# On the general solutions of the Brinkman and the Stokes equation in spheroidal coordinates

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# Overview

- Creeping flow around and through porous particles
- Governing equations in the primitive variables
- Steady axisymmetric flow around spheroidal particles
- General solutions of the Stokes and Brinkman PDE's in the spheroidal stream function formulation
- Application: BV solutions to axisymmetric flow around porous or rigid spheroidal particles
- Numerical solution of the present problem by using a boundary integral equation formulation

- Creeping flow around and through porous particles
  - **Chemical and biological processes**
  - **Industrial and engineering applications**
    - **Flow of fluids through porous beds (fixed or fluidized),**
    - **Sedimentation of fine particulate suspensions,**
    - **Modeling of polymer macromolecule coils in a solvent,**
    - **Catalytic reactions, where porous pellets are used,**
    - **Floc settling processes, etc.**

- Governing equations in the primitive variables

**Stokes:** $$\mu \Delta \mathbf{v} = grad\, p \qquad (1)$$

**Brinkman:** $$\mu \Delta \mathbf{v} - \frac{\mu}{k}\mathbf{v} = grad\, p \qquad (2)$$

**Continuity:** $$div\, \mathbf{v} = 0 \qquad (3)$$

$\mu = const$ - dynamic viscosity

$k = const$ - permeability

# • Axisymmetric flow around spheroidal particles

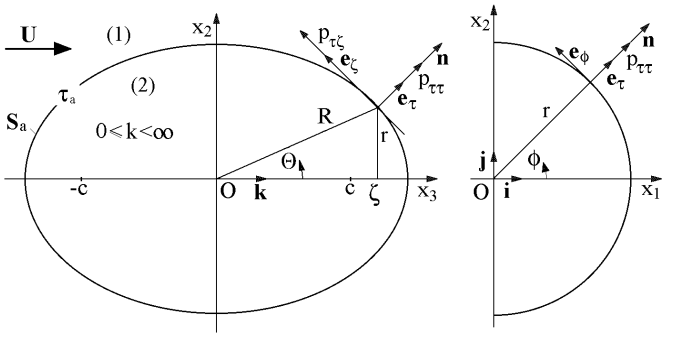


Fig. 1 Porous prolate spheroid

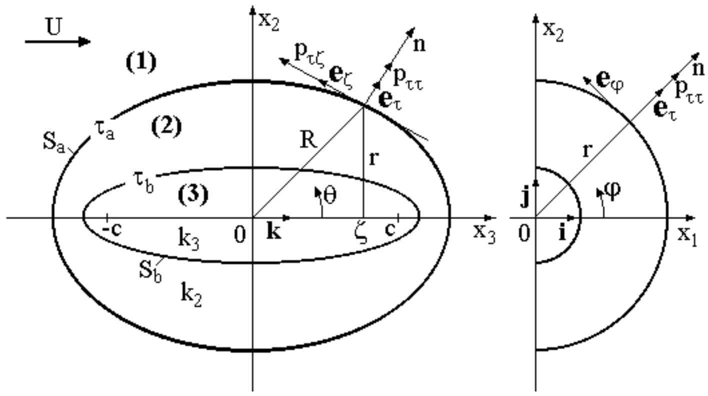


Fig. 2 Porous prolate spheroidal shell

$$
\begin{aligned}
x_1 &= c\sqrt{\tau^2-1}\sqrt{1-\zeta^2}\cos\varphi \\
x_2 &= c\sqrt{\tau^2-1}\sqrt{1-\zeta^2}\sin\varphi \qquad (4)\\
x_3 &= c\,\tau\zeta
\end{aligned}
$$

$$1\le\tau<\infty,\ \ -1\le\zeta\le 1,\ \ 0\le\varphi<2\pi$$

$$\frac{x_3^2}{a_{3,i}^2}+\frac{x_1^2+x_2^2}{a_{1,i}^2}=1 \qquad (5)$$

$$a_{3,i}=c\,\tau_i,\ \ a_{1,i}=c\sqrt{\tau_i{}^2-1},\ \ i=1,2,\dots,m-1$$

# • Stream function formulation

- **Axisymmetric flow:**

$$v_\varphi = 0,\quad \partial/\partial\varphi = 0, \qquad \mathbf{v} = v_\tau \mathbf{e}_\tau + v_\zeta \mathbf{e}_\zeta$$

