A SYMMETRIC COLOCATION METHOD WITH FAST EVALUATION

Michael J. Johnson

Kuwait University

ftp://ftp.cs.wisc.edu/Approx/

1. Minimal Norm Interpolation

Theorem. *Let H be an inner-product space and let $x_1, x_2, \dots, x_n$ be linear independent elements of H. Given real values $\{y_i\}$ there exists a unique $s \in H$, of minimal norm, which satisfies the interpolation equations*

$$\langle s, x_i \rangle_H = y_i, \quad i = 1, 2, \dots, n.$$

Moreover, s can be written (uniquely) in the form

$$s = \sum_{i=1}^{n} \lambda_i x_i.$$

Definition. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be even, continous, and integrable and assume that its Fourier transform satisfies
a) $\widehat{\phi}(w) > const \frac{1}{1+|w|^N}$ for some constant N,
b) $\int_{\mathbb{R}^d} \widehat{\phi}(w)\, dw < \infty$.
We define the inner product

$$\langle f, g \rangle_\phi := \int_{\mathbb{R}^d} \widehat{f}(w)\overline{\widehat{g}(w)}/\widehat{\phi}(w)\, dw,$$

and subsequently the space H_ϕ is the set of all $f \in L_2(\mathbb{R}^d)$ for which

$$\|f\|_\phi := \sqrt{\langle f, f \rangle_\phi} < \infty.$$

For any point $\xi \in \mathbb{R}^d$, the function $\phi(\cdot - \xi)$ belongs to H_ϕ and represents the linear functional evaluation at ξ in the sense that

$$f(\xi) = \langle f, \phi(\cdot - \xi)\rangle_\phi, \quad f \in H.$$

When applied to the problem of function value interpolation, the minimal norm interpolation theorem yields the following

Theorem. *Given distinct points* $\xi_1, \xi_2, \ldots, \xi_n$ *in* $\mathbb{R}^d$ *and corresponding values* $\{y_i\}$, *there exists a unique function* $s \in H_\phi$ *having minimal norm while satisfying the interpolation conditions*

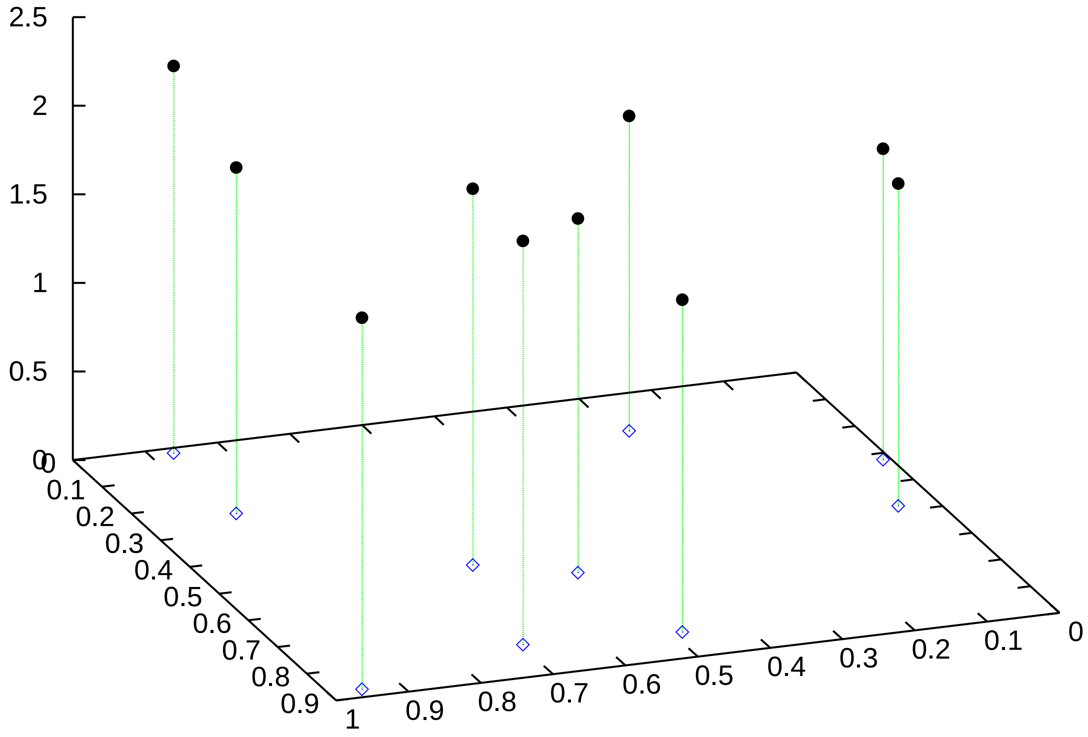
$$s(\xi_i) = y_i, \quad i = 1, 2, \ldots, n \quad ie\ \langle s, \phi(\cdot - \xi_i)\rangle_\phi = y_i.$$

Moreover, the function s *has the form*

$$s = \sum_{i=1}^{n} \lambda_i \phi(\cdot - \xi_i).$$

The coefficients $\{\lambda_i\}$ are determined by the linear system $A\lambda = y$, where $A = [\phi(\xi_i - \xi_j)]$ is symmetric positive definite.

The above interpolation method is known in the literature as radial basis function interpolation and has been developed in part by Duchon, Madych, Nelson, Schaback, Dyn, Beatson, Ward, Narcowich, Light, Levesley, Powell, Buhmann, and Wendland.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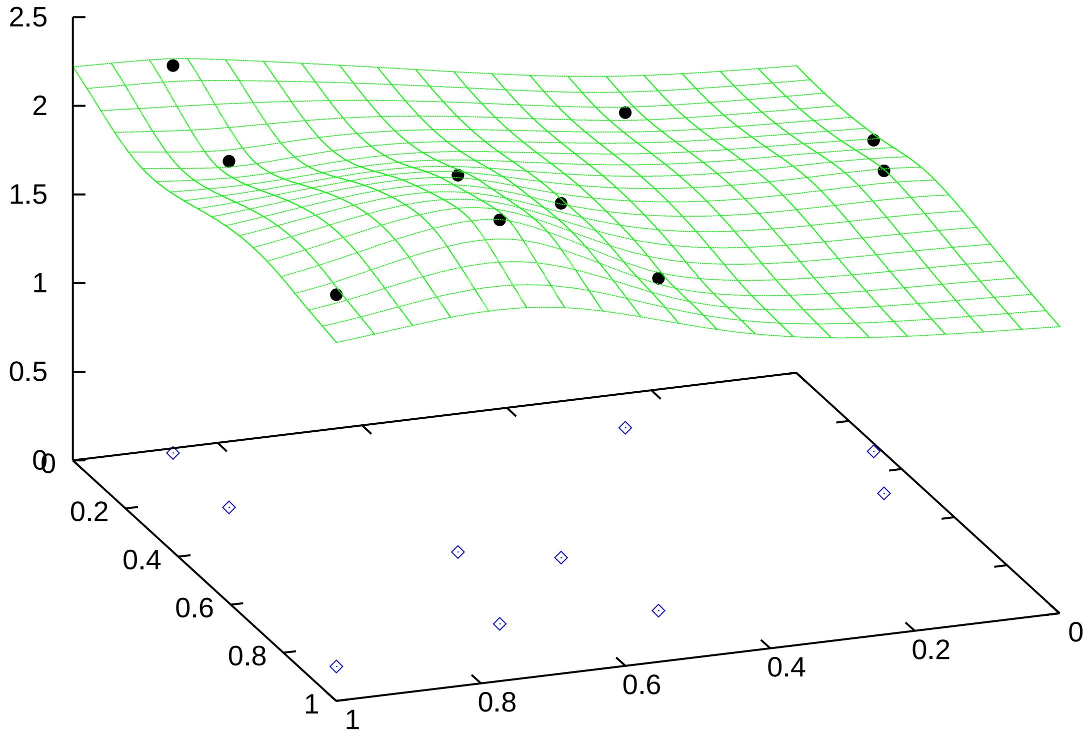
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2. Symmetric Colocation

When the minimal norm interpolation theorem is applied to solving PDEs, the method is generally refered to as symmetric colocation. Some of the people who have worked with this method are Kansa, Schaback, Franke, Wendland, Fasshauer and C.S. Chen.

