u}{\partial^2 x \partial^2 y} + y^2 \frac{\partial^4 u}{\partial^2 x \partial^2 z} + x^2 \frac{\partial^4 u}{\partial^2 y \partial^2 z} = f(x, y, z)$$

$$\downarrow$$

$$\psi(z) \frac{\partial^4 u}{\partial^2 x \partial^2 y} + \psi(y) \frac{\partial^4 u}{\partial^2 x \partial^2 z} + \psi(x) \frac{\partial^4 u}{\partial^2 y \partial^2 z} = f(x, y, z)$$





## More precise definition of the coarse grid and $\psi$

$\xi_{s,i} = \eta_i = ih$, for $s = 1, 2, 3,\ i = 0, 1, \ldots, N+1$

$\overline{\xi}_{s,l} = \zeta_l$,

$q > 1$ and $n_0 \geq 1$ are two parameters of the coarse grid

$$\zeta_0 = 0, \quad \zeta_{l_0} = 1, \quad \zeta_l = \eta_i,\ \text{for } i = \theta(l) := \text{int}\,[(q^l - 1)n_0],$$
$$l = 1, 2, \ldots, l_0 - 2,$$
$$\zeta_{l_0 - 1} = \begin{cases} \eta_{\theta(l_0-1)}, & \text{if} \quad \eta_{\theta(l_0)} = 1, \\ \eta_i, i = \text{int}\,[0.5(\theta(l_0 - 2) + N + 1)], & \text{if} \quad \eta_{\theta(l_0)} > 1, \end{cases}$$

where $l_0 = \text{int}\,[\ln\left(N/n_0 + 1\right) / \ln q]$.

$$\psi(\zeta) = \begin{cases} \psi_l \equiv \frac{1}{2}(\zeta_{l-1}^2 + \zeta_l^2), & \zeta \in (\zeta_{l-1}, \zeta_l)\,, \\ \psi_1 \equiv \frac{1}{2}(h^2 + \zeta_1^2), & \zeta \in (\zeta_0, \zeta_1) \end{cases}$$

## Spectral equivalent preconditioner for $\boldsymbol{\Lambda}$

$$z^2\frac{\partial^4 u}{\partial^2 x \partial^2 y} + y^2\frac{\partial^4 u}{\partial^2 x \partial^2 z} + x^2\frac{\partial^4 u}{\partial^2 y \partial^2 z} = f(x,y,z)$$

$$\downarrow$$

Finite-difference matrix $\boldsymbol{\Lambda}$

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$$\psi(z)\frac{\partial^4 u}{\partial^2 x \partial^2 y} + \psi(y)\frac{\partial^4 u}{\partial^2 x \partial^2 z} + \psi(x)\frac{\partial^4 u}{\partial^2 y \partial^2 z} = f(x,y,z)$$

$$\downarrow$$

Finite-difference matrix $\mathbf{B}$

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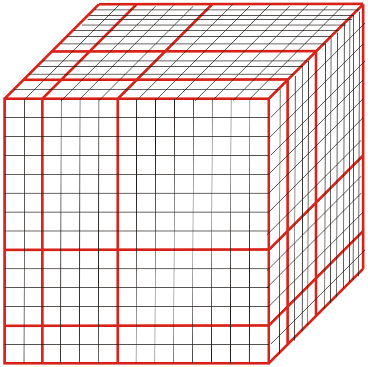
$\mathbf{B} \prec \boldsymbol{\Lambda} \prec \mathbf{B}$ $\Rightarrow$ We need fast solver for $\mathbf{B}$

## Secondary decomposition

$$\mathbf{B}\mathbf{v} = \mathbf{d}\,,\mathbf{v},\mathbf{d} \in U$$



$$U = U_I \oplus U_F \oplus U_W$$

$U_I = \biguplus_{s,n_s=1}^{3,l_0} U_{I,\mathbf{n}}$ – internal unknowns for all subdomains $\delta_{\mathbf{n}}$

$U_F$ – face unknowns

$U_W$ – wire-basket unknowns

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_I & \mathbf{B}_{IF} & \mathbf{B}_{IW} \\ \mathbf{B}_{FI} & \mathbf{B}_F & \mathbf{B}_{FW} \\ \mathbf{B}_{WI} & \mathbf{B}_{WF} & \mathbf{B}_W \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} \mathbf{v}_I \\ \mathbf{v}_F \\ \mathbf{v}_W \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} \mathbf{d}_I \\ \mathbf{d}_F \\ \mathbf{d}_W \end{pmatrix},$$

$$\mathbf{B}_I = \mathrm{diag}\left[\mathbf{B}_{I,\mathbf{n}}\right]$$

## Main components of DD-algorithm

1) $\mathbf{B}_{I,\mathbf{n}}$ – FFT $(\mathcal{O}(N^3 \log N)\,)$ or multilevel method $(\mathcal{O}(N^3))$