$$v_\tau = \frac{1}{c^2\sqrt{\tau^2-\zeta^2}\sqrt{\tau^2-1}}\frac{\partial\Psi}{\partial\zeta},$$

$$v_\zeta = \frac{-1}{c^2\sqrt{\tau^2-\zeta^2}\sqrt{1-\zeta^2}}\frac{\partial\Psi}{\partial\tau}$$

- **Elimination of pressure:**

**Stokes:** $$E^4\Psi = 0 \qquad (6)$$

**Brinkman:** $$E^4\Psi - K^2E^2\Psi = 0 \qquad (7)$$

**Pressure:** $$\Delta p = 0 \qquad (8)$$

$$\Psi = \Psi(\tau,\zeta), \qquad p = p(\tau,\zeta), \qquad K = \frac{L}{\sqrt{k}},$$

$$E^2 = \frac{1}{c^2(\tau^2-\zeta^2)}\left[(\tau^2-1)\frac{\partial^2}{\partial\tau^2} + (1-\zeta^2)\frac{\partial^2}{\partial\zeta^2}\right] \qquad (9)$$

- **Boundary and far-field conditions:**

$$v_\tau^{(i)}(\tau_i,\zeta) = v_\tau^{(i+1)}(\tau_i,\zeta), \qquad v_\zeta^{(i)}(\tau_i,\zeta) = v_\zeta^{(i+1)}(\tau_i,\zeta) \qquad (10a)$$

$$p^{(i)}(\tau_i,\zeta) = p^{(i+1)}(\tau_i,\zeta), \qquad p_{\tau\zeta}^{(i)}(\tau_i,\zeta) = p_{\tau\zeta}^{(i+1)}(\tau_i,\zeta) \qquad (10b)$$

$$\lim_{\tau\to\infty} v_\tau^{(1)} = \zeta, \qquad \lim_{\tau\to\infty} v_\zeta^{(1)} = \sqrt{1-\zeta^2} \qquad (10c)$$

# • General solutions of the Brinkman equation:

$$E^4\Psi - K^2E^2\Psi = 0$$

$$\left(E^2 - K^2\right)E^2\Psi = 0, \quad E^2\left(E^2\Psi - K^2\Psi\right) = 0 \tag{11}$$

$$\Psi = \Psi_1 + \Psi_2 \tag{12}$$

$$E^2\Psi_1 = 0, \qquad E^2\Psi_2 - K^2\Psi_2 = 0 \tag{13}$$

$$\Psi_j = T_j(\tau)Z_j(\zeta), \quad j = 1,2 \tag{14}$$

$$\left(1-\zeta^2\right)Z_1'' + n(n-1)Z_1 = 0, \tag{15a}$$

$$\left(1-\zeta^2\right)T_1'' + n(n-1)T_1 = 0 \tag{15b}$$

$$\left(1-\zeta^2\right)Z_2'' + \left(q^2\zeta^2 + \lambda\right)Z_2 = 0 \tag{16a}$$

$$\left(1-\zeta^2\right)T_2'' + \left(q^2\zeta^2 + \lambda\right)T_2 = 0 \tag{16b}$$

$$q = cK$$

$$\Psi_1 = \sum_{n=0}^{\infty}\left\{\left[a_{1,n}G_n(\tau) + b_{1,n}H_2(\tau)\right]G_n(\zeta) + \left[c_{1,n}G_n(\tau) + d_{1,n}H_2(\tau)\right]H_n(\zeta)\right\}$$

$G_n(x)$, $H_n(x)$ - Gegenbauer functions of first, resp. second kind, of order $n$ and degree $-\frac{1}{2}$

**Symmetry reasons and regularity on the $x_3$-axis:**

$$\Psi_1 = \sum_{n=2,4,\ldots}^{\infty}\left[A_nG_n(\tau) + B_nH_n(\tau)\right]G_n(\zeta) \tag{18}$$

- **Eigenvalue problem:** Find the eigenvalues $\lambda$ for which the equation

$$\left(1-\zeta^2\right)Z_2''+\left(q^2\zeta^2+\lambda\right)Z_2=0 \tag{19}$$

has eigensolutions bounded in the interval $0\le\zeta\le 1$ and, additionally, $Z_2(1)=0$.

