Finite strip method for biharmonic equation

Ivan Lirkov and Svetozar Margenov

ivan@parallel.bas.bg, margenov@parallel.bas.bg

Institute for Parallel Processing, Bulgarian Academy of Sciences

Acad. G. Bonchev, bl. 25A, 1113 Sofia, BULGARIA

http://parallel.bas.bg/~ivan/

http://parallel.bas.bg/~margenov/

Introduction

The small displacement plate bending is described by the biharmonic equation. The related fourth order boundary value problem reads as follows:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = f(x,y) \qquad (x,y) \in \Omega$$

with boundary conditions

$$w|_{\Gamma_\alpha} = \left.\frac{\partial w}{\partial n}\right|_{\Gamma_\alpha} = 0, \qquad w|_{\Gamma_\beta} = \left.\frac{\partial^2 w}{\partial n^2}\right|_{\Gamma_\beta} = 0,$$

$$\left.\left(\frac{\partial^2 w}{\partial n^2} + \nu\frac{\partial^2 w}{\partial \tau^2}\right)\right|_{\Gamma_\gamma} = \left.\left(\frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n \partial \tau^2}\right)\right|_{\Gamma_\gamma} = 0,$$

Introduction

The computational domain Ω is a polygonal domain in $\mathbb{R}^2$, (n, τ) are the external unit normal and unit tangential vectors to $\partial\Omega$, and $\nu \in (0, \frac{1}{2})$ is the Poisson ratio. The boundary $\partial\Omega = \Gamma_\alpha \cup \Gamma_\beta \cup \Gamma_\gamma$ is split into three parts where the boundary conditions correspond to clamped, joint and free edges.

$$w|_{\Gamma_\alpha} = \left.\frac{\partial w}{\partial n}\right|_{\Gamma_\alpha} = 0, \qquad w|_{\Gamma_\beta} = \left.\frac{\partial^2 w}{\partial n^2}\right|_{\Gamma_\beta} = 0,$$

$$\left.\left(\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial \tau^2}\right)\right|_{\Gamma_\gamma} = \left.\left(\frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n \partial \tau^2}\right)\right|_{\Gamma_\gamma} = 0,$$

Discretization

The weak formulation of the boundary value problem is: find

$$w \in \mathcal{V} = \left\{ v \in H^2(\Omega) : v|_{\Gamma_\alpha} = \left.\frac{\partial v}{\partial n}\right|_{\Gamma_\alpha} = 0,\ v|_{\Gamma_\beta} = 0 \right\},$$

such that

$$a(w,v) = \int_\Omega f v dx, \qquad \forall v \in \mathcal{V}$$

where

$$a(w,v) =$$

$$\int_\Omega \left[w_{xx} v_{xx} + w_{yy} v_{yy} + \nu (w_{xx} v_{yy} + w_{yy} v_{xx}) + 2(1-\nu) w_{xy} v_{xy} \right] d\Omega.$$

Discretization

The weak formulation of the boundary value problem is: find

$$w \in \mathcal{V} = \left\{ v \in H^2(\Omega) : v|_{\Gamma_\alpha} = \frac{\partial v}{\partial n}\bigg|_{\Gamma_\alpha} = 0,\, v|_{\Gamma_\beta} = 0 \right\},$$