We will consider a 2nd order PDE, so we need to assume that

$$\int_{\mathbb{R}^2} (1 + |w|^2)\widehat{\phi}(w)\, dw < \infty.$$

The following functions, which belong to H_ϕ, represent the indicated linear functional:

$$\begin{array}{rl}
\phi : & f(\xi) = \langle f, \phi(\cdot - \xi)\rangle_\phi \\
\phi_x : & f_x(\xi) = \langle f, -\phi_x(\cdot - \xi)\rangle_\phi \\
\phi_y : & f_y(\xi) = \langle f, -\phi_y(\cdot - \xi)\rangle_\phi \\
\phi_{xx} : & f_{xx}(\xi) = \langle f, \phi_{xx}(\cdot - \xi)\rangle_\phi \\
\phi_{xy} : & f_{xy}(\xi) = \langle f, \phi_{xy}(\cdot - \xi)\rangle_\phi \\
\phi_{yy} : & f_{yy}(\xi) = \langle f, \phi_{yy}(\cdot - \xi)\rangle_\phi
\end{array}$$

We consider Poisson's equation with Dirichlet boundary conditions

$$\begin{aligned} u_{xx} + u_{yy} &= f \text{ on } \Omega, \\ u &= g \text{ on } \partial\Omega, \end{aligned}$$

where Ω is a nice domain in $\mathbb{R}^2$. Let $\xi_1, \xi_2, \ldots, \xi_n$ be points in Ω where we wish to impose the PDE and let $\eta_1, \eta_2, \ldots, \eta_k$ be points on $\partial\Omega$ where we wish to impose the boundary conditions.

We seek the minimal norm function $s \in H_\phi$ which satisfies

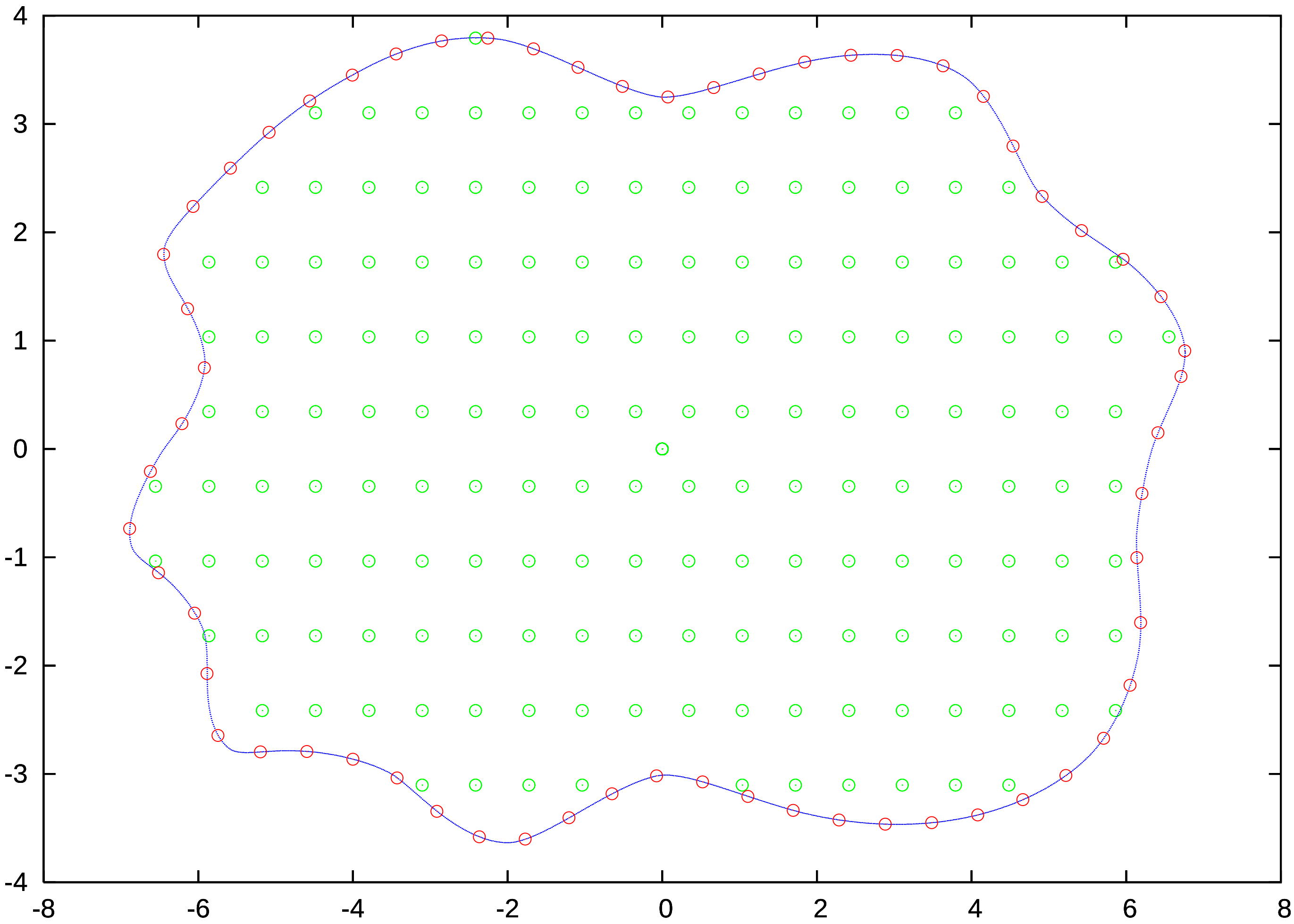
$$\begin{aligned} s_{xx}(\xi_i) + s_{yy}(\xi_i) &= f(\xi_i), \quad i = 1, 2, \ldots, n, \\ s(\eta_i) &= g(\eta_i), \quad i = 1, 2, \ldots, k. \end{aligned}$$

These conditions can be expressed as

$$\begin{aligned} \langle s, \phi_{xx}(\cdot - \xi_i) + \phi_{yy}(\cdot - \xi_i)\rangle_\phi &= f(\xi_i) \\ \langle s, \phi(\cdot - \eta_i)\rangle_\phi &= g(\eta_i). \end{aligned}$$

The minimal norm interpolation theorem then tells us that s has the form

$$s = \sum_{i=1}^{n} \lambda_i [\phi_{xx}(\cdot - \xi_i) + \phi_{yy}(\cdot - \xi_i)] + \sum_{i=1}^{k} \mu_i \phi(\cdot - \eta_i).$$



The colocation equations:

$$\begin{aligned} s_{xx}(\xi_i) + s_{yy}(\xi_i) &= f(\xi_i) \\ s(\eta_i) &= g(\eta_i). \end{aligned}$$

The unknowns $\lambda := [\lambda_1, \lambda_2, \ldots, \lambda_n]^T$ and $\mu = [\mu_1, \mu_2, \ldots, \mu_k]^T$ are determined by the linear system

$$A \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} f(\{\xi_i\}) \\ g(\{\eta_i\}) \end{bmatrix},$$

where A is symmetric positive definite and has the block structure

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}.$$

Here

$$A_{11}(i,j) = \phi_{xxxx}(\xi_i - \xi_j) + 2\phi_{xxyy}(\xi_i - \xi_j) + \phi_{yyyy}(\xi_i - \xi_j)$$
$$A_{12}(i,j) = \phi_{xx}(\xi_i - \eta_j) + \phi_{yy}(\xi_i - \eta_j)$$
$$A_{22}(i,j) = \phi(\eta_i - \eta_j)$$

3. Evaluating the function s

$$s = \sum_{i=1}^{n} \lambda_i [\phi_{xx}(\cdot - \xi_i) + \phi_{yy}(\cdot - \xi_i)] + \sum_{i=1}^{k} \mu_i \phi(\cdot - \eta_i).$$

As written, each evaluation of s (or of $s_{xx} + s_{yy}$) requires $O(n + k)$ flops.
To evaluate the residual (ie to evaluate $s_{xx}(\xi_i) + s_{yy}(\xi_i)$, $1 \leq i \leq n$ and $s(\eta_i)$, $1 \leq i \leq k$), requires $O(n + k)^2$ flops.

The high cost of evaluating s is the major drawback to the symmetric colocation method.