Ansatz:

$$Z_2(\zeta)=\sum_{k=2,4,\dots}^{\infty} a_k G_k(\zeta) \tag{20}$$

Recursive formulas for $a_k^{(i)}$ obtained. **(Details must be omitted here):**

$$a_2=1,\quad a_4=\frac{2-\lambda\ -\frac{1}{5}q^2}{q^2\alpha_4},\quad a_6=\frac{\left[12-\lambda\ -q^2\gamma_4\right]a_4-\frac{4}{5}q^2}{q^2\alpha_6},$$

$$a_k=\frac{\left[(k-2)(k-3)-\lambda\ -q^2\gamma_{k-2}\right]a_{k-2}-q^2\beta_{k-4}a_{k-4}}{q^2\alpha_k},\quad k=8,10,\dots \tag{21}$$

$$\alpha_k=\frac{(k-3)(k-2)}{(2k-3)(2k-1)};\ \beta_k=\frac{(k+1)(k+2)}{(2k-1)(2k+1)};\ \gamma_k=\frac{2k^2-2k-3}{(2k+1)(2k-3)},\quad k=4,6,\dots$$

- Analytical calculation of the eigenvalues $\lambda_n=\lambda_n(q)$, $n=2,4,\cdots$:

$$\lim_{m\to\infty}a_m=\lim_{m\to\infty}\sum_{i=0}^{m/2-1}S_{i,m}(q)\lambda^i=0,\quad m=2,4,\cdots$$

$$P_{N/2}(\lambda)=\sum_{i=0}^{N/2-1}S_{i,N}(q)\lambda^i \tag{22}$$

Recursive expressions for $S_{i,N}(q)$ derived (not shown here).

Example $q=1$:

$P_1(\lambda)=C_1(\lambda-1.8)$

$P_2(\lambda)=C_2(\lambda-1.795305590)(\lambda-11.538027742)$

$P_3(\lambda)=C_3(\lambda-1.795304587)(\lambda-11.534818696)(\lambda-29.516030563)$

$P_4(\lambda)=C_4(\lambda-1.795304587)(\lambda-11.534818451)(\lambda-29.513713150)(\lambda-55.509104988)$, etc.

