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Multiresolution Expansions of Tempered Distributions[1]

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The space of tempered distributions of order r, $r \in \mathbf{N}_0$, $\mathcal{S}'_r$ is the dual space of the space

$$\mathcal{S}_r = \{f \in C^r \ ; \ |f^{(\alpha)}(x)| \le C_\alpha (1+|x|^2)^{-r/2}, \\ x \in \mathbf{R}, \ 0 \le \alpha \le r\}.$$

A sequence $\{f_k\}_{k\in\mathbf{N}}$, $f_k \in \mathcal{S}_r$, converges to 0 in $\mathcal{S}_r$ as $k \to \infty$ if and only if for every α, $0 \le \alpha \le r$,

$$\lim_{k\to\infty} \sup_{x\in\mathbf{R}} (1+|x|^2)^{r/2} |f_k^{(\alpha)}(x)| = 0.$$

The Schwartz space $\mathcal{S}$ is the projective limit of $\mathcal{S}_r$ as r tends to infinity. Its strong dual is called the space of tempered distributions $\mathcal{S}'$. $\mathcal{S}' = \cup_{r\in\mathbf{N}_0} \mathcal{S}'_r$, and the order of a distribution is the smallest $r \in \mathbf{N}_0$, such that $f \in \mathcal{S}'_r$.

Lemma 1. *Let there be given a sequence $\{\sigma_n\}_{n\in\mathbf{N}}$, $\sigma_n \in \mathcal{S}_{r+1}$, $n \in \mathbf{N}$. The sequence $\{\sigma_n\}_{n\in\mathbf{N}}$ converges to σ in the norm of $\mathcal{S}_r$ if and only if*

a) for every compact set $K \subset \mathbf{R}$ and every α, $\alpha = 0, 1, \ldots, r$, $\sigma_n^{(\alpha)}$ converges uniformly to $\sigma^{(\alpha)}$,

b) $\sup_{x\in\mathbf{R}, n\in\mathbf{N}} (1+|x|^2)^{(r+1)/2} |\sigma_n^{(\alpha)}(x)| \le C_\alpha$, $\alpha = 0, 1, \ldots, r+1$. That is, the sequence $\{\sigma_n\}_{n\in\mathbf{N}}$ is bounded in $\mathcal{S}_{r+1}$.

A multiresolution approximation of L^2 (MRA) is an increasing sequence of closed linear subspaces $V_j, j \in \mathbf{Z}$, of L^2, such that, for all $f \in L^2$ and all $j, k \in \mathbf{Z}$,

$$\bigcap_{-\infty}^{\infty} V_j = \{0\} \quad \text{and} \quad \bigcup_{-\infty}^{\infty} V_j \text{ is dense in } L^2; \tag{1}$$

$$f(x) \in V_j \iff f(2x) \in V_{j+1}; \tag{2}$$

$$f(x) \in V_0 \iff f(x-k) \in V_0; \tag{3}$$

$$\begin{gathered} \exists \phi \in V_0, \text{ such that} \\ \{\phi(x-k),\ k \in \mathbf{Z}\} \text{ is an ONB of } V_0. \end{gathered} \tag{4}$$

The function ϕ given by (4) is called scaling function.

A multiresolution approximation, $V_j,\ j \in \mathbf{Z}$, is *r-regular* ($r \in \mathbf{N}_0$), if and only if for every $m \in \mathbf{N}$ there exists $C_m > 0$ such that

$$\left|\phi^{(q)}(x)\right| \leq C_m(1+|x|^2)^{-m/2}, \quad x \in \mathbf{R}, \tag{5}$$

$0 \leq q \leq r$. We write $\phi \in \tilde{\mathcal{S}}_r$ for short. (Obviously $\tilde{\mathcal{S}}_r \subset \mathcal{S}_r$.) It is well known that for every $r \in \mathbf{N}$ there exists an $r-$regular MRA, i.e., a function ϕ which satisfies conditions (4) and (5). This fact enables the analysis of the space of tempered distributions $\mathcal{S}'$.

Let there be given an $r-$regular MRA of L^2 ($r \in \mathbf{N}_0$) and let ϕ be a scaling function with properties (4) and (5). Operator E_0, the orthogonal projection of L^2 onto V_0, is defined by the kernel

$$E(x,y) = \sum_{k \in \mathbf{Z}} \phi(x-k)\phi(y-k), \quad x, y \in \mathbf{R}, \tag{6}$$

as follows: If $h \in L^2$ then

$$E_0 h(x) = \langle h(y), E_0(x,y) \rangle = \int E(x,y) h(y) dy,$$

$x \in \mathbf{R}$. Let $j \in \mathbf{Z}$. Then the kernel of the projection operator onto V_j is

$$E_j(x,y) = 2^j E(2^j x, 2^j y), \quad x, y \in \mathbf{R}, \tag{7}$$

and the projection of $h \in L^2$ onto V_j is given by

$$E_j h(x) = \int h(y) E_j(x,y) dy, \quad x \in \mathbf{R}. \tag{8}$$

Functions $E_j(x,y)$, $x, y \in \mathbf{R}$, are the reproducing kernels for V_j, $j \in \mathbf{Z}$. We use the notation $h_j = E_j h$. Since we are interested in the properties of the kernel of integral transform (8) we may leave the MRA framework and allow j to be a real number, not necessarily an integer.

