Numerical approximation of output functionals for Maxwell equations

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MAXWELL EQUATIONS

Assumption:
electric field ($\sim$ electromagnetic wave)
consists of only one frequency
"Time harmonic" Maxwell equation:

$$\begin{aligned} \mathcal{L}(E) = \text{curl}(\text{curl}\, E) \quad - \quad k^2 E = F \quad & \text{in } \Omega \subset \mathbf{R}^3 \\ \nu \times E \;\; = \;\; 0 \quad & \text{on } \Gamma = \partial\Omega \end{aligned} \qquad (1)$$

Notation:

$$\text{curl}\, E \equiv \nabla \times E \equiv \text{rot}\, E$$

curl : $[L_2(\Omega)]^3 \to [L_2(\Omega)]^3$
$\text{curl}(u_1, u_2, u_3) =$
$= (\partial_2 u_3 - \partial_3 u_2, -\partial_1 u_3 + \partial_3 u_1, \partial_1 u_2 - \partial_2 u_1)$

- $\Omega \subset \mathbf{R}^3$ - usually a bounded Lipschitz domain

- $E : \Omega \to \mathbf{R}^3$ - electric field (unknown)

- $F : \Omega \to \mathbf{R}^3$ - current forces

- ν - unit outward normal of Γ
in (1) perfect conducting boundary condition

- Assumption (on k): $F = 0 \Rightarrow E = 0$.

PRELIMINARIES

$H(\mathsf{curl}, \Omega) = \{u \in [L_2(\Omega)]^3 : \mathsf{curl}\, u \in [L_2(\Omega)]^3\}$

completed with

$$||u||_{\mathsf{curl}} = (||u||^2_{[L_2(\Omega)]^3} + ||\mathsf{curl}\, u||^2_{[L_2(\Omega)]^3})^{1/2}$$

and

$$H_0(\mathsf{curl}, \Omega) = \{u \in H(\mathsf{curl}, \Omega) : u \times \nu|_\Gamma = 0\},$$

For $\Psi \in [H^{1/2}(\Gamma)]^3$

$$H_\Psi(\mathsf{curl}, \Omega) = \{u \in H(\mathsf{curl}, \Omega) : u \times \nu|_\Gamma = \Psi\}.$$

(Girault & Raviart '86, Dautray & Lions '90.)
$\gamma_\tau : H(\mathsf{curl}, \Omega) \to H^{-1/2}(\Gamma), \gamma_\tau(u) = u \times \nu|_\Gamma$
tangential trace operator
Range for a gen. Lipschitz polyhedron?
(Buffa '01)

VARIATIONAL FORMULATION

$$B : H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega) \to \mathbf{R}$$

with

$$B(u, v) = (\mathrm{curl}\, u,\ \mathrm{curl}\, v) - k^2(u, v).$$

- continuous, bilinear

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$$\sup_{v \in H_0(\mathrm{curl},\Omega)} \frac{|B(u, v)|}{||v||_{H(\mathrm{curl},\Omega)}} \geq C||u||_{H(\mathrm{curl},\Omega)}.$$

Variational form of (1):
Find $E \in H_0(\mathrm{curl}, \Omega)$ such that

$$B(E, v) = (F, v) \qquad \forall\, v \in H_0(\mathrm{curl}, \Omega). \tag{2}$$

Finite element (FE) approximation:
"edge" elements (Nédélec '80, Monk '03.)
Difficulty:
"follow" the decomposition of $H_0(\mathrm{curl}, \Omega)$

"OUTPUT" FUNCTIONALS

Standard case:
Find $J(E)$, where $J : H_0(\mathrm{curl}, \Omega) \to \mathbf{R}$ is continuous.
Idea: seek for a "representing" element of J, do not compute E - sol. of (1).
An example:

$$J(E) = J_\psi(E) = \int_\Gamma \Psi \cdot \mathrm{curl}\, E.$$

Meaning: total amount of the variation in the magnetic field
Treatment: adjoint formalism (also discrete)
Further aims: Error control - adaptive method (Giles & Süli '02)