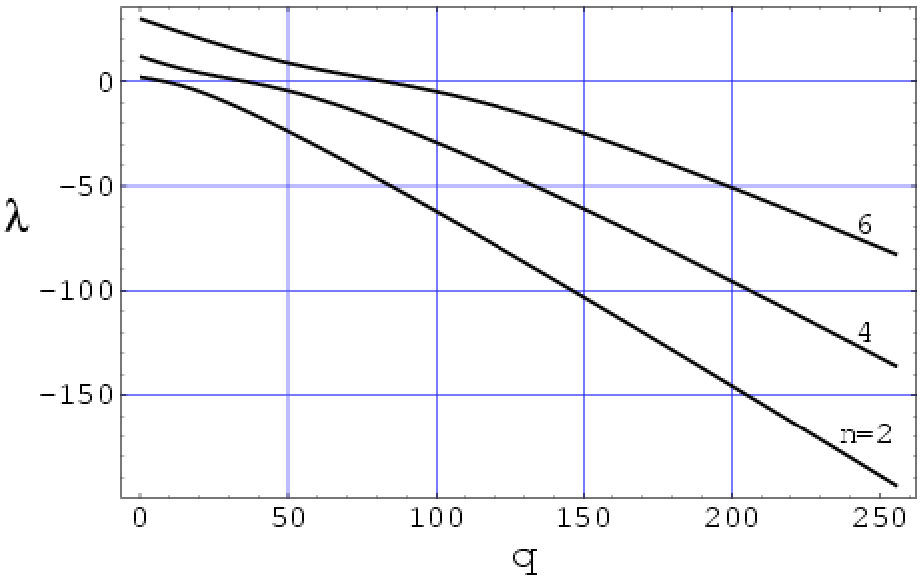


Fig. 3 The first three eigenvalues

- **General solution of the $\tau$-dependent equation:**

$$\left(1-\zeta^2\right)T_2''+\left(q^2\zeta^2+\lambda\right)T_2=0$$

Two linearly independent solutions as series expansions in terms of $\left(\tau-\tau_M\right)$, where in the case of one-layer porous shell $\tau_M=\left(\tau_a+\tau_b\right)/2$:

$$f_n(\tau)=\sum_{l=0}^{\infty}\hat{a}_{n,l}\left(\tau-\tau_M\right)^l, \qquad 1\le\tau_b\le\tau\le\tau_a \tag{23}$$

$$g_n(\tau)=\sum_{l=1}^{\infty}\hat{b}_{n,l}\left(\tau-\tau_M\right)^l, \qquad \tau_b<\tau\le\tau_a\le\infty \tag{24}$$

- Recursive formulas for the series coefficients $\hat{a}_{n,l}$ and $\hat{b}_{n,l}$ derived (not shown):
- **Complete general solution of $\Psi^{(i)}(\tau,\zeta)$, regular in the porous shell regions:**

$$\Psi^{(i)}(\tau,\zeta)=\sum_{k=2,4,\dots}^{\infty}\left\{A_k^{(i)}G_k(\tau)+B_k^{(i)}H_k(\tau)+\sum_{n=2,4,\dots}^{\infty}\left[C_n^{(i)}f_n^{(i)}(\tau)+D_n^{(i)}g_n^{(i)}(\tau)\right]a_{n,k}^{(i)}\right\}G_k(\zeta)$$

$i=2,3$

If porous core region or single porous spheroid, then $B_k^{(3)}=0$ and the second term of the above equation shall be replaced by the one general solution of $E^2\Psi_2-K^2\Psi_2=0$ that is regular on the $x_3$-axis.

## • General solutions of the Stokes equation $E^4\Psi(\tau,\zeta)=0$ (New Results)

Carrying out a limiting analysis of the general solution to the Brinkman equation when $q\to 0$, we were able to produce the following very simple complete general solution of the Stokes equation for axisymmetric flow in spheroidal coordinates:

$$\begin{aligned}\Psi(\tau,\zeta)=\sum_{n=0}^{\infty}\Big\{&\Big[\Big(A_n^{(1)}+B_n^{(1)}\left(\tau^2+\zeta^2\right)\Big)G_n(\tau)+\Big(A_n^{(3)}+B_n^{(3)}\left(\tau^2+\zeta^2\right)\Big)H_n(\tau)\Big]G_n(\zeta)\\&+\Big[\Big(A_n^{(2)}+B_n^{(2)}\left(\tau^2+\zeta^2\right)\Big)G_n(\tau)+\Big(A_n^{(4)}+B_n^{(4)}\left(\tau^2+\zeta^2\right)\Big)H_n(\tau)\Big]H_n(\zeta)\Big\}\end{aligned}$$

A consequence of the above is the following statement that turns out to be valid both in spheroidal and spherical coordinates:

If $E^2\Psi_1(\tau,\zeta)=0$, then $E^4\left(R^2\Psi_1(\tau,\zeta)\right)=0$, (27)

where $R=\sqrt{\tau^2+\zeta^2-1}$ is the distance from the origin.