2) Change basis in $U$: $U = U_I \oplus U_{F,\mathrm{tr}} \oplus U_W$, $U_I \perp_{\mathbf{B}} U_{F,\mathrm{tr}}$ $(\cdot,\cdot)_{\mathbf{B}} := (\cdot, \mathbf{B}\cdot)$

On each face elements of $U_{F,\mathrm{tr}}$ are orthogonal to each other,

but for the adjacent faces their are not orthogonal

Their traces on faces are vectors $\boldsymbol{\mu}_{s,\alpha,\beta}$ with components

$\sin(\alpha i\pi/m_{\widehat{s}})\,\sin(\beta j\pi/m_{\overline{s}})$

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$$\mathbf{B}\mathbf{v} = \mathbf{d} \rightarrow \widetilde{\mathbf{B}}\,\widetilde{\mathbf{v}} = \widetilde{\mathbf{d}}$$

$$\widetilde{\mathbf{B}} = \begin{pmatrix} \mathbf{B}_I & \mathbf{O} & \mathbf{B}_{IW} \\ \mathbf{O} & \mathbf{S}_F & \mathbf{B}_{FW,\mathrm{tr}} \\ \mathbf{B}_{WI} & \mathbf{B}_{WF,\mathrm{tr}} & \mathbf{B}_W \end{pmatrix}, \quad \widetilde{\mathbf{v}} = \begin{pmatrix} \mathbf{v}_I \\ \mathbf{v}_{F,\mathrm{tr}} \\ \mathbf{v}_W \end{pmatrix}, \quad \widetilde{\mathbf{d}} = \begin{pmatrix} \mathbf{d}_I \\ \mathbf{d}_{F,\mathrm{tr}} \\ \mathbf{d}_W \end{pmatrix}$$

# Main components of DD-algorithm

$$\widetilde{\mathbf{B}} = \begin{pmatrix} \mathbf{B}_I & \mathbf{O} & \mathbf{B}_{IW} \\ \mathbf{O} & \mathbf{S}_F & \mathbf{B}_{FW,\mathrm{tr}} \\ \mathbf{B}_{WI} & \mathbf{B}_{WF,\mathrm{tr}} & \mathbf{B}_W \end{pmatrix} \qquad \widetilde{\mathbf{B}}_\Gamma = \begin{pmatrix} \mathbf{S}_F & \mathbf{B}_{FW,\mathrm{tr}} \\ \mathbf{B}_{WF,\mathrm{tr}} & \mathbf{B}_W \end{pmatrix} - O(N^3)$$

On each iteration of PCG with $\mathbf{B}$

$$\mathbf{B}_I^{-1}\mathbf{w}_I - O(N^3) \qquad \mathbf{B}_I^{-1}\mathbf{B}_{I,W}\mathbf{w}_W - O(N^3)$$

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How to obtain fast solver for

$$\mathbf{S}_\Gamma = \begin{pmatrix} \mathbf{S}_F & \mathbf{B}_{FW,\mathrm{tr}} \\ \mathbf{B}_{WF,\mathrm{tr}} & \mathbf{B}_\Delta \end{pmatrix}, \qquad \mathbf{B}_\Delta = \mathbf{B}_W - \mathbf{B}_{WI}\mathbf{B}_I^{-1}\mathbf{B}_{IW}\,.$$