Some properties: $\forall m \in \mathbf{N}\ \exists C_m > 0$

$$\sup_{x,y\in\mathbf{R}} \sup_{0\le\alpha,\beta\le r} (1+|x-y|)^m \left|\frac{d^\alpha}{dx^\alpha}\frac{d^\beta}{dy^\beta}E(x,y)\right| \le C_m. \tag{9}$$

$$E(x+k,y+k) = E(x,y), \quad x,y\in\mathbf{R}, \quad k\in\mathbf{Z}.$$

$$E(x,y) = E(y,x).$$

$$\int E(x,y)x^\alpha dx = y^\alpha, \quad y\in\mathbf{R}, \quad 0\le\alpha\le r. \tag{10}$$

Thus, if the scaling function ϕ is $r-$regular, all polynomials up to the order r belong to V_0 and, therefore, to V_j, $j \ge 0$. The proof of the following theorem is based on this property.

THEOREM 1. *Let V_j, $j\in\mathbf{Z}$, be an $r-$regular MRA of L^2. Then, for every α, $0\le\alpha\le r$, there exists $R_\alpha \in L^\infty(\mathbf{R}\times\mathbf{R})$, satisfying*

$$|R_\alpha(x,y)| \le C_m(1+|x-y|)^{-m}, \quad x,y\in\mathbf{R}, \quad m\in\mathbf{N},$$

$$\int R_\alpha(x,y)dy = 0, \quad x\in\mathbf{R},$$

and if $f\in C^r$, then

$$\frac{d^\alpha}{dx}E_jf(x) = 2^j\int g(2^j(x-y))\frac{d^\alpha}{dy}f(y)dy$$

$$+2^j\int R_\alpha(2^jx,2^jy)\frac{d^\alpha}{dy}f(y)dy, \qquad x\in\mathbf{R},$$

where $g\in\mathcal{D}$, $\int g = 1$.

THEOREM 2. *Let $h \in C^r$, such that $h^{(\alpha)}$ is of polynomial growth for every α, $0 \leq \alpha \leq r$ and let $h_j(\cdot) = \langle h(y), E_j(\cdot, y)\rangle$, $j \in \mathbf{N}$. Then the sequence of derivatives $\{h_j^{(\alpha)}\}_{j\in\mathbf{N}}$ converges pointwisely and, moreover, uniformly on compact sets to $h^{(\alpha)}$ as $j \to \infty$, for any α, $0 \leq \alpha \leq r$.*

By Theorem 1 we only need to prove that

$$\lim_{j\to\infty} 2^j \int R_\alpha(2^j x, 2^j y) h^{(\alpha)}(y) dy = 0,$$

uniformly on compact sets.

THEOREM 3. *Let $\phi \in \tilde{\mathcal{S}}_r$. Let $E_j(x, y)$, $j \in \mathbf{Z}$, denote corresponding kernels given by (6) and (7). Let $\sigma \in \mathcal{S}_r$. Then*

i) the sequence $\langle kE_j(kx, ky), \sigma(x)\rangle$ converges to $\sigma(y)$ in $\mathcal{S}_r$ as $k \to \infty$, for any given $j \in \mathbf{Z}$.

ii) the sequence $\langle kE_k(kx, ky), \sigma(x)\rangle$ converges to $\sigma(y)$ in $\mathcal{S}_r$ as $k \to \infty$.

iii) the sequence $\langle E_j(x, y), \sigma(x)\rangle$ converges to $\sigma(y)$ in $\mathcal{S}_r$ as $j \to \infty$.

Step 1. Let there be given $j \in \mathbf{N}$. We first show that the family

$$\{\langle kE_j(kx, k\cdot), \sigma(x)\rangle; \quad k > 0 \ \}$$

is bounded in $\mathcal{S}_{r+1}$. We will prove this by showing that $|a(y) - b(y)| \leq C_1$ and $|b(y)| \leq C_2$, $y \in \mathbf{R}$, where

$$a(y) := (1+y^2)^{\frac{r+1}{2}} \frac{d^p}{dy^p} \int_{-\infty}^{\infty} kE_j(kx, ky)\sigma(x)dx,$$

$$b(y) := (1+y^2)^{\frac{r+1}{2}} \sigma(y) \frac{d^p}{dy^p} \int_{-\infty}^{\infty} kE_j(kx, ky)dx,$$

$y \in \mathbf{R}$, $p \leq r+1$.