The complete, in the $\zeta$-dependence regular on the $x_3$-axis, general solution spectrum of the Stokes stream function, that we derived from the above general solution and that is very useful in solving BVP to axisymetric Stokes flow in the spheroidal geometry, is the following:

$$\Psi_0^{(1)}(\tau,\zeta)=C_0^{(1)}G_2(\tau)G_2(\zeta)$$
$$\Psi_1^{(1)}(\tau,\zeta)=-D_1^{(1)}G_2(\tau)G_2(\zeta)$$
$$\Psi_2^{(1)}(\tau,\zeta)=-3D_2^{(1)}G_3(\tau)G_3(\zeta)-\tfrac{2}{25}C_2^{(1)}\left[2G_2(\tau)+3G_4(\tau)\right]\left[2G_2(\zeta)+3G_4(\zeta)\right]$$
$$\Psi_n^{(1)}(\tau,\zeta)=-\frac{1}{2}n(n+1)D_n^{(1)}G_{n+1}(\tau)G_{n+1}(\zeta)$$
$$-C_n^{(1)}\Big\{(n-1)(n+2)\big/2(2n+1)^2\Big[n^2\,G_n(\tau)G_n(\zeta)+(n+1)^2\,G_{n+2}(\tau)G_{n+2}(\zeta)$$
$$+n(n+1)\big(G_{n+2}(\tau)G_n(\zeta)+G_n(\tau)G_{n+2}(\zeta)\big)\Big]\Big\},\quad n=3,4,\cdots \tag{28}$$

$$\Psi_0^{(3)}(\tau,\zeta)=C_0^{(3)}\big(H_2(\tau)+G_1(\tau)\big)G_2(\zeta)-\tfrac{1}{2}D_0^{(3)}G_1(\zeta)$$
$$\Psi_1^{(3)}(\tau,\zeta)=-D_1^{(3)}H_2(\tau)G_2(\zeta)-\tfrac{1}{3}C_1^{(3)}G_3(\zeta)-\tfrac{1}{6}C_1^{(3)}G_1(\zeta)$$
$$\Psi_2^{(3)}(\tau,\zeta)=-3D_2^{(3)}H_3(\tau)G_3(\zeta)-\tfrac{2}{25}C_2^{(3)}\left[2H_2(\tau)+3H_4(\tau)\right]\left[2G_2(\zeta)+3G_4(\zeta)\right]$$
$$\Psi_n^{(3)}(\tau,\zeta)=-\frac{1}{2}n(n+1)D_n^{(3)}H_{n+1}(\tau)G_{n+1}(\zeta)$$
$$-C_n^{(3)}\Big\{(n-1)(n+2)\big/2(2n+1)^2\Big[n^2\,H_n(\tau)G_n(\zeta)+(n+1)^2\,H_{n+2}(\tau)G_{n+2}(\zeta)$$
$$+n(n+1)\big(H_{n+2}(\tau)G_n(\zeta)+H_n(\tau)G_{n+2}(\zeta)\big)\Big]\Big\},\quad \text{n=3,4,}\ldots$$

- **General solution of the Laplace equation for the pressure**

The Laplace equation $\Delta p = 0$ for the pressure, which is valid both in the porous and in the free-fluid regions, is completely separable in the spheroidal coordinate systems. Thus, the general solution for the pressure reads:

$$p(\tau,\zeta)=\sum_{n=0}^{\infty}\Big\{\big[\tilde{A}_n P_n(\tau)+\tilde{B}_n Q_n(\tau)\big]P_n(\zeta)+\big[\tilde{C}_n P_n(\tau)+\tilde{D}_n Q_n(\tau)\big]Q_n(\zeta)\Big\}, \tag{29}$$

where $P_n(x),\ Q_n(x)$ are the Legendre functions of the first and second kind, respectively, and of order $n$.