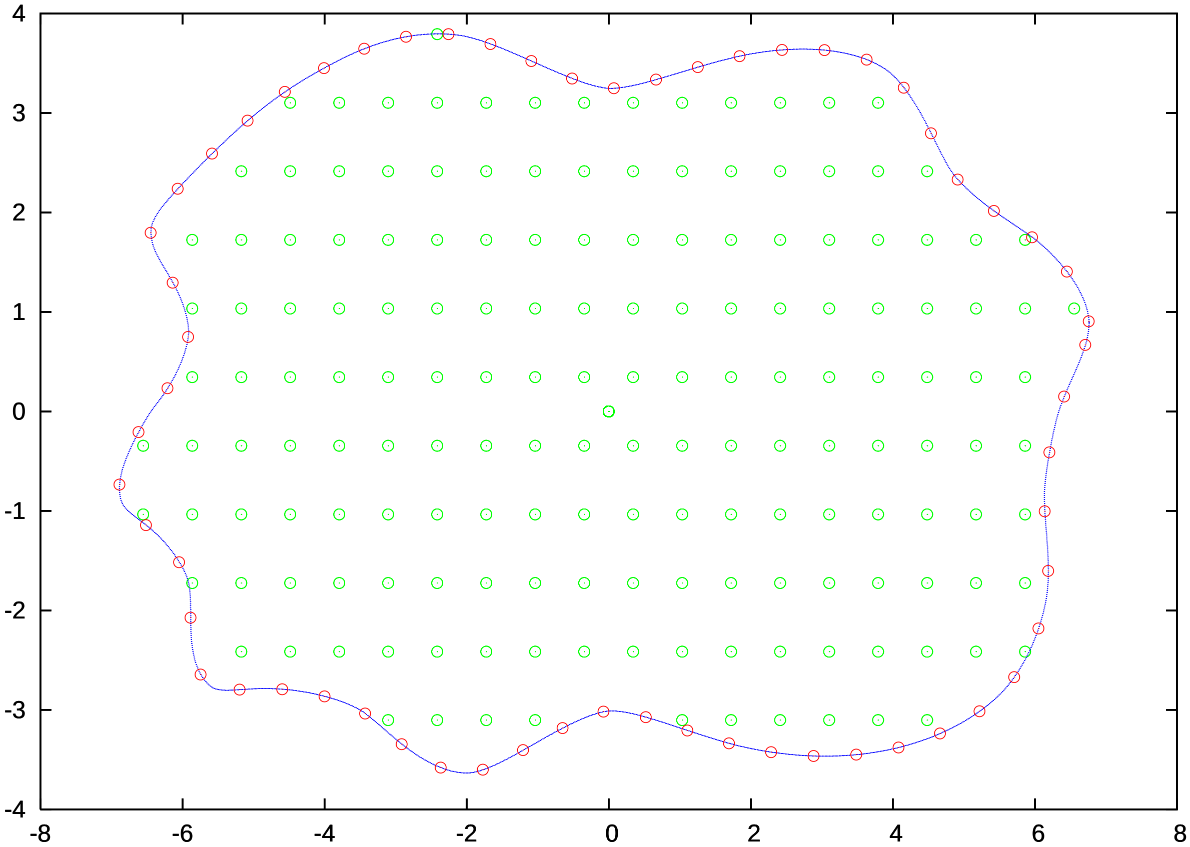
4. Fast evaluation of s

The following basic evaluation task appears repeatedly whenever an interpolation matrix is constructed or when an obtained function s is evaluated.

Basic Evaluation Task. Given a function of the form

$$h(x) := \sum_{i=1}^{n} c_i \psi(x - \xi_i), \quad x \in \mathbb{R}^d,$$

and points $\{z_1, z_2, \ldots, z_m\} \subset \mathbb{R}^d$, compute the values $h(z_j)$, $j = 1, 2, \ldots, m$.



For example, to evaluate our function

$$s = \sum_{i=1}^{n} \lambda_i [\phi_{xx}(\cdot - \xi_i) + \phi_{yy}(\cdot - \xi_i)] + \sum_{i=1}^{k} \mu_i \phi(\cdot - \eta_i).$$

at the points $\{\eta_j\}$ we have

$$s(\eta_j) = \sum_{i=1}^{n} \lambda_i \phi_{xx}(\eta_j - \xi_i) + \sum_{i=1}^{n} \lambda_i \phi_{yy}(\eta_j - \xi_i) + \sum_{i=1}^{k} \mu_i \phi(\eta_j - \eta_i).$$

Basic Evaluation Task. Given a function of the form

$$h(x) := \sum_{i=1}^{n} c_i \psi(x - \xi_i), \quad x \in \mathbb{R}^d,$$

and points $\{z_1, z_2, \ldots, z_m\} \subset \mathbb{R}^d$, compute the values $h(z_j)$, $j = 1, 2, \ldots, m$.

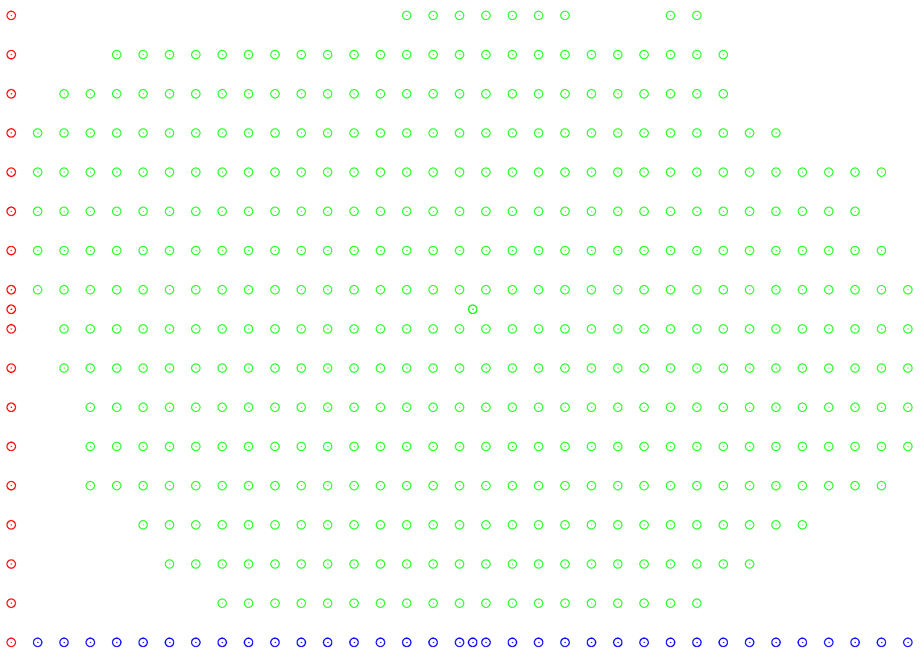
Direct (or naive) evaluation requires $O(nm)$ flops; however, if the function ψ is a tensor product piecewise-polynomial, then there is room for improvement.

Let us consider the case $d = 2$ with

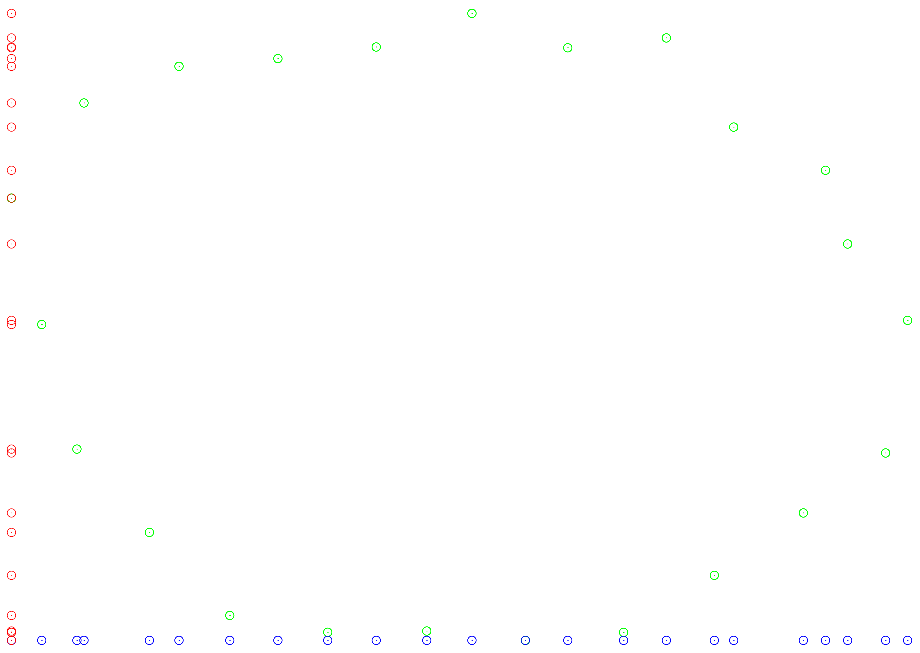
$$\psi(x, y) = \psi_1(x)\psi_2(y), \quad (x, y) \in \mathbb{R}^2,$$