# Preconditioner-solver for $\mathbf{S}_F$

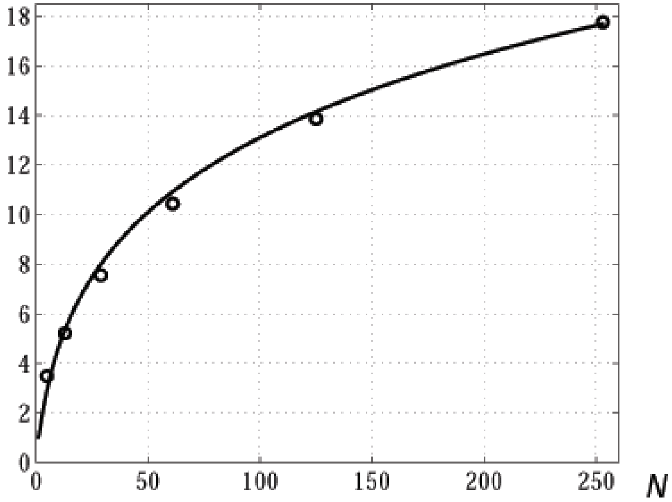
$$\mathbf{S}_\Gamma = \begin{pmatrix} \mathbf{S}_F & \mathbf{B}_{FW,\mathrm{tr}} \\ \mathbf{B}_{WF,\mathrm{tr}} & \mathbf{B}_\Delta \end{pmatrix}$$

$\mathbf{S}_F \longrightarrow$ blocks for adjacent faces $= 0 \longrightarrow \boldsymbol{\mathcal{S}}_F$

$\boldsymbol{\mathcal{S}}_F{}^{-1} - \mathcal{O}(\mathrm{N}^2 \log N)$

$\mathrm{cond}\,[\boldsymbol{\mathcal{S}}_F^{-1}\mathbf{S}_F] \le c(\log N)^2$

$1/\min_{\mathbf{n}} \lambda_{\min}\,[\boldsymbol{\mathcal{S}}_{F_{\mathbf{n}}}^{-1}\mathbf{S}_{F_{\mathbf{n}}}]$



Preconditioner-solver for $\mathbf{S}_\Gamma = \begin{pmatrix} \mathbf{S}_F & \mathbf{B}_{FW,\mathrm{tr}} \\ \mathbf{B}_{WF,\mathrm{tr}} & \mathbf{B}_\Delta \end{pmatrix}$

$$\boldsymbol{\mathcal{S}}_\Gamma{}^{-1} = \begin{pmatrix} \mathbf{I}_F & \mathbf{O} \\ -\mathbf{B}_{\mathrm{tr},WF}\boldsymbol{\mathcal{S}}_{F,\mathrm{it}}{}^{-1} & \mathbf{I}_W \end{pmatrix} \begin{pmatrix} \mathbf{S}_{F,\mathrm{it}}{}^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{B}_\delta{}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}_F & -\boldsymbol{\mathcal{S}}_{F,\mathrm{it}}{}^{-1}\mathbf{B}_{tr,FW} \\ \mathbf{O} & \mathbf{I}_W \end{pmatrix}$$

where $\mathbf{B}_\delta = \mathbf{B}_\Delta - \mathbf{B}_{WF,\mathrm{tr}}\mathbb{S}_{F,\mathrm{it}}{}^{-1}\mathbf{B}_{FW,\mathrm{tr}}$

$$\begin{aligned} \mathbf{S}_{F,\,\mathrm{it}}{}^{-1} &= \left[\mathbf{I} - \textstyle\prod_{k=1}^{k_1}(\mathbf{I} - \sigma_k \boldsymbol{\mathcal{S}}_F{}^{-1}\mathbf{S}_F)\right]\mathbf{S}_F^{-1} \\ \boldsymbol{\mathcal{S}}_{F,\,\mathrm{it}}{}^{-1} &= \left[\mathbf{I} - \textstyle\prod_{k=1}^{k_2}(\mathbf{I} - \sigma_k \boldsymbol{\mathcal{S}}_F{}^{-1}\mathbf{S}_F)\right]\mathbf{S}_F^{-1} \\ \mathbb{S}_{F,\,\mathrm{it}}{}^{-1} &= \left[\mathbf{I} - \textstyle\prod_{k=1}^{k_3}(\mathbf{I} - \sigma_k \boldsymbol{\mathcal{S}}_F{}^{-1}\mathbf{S}_F)\right]\mathbf{S}_F^{-1} \end{aligned}$$

- ✓ $k_1, k_2, k_3 = \mathcal{O}(\log N)$ iterations ⇒ spectral equivalence
  (Korneev,Langer,Xanthis 2003)

- ✓ comp. cost – $\mathcal{O}(N^2(\log N)^2)$ (only if to avoid $\mathbf{S}_F\mathbf{w}_F$)

$$\mathbf{S}_F = \boldsymbol{\mathcal{F}}_F\left(\mathbf{B}_F - \mathbf{B}_{FI}\mathbf{B}_I^{-1}\mathbf{B}_{IF}\right)\boldsymbol{\mathcal{F}}_F$$