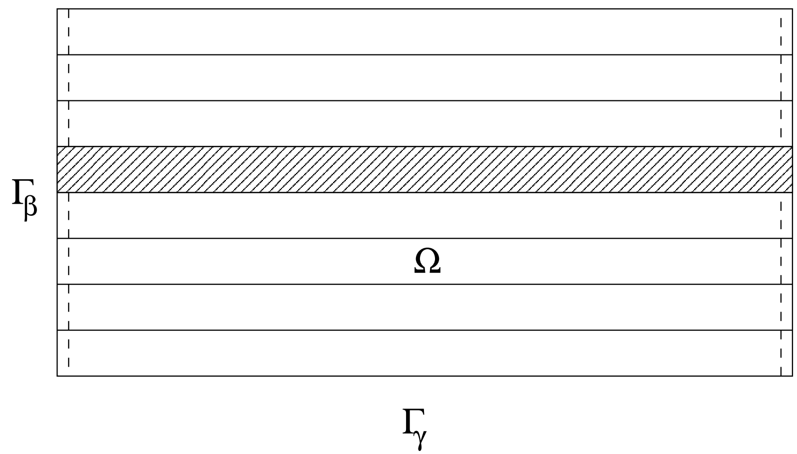
such that

$$a(w, v) = \int_\Omega f v dx, \qquad \forall v \in \mathcal{V}$$

The variational formulation is next discretized, i.e., the continuous space $\mathcal{V}$ is replaced by a finite dimensional subspace.

Finite strip method

Let us consider the model problem in $\Omega = (0, l_1) \times (0, l_2)$



The FSM is associated with the semi-discrete space

$$\mathcal{V}^{(n)} = \left\{ v(x, y) = \sum_{k=1}^{n} v_k(y) \frac{2}{l_1} \sin \frac{k\pi x}{l_1} : v_k(y) \in H^2(0, l_2) \right\}.$$

Finite strip method

The classical FSM is now obtained if $H^2(0, l_2)$ is substituted by the standard finite element space $\mathcal{V}_h(0, l_2)$ of Hermitian cubic splines. This leads to the discrete space

$$\mathcal{V}_h^{(n)} = \left\{ v_h(x, y) = \sum_{k=1}^{n} v_{h,k}(y) \frac{2}{l_1} \sin \frac{k\pi x}{l_1} : v_{h,k}(y) \in \mathcal{V}_h(0, l_2) \right\}.$$

Finite strip method

Using the L_2 orthogonality of the basis $\left\{\sin\frac{k\pi x}{l_1}\right\}_{k=1}^{n}$ we get n independent FEM subproblems:

find a function $W_{h,k} \in \mathcal{V}_h(0.l_2)$, satisfying

$$\mathcal{A}_{h,k}(W_{h,k}, V_h) = \int_\Omega f V_h \frac{2}{l_1} \sin\frac{k\pi x}{l_1} d\Omega, \qquad \forall V_h \in \mathcal{V}_h(0, l_2),$$

where

$$\begin{aligned}\mathcal{A}_{h,k}(W_{h,k}, V_h) &= \int_0^{l_2} \left\{ \frac{k^4\pi^4}{l_1^4} W_{h,k} V_h + 2(1-\nu)\frac{k^2\pi^2}{l_1^2} W'_{h,k} V'_h \right. \\ &+ \left. W''_{h,k} V''_h + \nu \frac{k^2\pi^2}{l_1^2} \left[W_{h,k} V''_h + W''_{h,k} V_h \right] \right\} dy.\end{aligned}$$

Systems of linear algebraic equations

The standard computational procedure leads to the linear systems of equations

$$A_k \mathbf{w}_k = \mathbf{f}_k,$$

where A_k stands for the matrix that corresponds on the basis function $\{\sin \frac{k\pi x}{l_1}\}$. In the considered setting, the matrices A_k are seven-diagonal. Hence, applying the generalized Thomas algorithm we obtain a total computational cost

$$\aleph = \mathcal{O}\left(n \frac{l_2}{h}\right).$$

Parallel algorithm

The global matrix of the linear algebraic problem is a block-diagonal matrix. This allows for a scalable parallel implementation of the algorithm. The communications of the parallel algorithm are restricted to the final step of the algorithm where the approximate solution is computed as a sum of the terms in the related sine expansion.

$$w_h^{(n)}(x,y) = \sum_{k=1}^{n} W_{h,k}(y) \frac{2}{l_1} \sin \frac{k\pi x}{l_1}.$$

Future work

The proposed here direct solver for biharmonic equation will be used as a preconditioner for solution of fourth order boundary value problem with variable coefficients of elliptic type. The rate of convergence of such iterative method will depend only on the ratio between maximal and minimal values of the coefficients. This solver will remain highly efficient and with the same parallel features.