where ψ_1 and ψ_2 are each univariate piecewise polynomials. Furthermore, we employ

$$\{\xi_1, \xi_2, \ldots, \xi_n\} \subset \{x_1, x_2, \ldots, x_{n_1}\} \times \{y_1, y_2, \ldots, y_{n_2}\}$$
$$\{z_1, z_2, \ldots, z_m\} \subset \{x'_1, x'_2, \ldots, x'_{m_1}\} \times \{y'_1, y'_2, \ldots, y'_{m_2}\}.$$



$$\{\xi_1, \xi_2, \ldots, \xi_m\} \subset \{x_1, x_2, \ldots, x_{n_1}\} \times \{y_1, y_2, \ldots, y_{n_2}\}$$



$$\{z_1, z_2, \ldots, z_N\} \subset \{x'_1, x'_2, \ldots, x'_{m_1}\} \times \{y'_1, y'_2, \ldots, y'_{n_2}\}$$

We now write

$$h(x,y)=\sum_{j=1}^{n_2}\sum_{i=1}^{n_1} c_{i,j}\psi_1(x-x_i)\psi_2(y-y_j),$$

and consider the task of evaluating

$$h(x'_q,y'_r),\quad 1\le q\le m_1,\quad 1\le r\le m_2.$$

For each j, the sum $G_j(x) := \sum_{i=1}^{n_1} c_{i,j}\psi_1(x-x_i)$ can be evaluated at the points $\{x'_q\}$ with only $O(m_1+n_1)$ flops; the values $\{G_j(x'_q) : 1\le q\le m_1, 1\le j\le n_2\}$ can thus be obtained with only $O(n_2(m_1+n_1))$ flops.

We can then write, for fixed q, $h(x'_q,y) = \sum_{j=1}^{n_2} G_j(x'_q)\psi_2(y-y_j)$ as a univariate function of y which can be evaluated at points $\{y'_r\}$ with only $O(n_2+m_2)$ flops.

It works out that the values $\{h(x'_q,y'_r)\}$ can be obtained with only $O(n_1+m_1)(n_2+m_2)$ flops.

Remark. The constant in $O(n_1+m_1)(n_2+m_2)$ depends linear on the number of pieces and quadratically on the degree of the piecewise-polynomials ψ_1 and ψ_2.

Question. What are the implications in terms of our example with Poisson's equation?

Recall (see slide 8) that we impose the PDE at the points $\{\xi_1, \xi_2, \ldots, \xi_n\}$ and we impose the boundary conditions at the points $\{\eta_1, \eta_2, \ldots, \eta_k\}$. Since the points $\{\xi_i\}$ can be chosen on a grid it is reasonable to assume that they are contained in a $\sim \sqrt{n} \times \sqrt{n}$ grid. Since the points $\{\eta_i\}$ are only on the boundary of Ω, it is reasonable to assume that $k \sim \sqrt{n}$. With

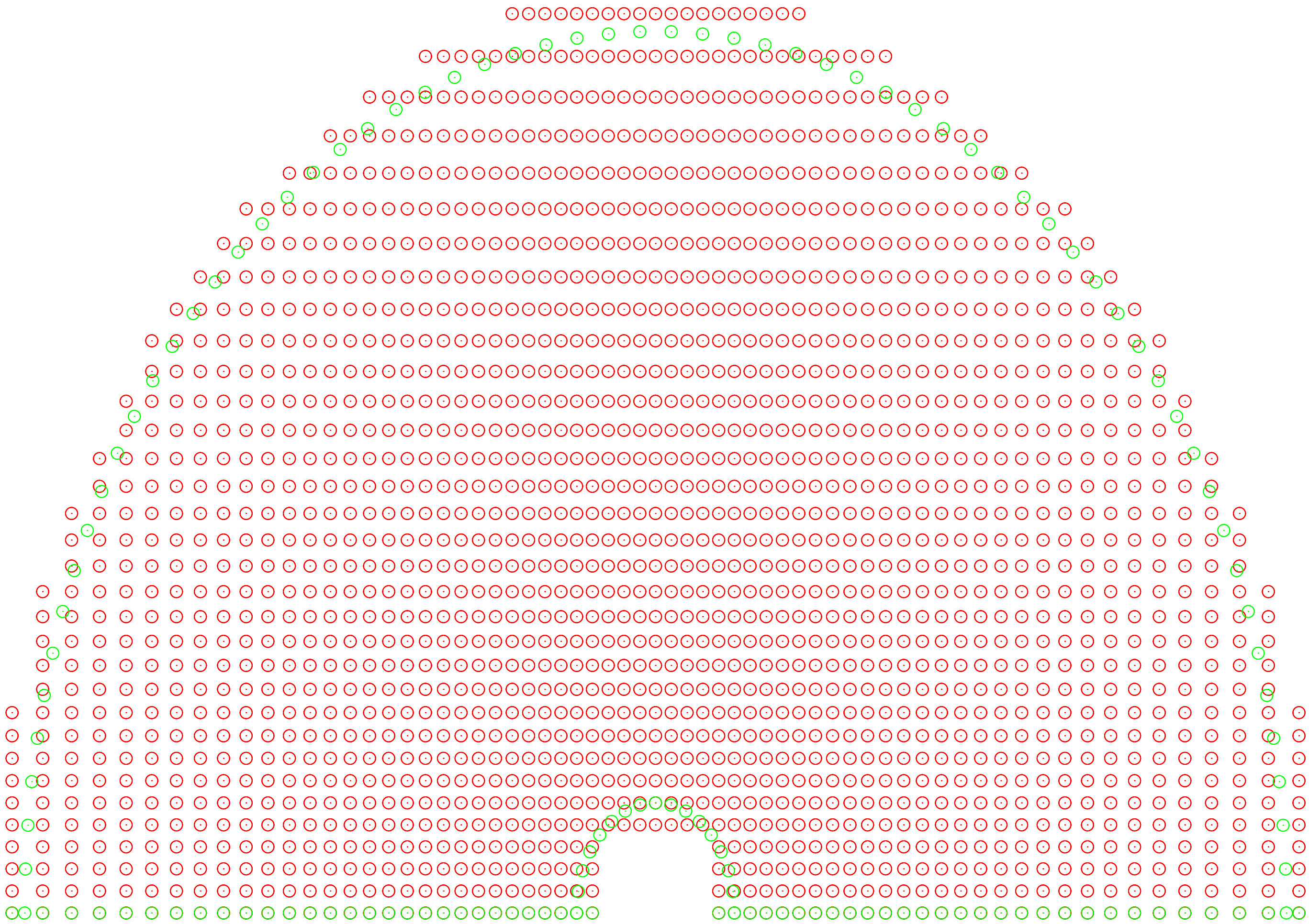
$$s = \sum_{i=1}^{n} \lambda_i [\phi_{xx}(\cdot - \xi_i) + \phi_{yy}(\cdot - \xi_i)] + \sum_{i=1}^{k} \mu_i \phi(\cdot - \eta_i),$$