- **Application: Porous prolate spheroidal shell (with porous, cavity or solid core) in uniform far-field flow**

▪ **Boundary conditions:**

These comprise:

– continuity of velocity, pressure and tangential stresses across the interfaces $S_a$ and $S_b$, which separate the flow regions (1), (2) and (3), and may be expressed via the stream function and the pressure as follows:

$$\Psi^{(1)}(\tau_a,\zeta)=\Psi^{(2)}(\tau_a,\zeta),\quad \Psi^{(1)}_{\tau}(\tau_a,\zeta)=\Psi^{(2)}_{\tau}(\tau_a,\zeta) \tag{30a}$$

$$\Psi^{(1)}_{\tau\tau}(\tau_a,\zeta)=\Psi^{(2)}_{\tau\tau}(\tau_a,\zeta),\quad p^{(1)}(\tau_a,\zeta)=p^{(2)}(\tau_a,\zeta) \tag{30b}$$

$$\Psi^{(1)}(\tau_b,\zeta)=\Psi^{(2)}(\tau_b,\zeta),\quad \Psi^{(1)}_{\tau}(\tau_b,\zeta)=\Psi^{(2)}_{\tau}(\tau_b,\zeta) \tag{30a}$$

$$\Psi^{(1)}_{\tau\tau}(\tau_b,\zeta)=\Psi^{(2)}_{\tau\tau}(\tau_b,\zeta),\quad p^{(1)}(\tau_b,\zeta)=p^{(2)}(\tau_b,\zeta) \tag{30b}$$

– Far-field conditions:

$$\lim_{\tau\to\infty} v^{(1)}_{\tau}=\zeta,\quad \lim_{\tau\to\infty} v^{(1)}_{\zeta}=\sqrt{1-\zeta^2} \tag{31}$$

– Regularity of velocity and pressure in the entire flow field in both variables.

▪ System of $4N$ linear algebraic equations with as many unknowns:

$$B^{(1)}_k, D^{(1)}_k, A^{(2)}_k, B^{(2)}_k, C^{(2)}_k, D^{(2)}_k, A^{(3)}_k, C^{(3)}_k,\ k=2,4,\dots,N,$$

$N$ - Truncation order

- **Total drag force acting on the shell**

$$F_D=-\frac{2\pi}{3c}D^{(1)}_2 \tag{32}$$

# ♦ Computed results

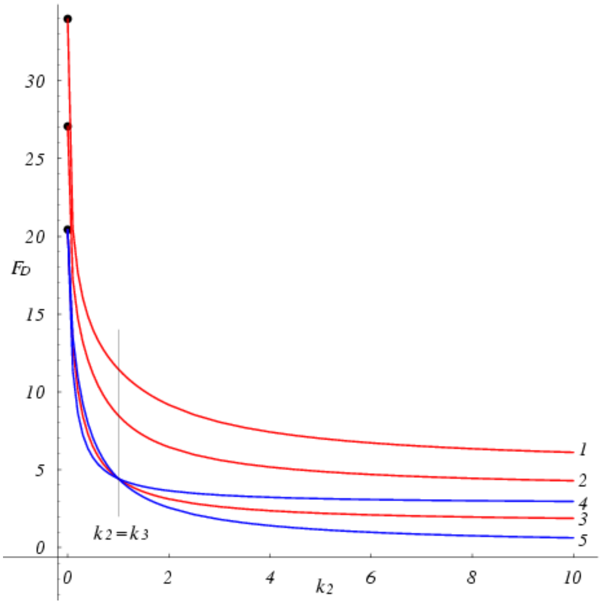


**Fig. 4** Drag force $F_D$ versus shell-medium permeability $k_2$ for the case of one-layer porous prolate spheroidal shell with porous core of a given permeability $k_3 = 1$. Varying parameters are the semifocal distance $c$ and the inner minor semiaxes $a_{1b}$ of the shell.

Curves 1, 2 and 3: $a_{1b} = 0.6$, $c = 5,\ 3,$ and $1$;

Curves 3 and 5: $a_{1b} = 0.8$ and $0.1$.