PRIMAL PROBLEM (in case of J_Ψ):
Find $E \in H_0(\text{curl}, \Omega)$ and $J_p \in \mathbf{R}$:

$$J_p = B(E, v) - (F, v) \; \forall \, v \in H_\Psi(\text{curl}, \Omega). \quad (3)$$

Discretized PRIMAL PROBLEM:
Find $(E^h, J_p^h) \in (H_0^h(\text{curl}, \Omega) \times \mathbf{R})$:

$$J_p^h = B(E^h, v^h) - (F, v^h) \; \forall \, v^h \in H_\Psi^h(\text{curl}, \Omega) \quad (4)$$

DUAL PROBLEM (in case of J_Ψ):
Find $z \in H_\Psi(\text{curl}, \Omega)$ and $J_d \in \mathbf{R}$:

$$J_d = (F, z) - B(w, z) \; \forall \, w \in H_\Psi(\text{curl}, \Omega). \quad (5)$$

Discretization – see (4)

EQUIVALENCE THEOREM:
$(E, J_p) \in (H_0(\text{curl}, \Omega) \times \mathbf{R})$ solves (3) iff
E solves (2) and then $J_p = J_d = J_\Psi(E)$.
(Giles & Süli '02.)

RESULTS

basic idea: the dual problem may have a smooth solution whenever E is not smooth

Convergence result for J_p^h:
Using the "edge" FE space of order l in (4) and assuming smoothness of z we obtain

$$|J_\Psi(E) - J_p^h(E_h)| \leq c|z|_{[H^{l+1}(\Omega)]^3} h^l ||E||_{\mathrm{curl}}.$$

Tools:

- Galerkin relation in (4)

- Approximation theorem (quasi uniform tetrahedral mesh with parameter h)

  $$||E - \Pi_l E||_{\mathrm{curl}} \leq ch^l |E|_{[H^l(\Omega)]^3}, (c \in \mathbf{R})$$

  if $E, \mathrm{curl}\, E \in [H^l(\Omega)]^3$
  Π_l – appropriate FE projection

Error analysis

Residual equation for the error:

$$\mathcal{L}(z - z^H) = f - \mathcal{L}z^H = r^H \text{ in } H_0(\text{curl}, K). \quad (6)$$

Boundary condition?
We may assume $\nu \times (z - z^H) = 0|_{\partial K}$.

- change the right hand side of (6)
- appl. of bubble function technique (Ainsworth & Oden '00)

Aims: estimate $\|z - z^H\|_{[L_2(\Omega)]^3}$
give the order of the error if $h \to 0$

Helmholtz-decomposition:

$$\begin{aligned} H_0(\mathrm{curl}, K) &= \nabla H_0^1(K) \oplus [\nabla H_0^1(K)]^\perp \\ z - z^H &= \mathrm{grad}\, p + u, \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}(u) = \Delta\, u + k^2 u &= -f - k^2 \mathrm{grad}\, p && \text{in } K \\ \nu \times u &= 0 && \text{on } \partial K \end{aligned} \tag{7}$$

and

$$\begin{aligned} \Delta\, p &= -\frac{1}{k^2} \mathrm{div}\, f && \text{in } K \\ p &= 0 && \text{on } \partial K. \end{aligned} \tag{8}$$

Remark:
$\mathrm{div}\, u = 0$ and $u \in H^\delta(\Omega)$ with $\delta \geq \frac{1}{2}$.

Error estimate:
If $K \subset h-$cube, where $h < \frac{k}{\pi}$. then for the solution u of (6)

$$||z - z^H||_{[L_2(\Omega)]^3} \leq \frac{h^2}{\pi^2} ||r^H||_{[L_2(\Omega)]^3}.$$

Tools of the proof:

- Estimate the right hand side of (7)

- Inverse of

  $\mathcal{H} : L_2(K) \to L_2(K) \quad \text{Dom}\, \mathcal{H} \subset H_0(\text{curl}, K)$

  $$\mathcal{H}v = \Delta v + k^2 v$$

  is compact.

- Eigenvalue structure of $\mathcal{H}$.

- Standard estimate of (8)

FUTURE WORKS:

- developing an adaptive solver
- rewrite a posteriori estimates of heat equation for Maxwell equations
- a sharper projection result – the FE projection is not defined on the whole of $H(\mathrm{curl}, \Omega)$

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