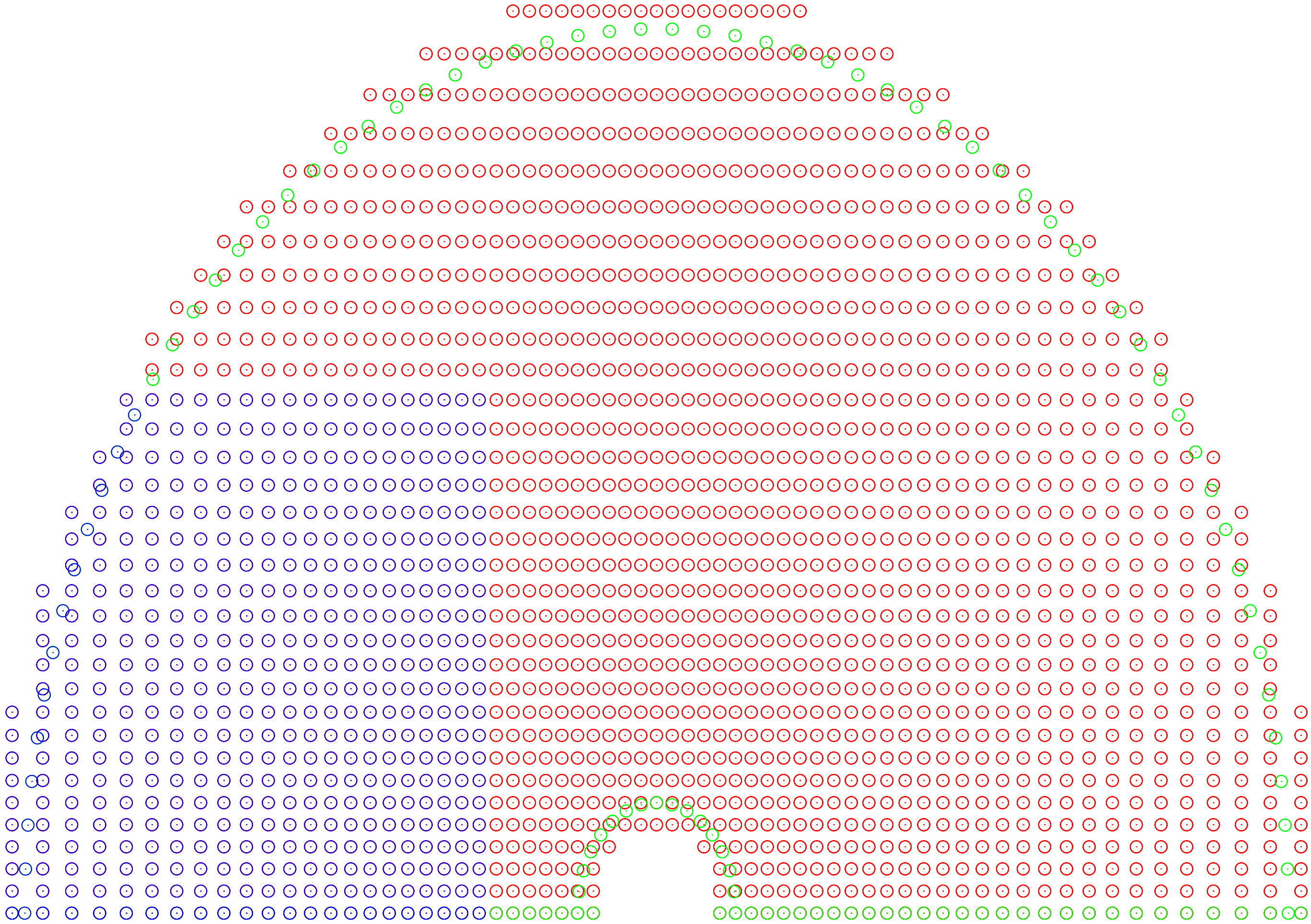
we have the following flop counts:

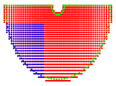
evaluation task	direct evaluation	fast evaluation
$\{s(\eta_i)\}$	$O(n^{3/2})$	$O(n)$
$\{s_{xx}(\xi_i) + s_{yy}(\xi_i)\}$	$O(n^2)$	$O(n)$