THEOREM 4. *Let there be given $\sigma \in \mathcal{S}_{r+1}$, and let $E(x, y)$ be the kernel of the projection operator given by (6) with $\varphi \in \tilde{\mathcal{S}}_r$ and $E_j(x, y)$ given by (7). Then for every p, $0 \leq p \leq r+1$ there is a constant $C > 0$ such that*

a) $\sup_{y \in \mathbf{R}} |a(y) - b(y)| \leq C$.

b) $\sup_{y \in \mathbf{R}} |b(y)| \leq C$.

Step 2. By Theorem 2, Lemma 1 and Theorem 4, $\langle kE_j(kx, ky), \sigma(x)\rangle$ converges to $\sigma(y)$ in $\mathcal{S}_r$ as $k \to \infty$.

Definition 1. Let there be given a tempered distribution $h \in \mathcal{S}'$ of order r. Choose an $r-$regular MRA. The *multiresolution expansion* of h is given by the sequence $(h_j)_{j\in\mathbf{Z}}$ defined by

$$\langle h_j, \sigma\rangle = \langle E_j h, \sigma\rangle := \langle h, E_j\sigma\rangle, \quad \sigma \in \mathcal{S}.$$

Corollary 1. *Let there be given a tempered distribution $h \in \mathcal{S}'$. Then there exists $r \in \mathbf{N}$, such that every $r-$regular MRA defines the multiresolution expansion of h which converges to h in $\mathcal{S}'$ as $j \to \infty$.*

Proof. For a given $h \in \mathcal{S}'$ there exists $r_0 \in \mathbf{N}$ such that h belongs to $\mathcal{S}'_{r_0}$, the dual space of $\mathcal{S}_{r_0}$. We take an $r_0 - 1$ - regular MRA. Definition 1 and Theorem 3 applied to the corresponding multiresolution expansion imply the result. ∎

Quasiasymptotic behavior of a distribution turned out to be more appropriate for Abelian and Tauberian type theorems for several integral transforms such as Fourier, Laplace, Stieltjes and Mellin transform, than some other types of asymptotic behaviors. It is also used as an asymptotic analysis tool.

Some functions have quasiasymptotic behavior different from their classical asymptotic behavior, or the last does not exist at all (e.g., e^{ix} at infinity, delta distribution as well as its derivatives). This is another motivation for the study of quasisaymptotic behavior.

Definition 2. Let $h \in \mathcal{S}'$ and let $c(x),\ x \in (a, \infty),\ \ a > 0$, be a continuous positive function. We say that h *has the quasiasymptotics at infinity (in $\mathcal{S}'$) related to* $c(k)$, if there exists $g \in \mathcal{S}', g \neq 0$, such that

$$\lim_{k\to\infty} \left\langle \frac{h(kx)}{c(k)}, \sigma(x) \right\rangle = \langle g(x), \sigma(x) \rangle, \quad \sigma \in \mathcal{S}.$$

In this case we write: $h \overset{q}{\sim} g$ at ∞ related to $c(k)$ in $\mathcal{S}'$.

THEOREM 5. *Let a distribution* $h \in \mathcal{S}'$ *have quasiasymptotics at infinity related to* ρ *equal to* $\gamma \neq 0$ $(h \overset{q}{\sim} \gamma)$. *Then there exists* $r > r_0$, *where* r_0 *is the order of* h, *such that for a scaling function* $\phi \in \tilde{\mathcal{S}}_r$, *and* E_j *given by (6) and (7),* $h_j(x) = \langle h(y), E_j(x,y)\rangle$, $j \in \mathbf{N}$, *has quasiasymptotics at infinity related to* ρ *equal to* $\gamma \neq 0$ $(h_j \overset{q}{\sim} \gamma)$.

THEOREM 6. *Let there be given* $h \in \mathcal{S}'$ *of order* r_0. *Let* $\phi \in \tilde{\mathcal{S}}_r$, *for* $r > r_0$, $h_j = E_j h$, E_j *being given by (6) and (7). Let* h_j, $j \in \mathbf{N}$, *have the quasiasymptotics at infinity equal to* γ_j, *and let* $\gamma_j \to \gamma \neq 0$ *as* $j \to \infty$. *Moreover, assume that the family* $\{h(ky)/\rho(k) |\ k \in \mathbf{N}\}$ *is bounded in* $\mathcal{S}_r$. *Then* h *has the quasiasymptotics at infinity equal to* γ.