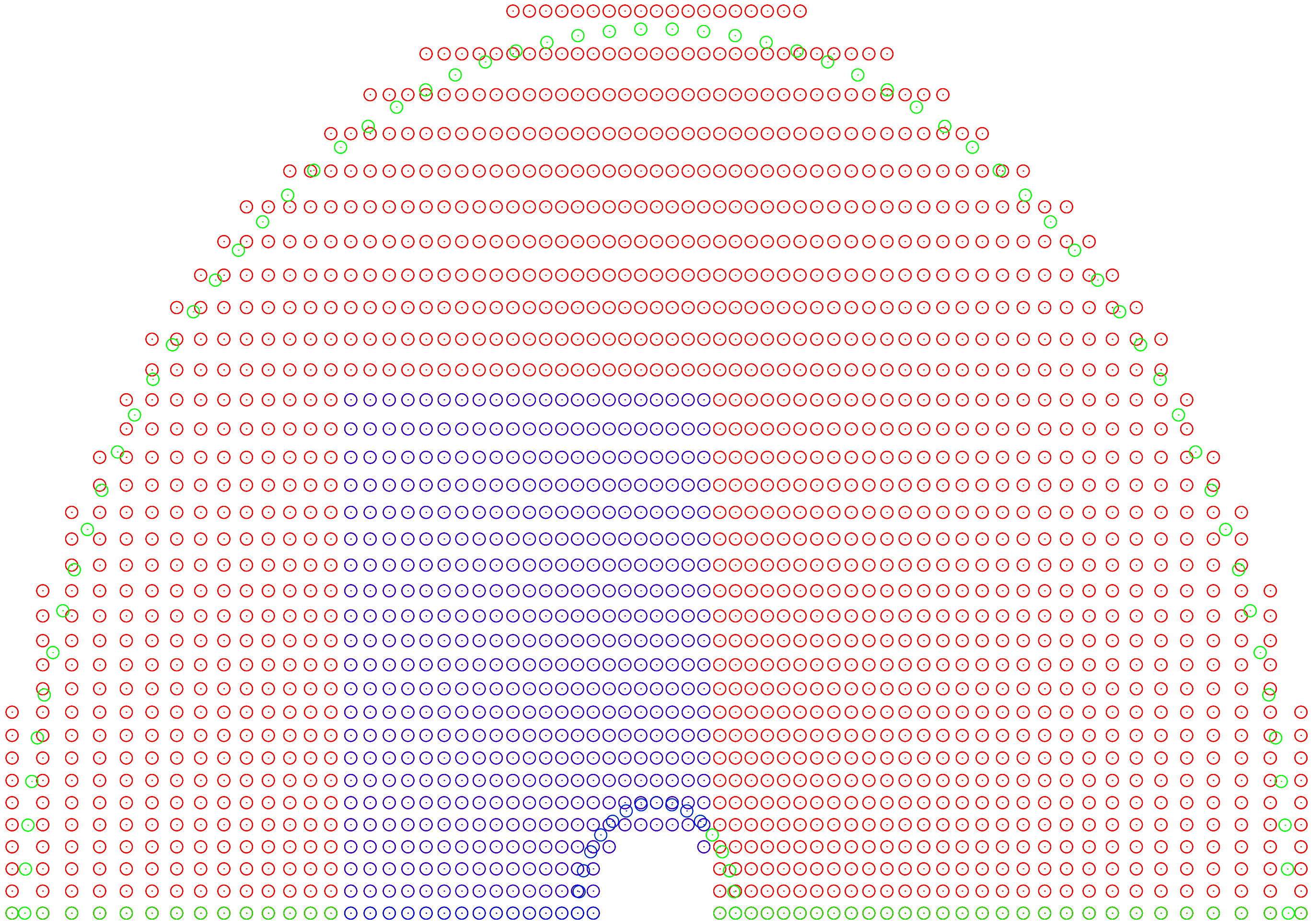
5. Experiments

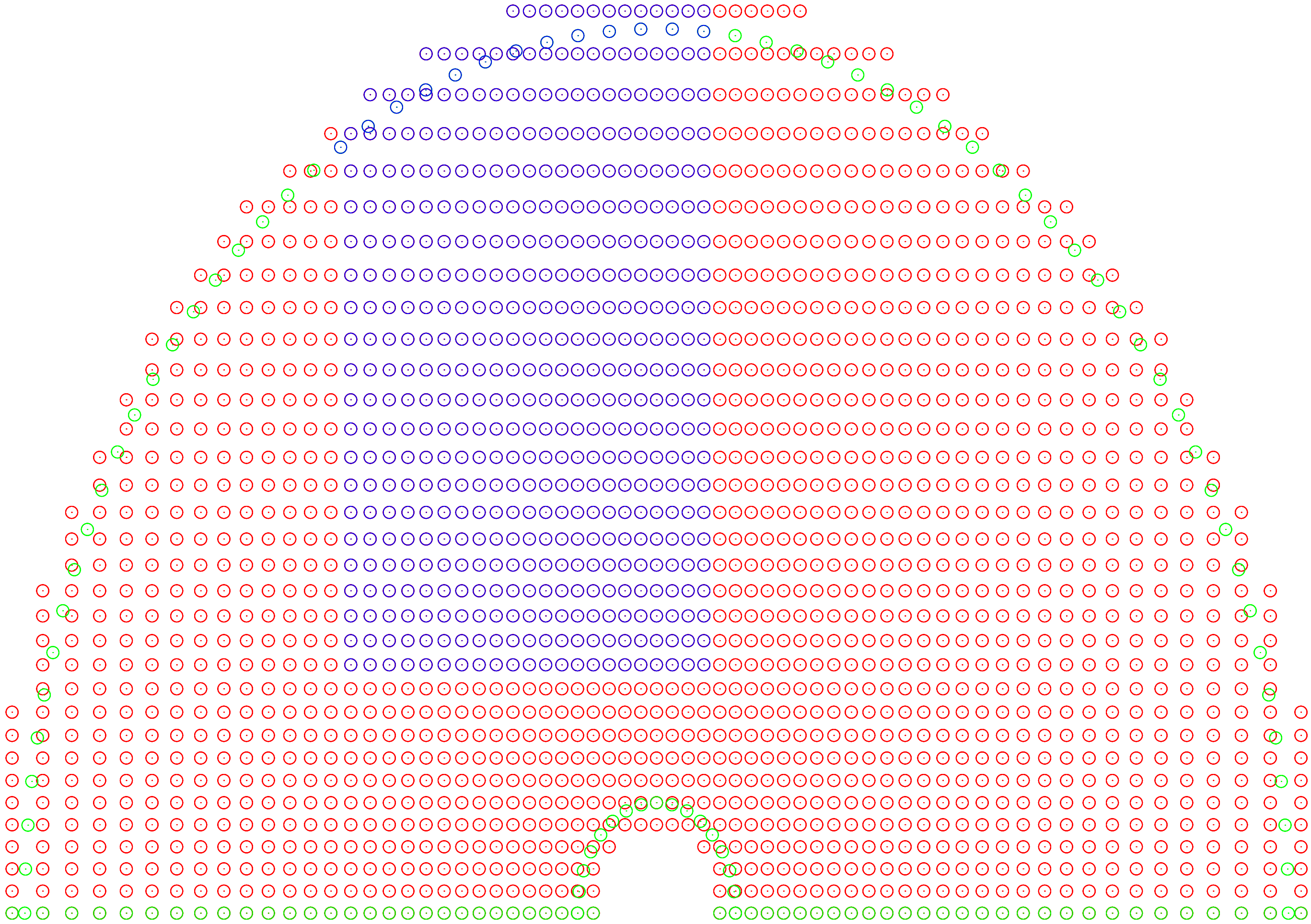
The experiments were performed using GNU's multiple precision library along with GNU Octave.

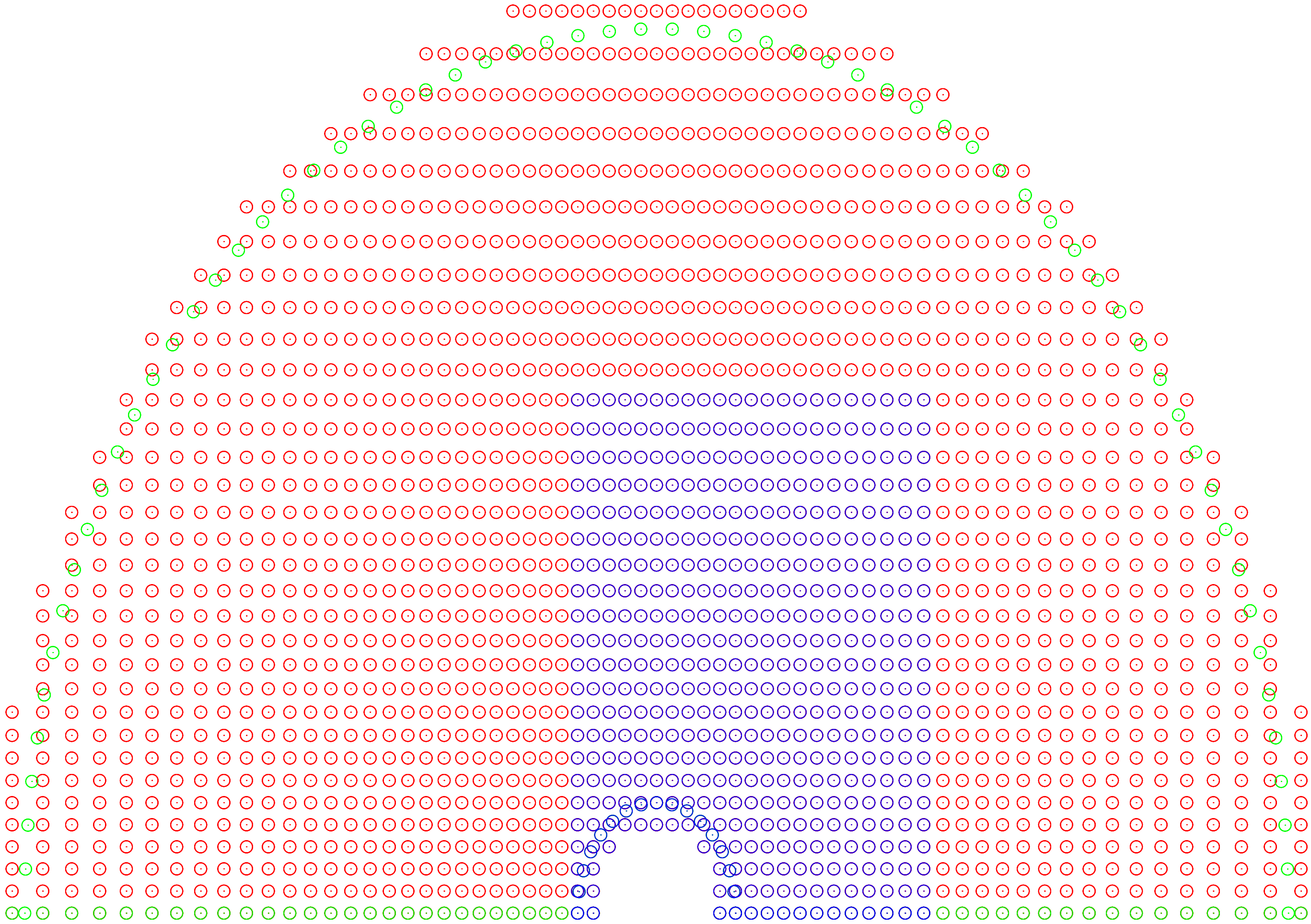


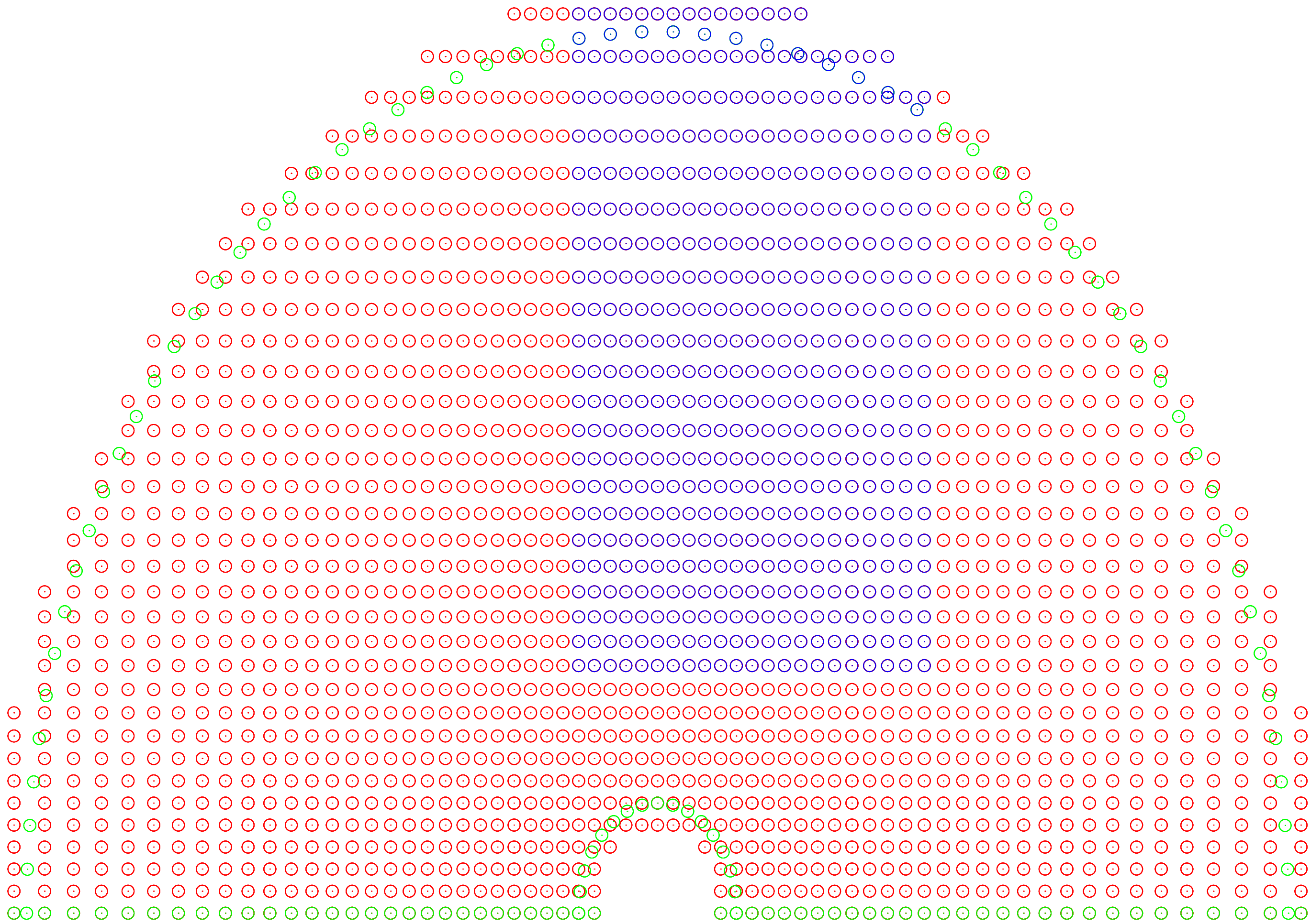


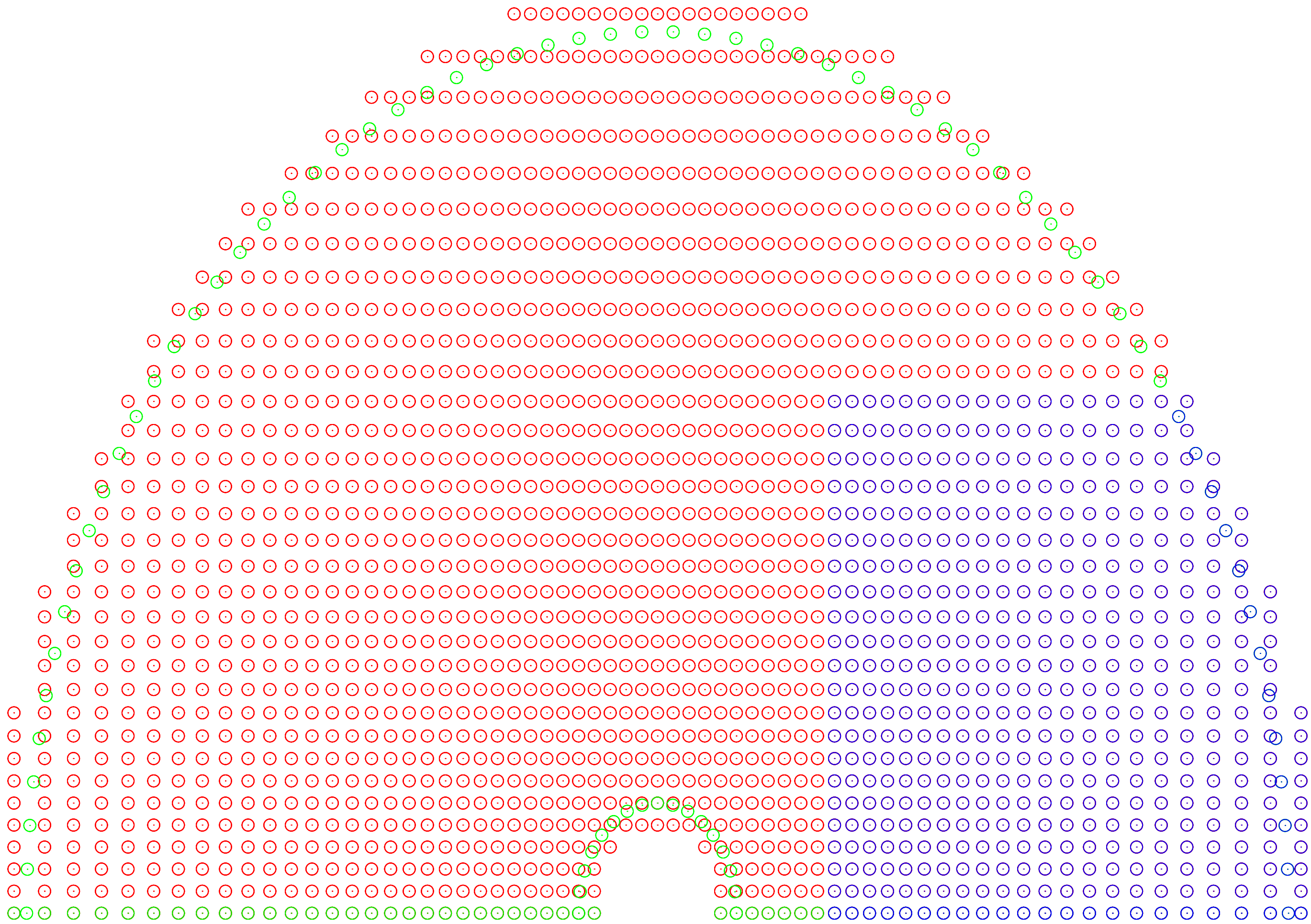


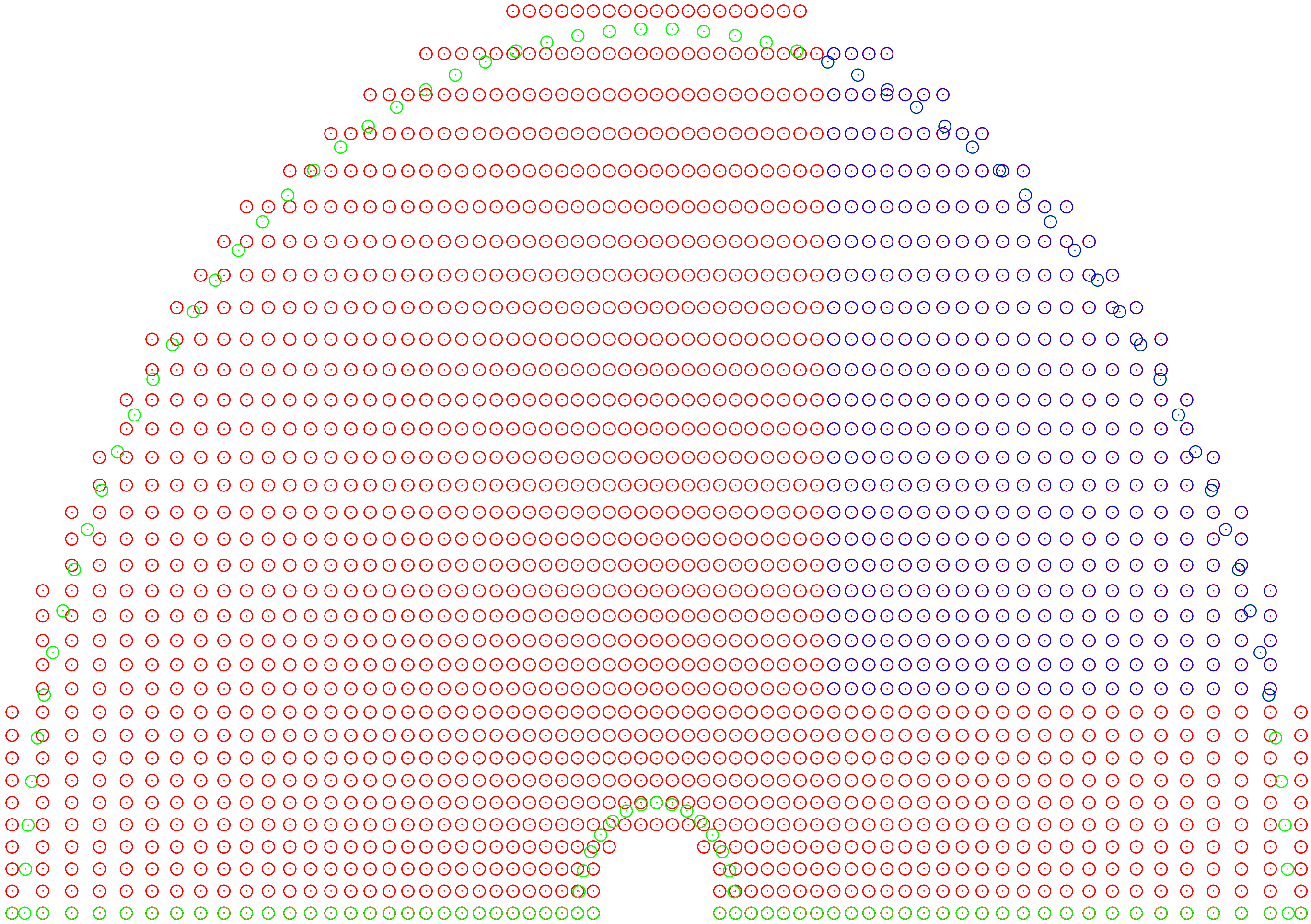












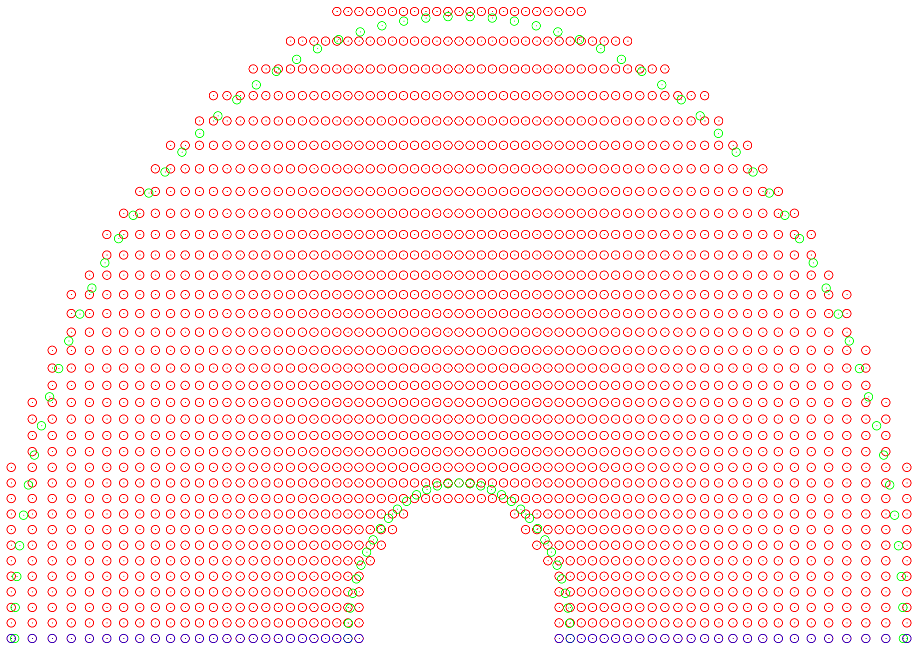
In the first set of experiments, we consider:
I. Function value interpolation
II. Poisson's equation
III. The Heat Equation
IV. The One-Way Wave Equation
V. The Wave equation.

In all cases
the domain was the square $[0,4]^2$,
the collocation points were chosen on a regular grid,
the data was random in $[0,1]$, and
the iteration was stopped when max-norm of the residual error is less than $10^{-10}\,\|s\|_\phi$.

In the following table, n denotes the number of grid points and κ denotes the required number of iterations.

n	$\kappa(I)$	$\kappa(II)$	$\kappa(III)$	$\kappa(IV)$	$\kappa(V)$
1089	6	8	8	15	19
3024	6	11	13	28	35
9801	7	17	23	53	64
$- - --$	$-$	$--$	$--$	$--$	$--$
$\kappa \sim$	n^0	$n^{0.34}$	$n^{0.48}$	$n^{0.57}$	$n^{0.55}$

6. Rate of Convergence



We consider Poisson's equation

$$\Delta u = 1 \text{ on } \Omega$$

$$u = 0 \text{ on } \partial\Omega \backslash x\text{-axis}$$

$$\frac{\partial u}{\partial y} = 0 \text{ on } \partial\Omega \cap x\text{-axis}$$

which has exact solution

$$u(x, y) = A + B \log(x^2 + y^2) + (x^2 + y^2)/4.$$

Our basic function is (with $\tau = 1.2$)

$$\phi(x,y) := \phi_{1,3}(x/\tau)\phi_{1,3}(y/\tau).$$

Using the error analysis of Franke and Schaback, assuming that it applies, one finds that

$\|u - s\|_{L_\infty(\Omega)} = O(h^2)$ and $\|1 - \Delta s\|_{L_2(\Omega)} = O(h^2)$.

Experiments show that

h	$\|u-s\|_{L_\infty(\Omega)}$	$\|1-\Delta s\|_{L_2(\Omega)}$
0.1	$2.70e-2$	$2.21e-1$
0.05	$2.23e-3$	$2.09e-2$
0.025	$1.71e-4$	$2.48e-3$
0.0125	$1.45e-5$	$3.53e-4$

which looks like

$\|u - s\|_{L_\infty(\Omega)} \sim 115\, h^{3.6}$ and $\|1 - \Delta s\|_{L_2(\Omega)} \sim 248\, h^{3.1}$.

7. Choosing the dilation parameter

Our basic function is

$$\phi(x, y) := \phi_{1,3}(x/\tau)\phi_{1,3}(y/\tau).$$

In this experiment we test the suggestion that the dilation parameter τ be chosen to minimize $\|1 - \Delta s\|_{L_2(\Omega)}$.

	τ	$\|u - s\|_{L_\infty(\Omega)}$	$\|1 - \Delta s\|_{L_2(\Omega)}$
$h = 0.1$	1.1	0.0430	0.331
	1.15	0.0321	0.249
	1.2	0.0270	0.221
	1.25	0.0205	0.176
	1.3	0.0178	0.164
	1.35	0.0155	0.141
	1.4	0.0157	0.139
	1.45	0.0148	0.128
	1.5	0.0156	0.131
	1.55	0.0156	0.130
	1.6	0.0271	0.245

	τ	$\|u-s\|_{L_\infty(\Omega)}$	$\|1-\Delta s\|_{L_2(\Omega)}$
$h = 0.05$	1.1	0.00346	0.0295
	1.15	0.00274	0.0244
	1.2	0.00223	0.0209
	1.25	0.00181	0.0198
	1.3	0.00173	0.0174
	1.35	0.00176	0.0170
	1.4	0.00185	0.0172
	1.45	0.00185	0.0172
	1.5	0.00219	0.0198
	1.55	0.00247	0.0247
	1.6	0.00275	0.0244

	τ	$\|u-s\|_{L_\infty(\Omega)}$	$\|1-\Delta s\|_{L_2(\Omega)}$
$h = 0.025$	1.1	0.000267	0.00323
	1.15	0.000214	0.00281
	1.2	0.000171	0.00248
	1.25	0.000150	0.00228
	1.3	0.000156	0.00225
	1.35	0.000169	0.00230
	1.4	0.000189	0.00242
	1.45	0.000217	0.00240
	1.5	0.000248	0.00291
	1.55	0.000290	0.00293
	1.6	0.000338	0.00331