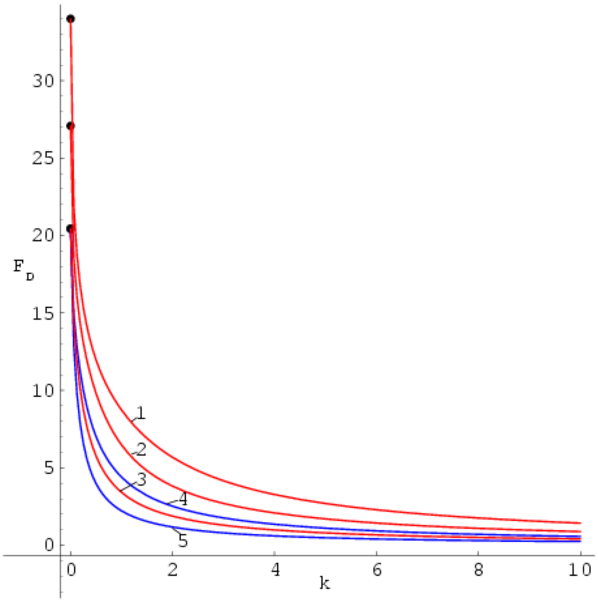


**Fig. 5** The same as in Fig. 4, but for a shell with cavity inside region ($k_3 = \infty$).

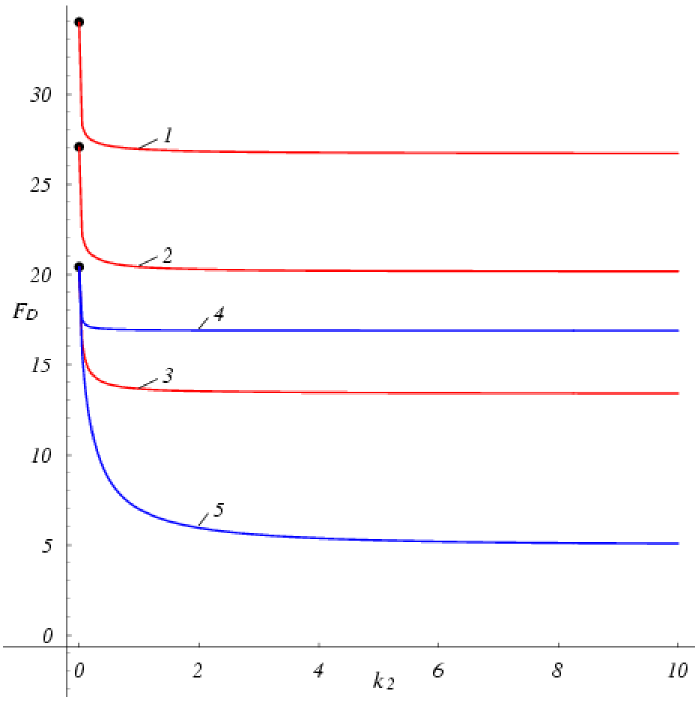


**Fig. 6** The same as in Fig. 4, but for a shell with a solid core ( $k_3 = 0$ ).

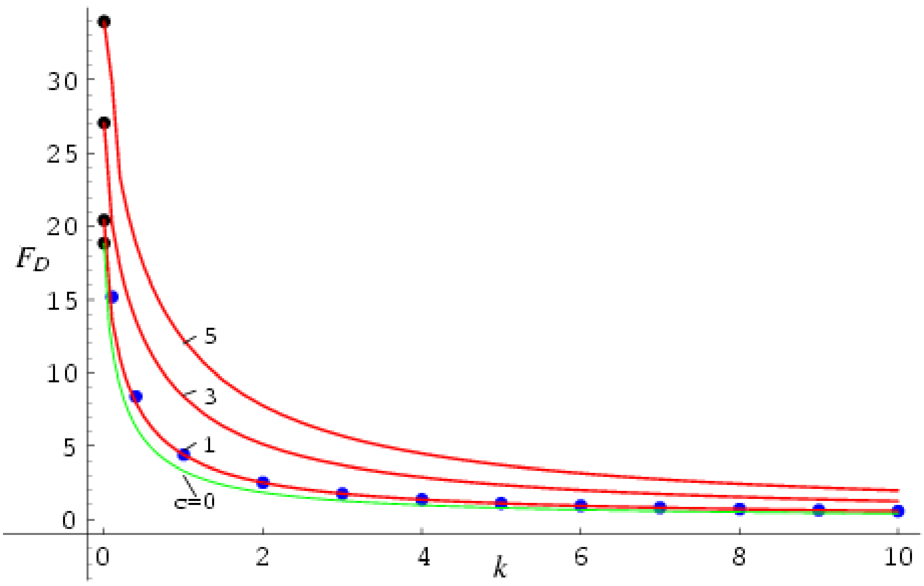


**Fig. 7** Drag force $F_D$ against permeability $k$ for a porous prolate spheroid; varying parameter is the semifocal distance $c = 0;\ 1;\ 2;\ 3;\ 5$.

Blue points: Numerical solution.

# ◆ Numerical solution by implementing an integral equation formulation for porous bodies of arbitrary shape

Stokes: $$\mu \frac{\partial^2 v(x)}{\partial x_i \partial x_j} = \frac{\partial p(x)}{\partial x_i}, \quad x \in \Omega_e, \qquad (1)$$

Brinkman: $$\mu \frac{\partial^2 v(x)}{\partial x_i \partial x_j} - \frac{\mu}{k} v_i = \frac{\partial p(x)}{\partial x_i}, \quad x \in \Omega_i, \qquad (2)$$

Continuity: $$\frac{\partial v_i(x)}{\partial x_i}, \qquad (3)$$

Integral representation formulas to (1) and (2) are well known (Ladyzhenskaya (1963), Higdon and Kojima (1981)).

Taking the limit of those formulas as the point $x$ from $\Omega_e$ or $\Omega_i$ approaches a point $\xi$ on the porous body surface $S$, the following boundary integral equations for the velocity $\mathbf{v}$ and the surface force $\mathbf{f}$ are obtained:

$$\tfrac{1}{2} v_i(\xi) = v_i^\infty(\xi) + \int_S \mathbf{K}_{ij}(\xi, y) v_j(y)\, dS_y - \int_S \mathbf{L}_{ij}(\xi, y) f_j(y)\, dS_y,$$

$$\tfrac{1}{2} v_i(\xi) = -\int_S \mathbf{K}_{\mathrm{B}ij}(\xi, y) v_j(y)\, dS_y + \int_S \mathbf{L}_{\mathrm{B}ij}(\xi, y) f_j(y)\, dS_y$$

$$\mathbf{K}_{ij}(x, y) = \frac{3}{4\pi} \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{r^5} n_k(y),$$

$$\mathbf{L}_{ij}(x, y) = \frac{3}{8\pi} \left( \frac{\delta_{ij}}{r} + \frac{(x_i - y_i)(x_j - y_j)}{r^3} \right)$$

$$\mathbf{K}\mathbf{B}_{ij}(x,y)=\frac{e^{-R}(1+R)}{4\pi}\left[\frac{\delta_{ik}(x_j-y_j)+\delta_{ij}(x_k-y_k)}{r^3}-\frac{2(x_i-y_i)(x_j-y_j)(x_k-y_k)}{r^5}\right]+\frac{1}{4\pi}\frac{\delta_{jk}(x_i-y_i)}{r^3}$$
$$+\frac{e^{-R}(6+6R+2R^2)-6}{4\pi R^2}\left[\frac{\delta_{ik}(x_j-y_j)+\delta_{ij}(x_k-y_k)+\delta_{jk}(x_i-y_i)}{r^3}-\frac{5(x_i-y_i)(x_j-y_j)(x_k-y_k)}{r^5}\right]n_k(y)$$

$$\mathbf{L}\mathbf{B}_{ij}(x,y)=\frac{e^{-R}}{4\pi}\left[\frac{\delta_{ij}}{r}-\frac{(x_i-y_i)(x_j-y_j)}{r^3}\right]+\frac{e^{-R}(1+R)-1}{4\pi R^2}\left[\frac{\delta_{ij}}{r}-\frac{3(x_i-y_i)(x_j-y_j)}{r^3}\right]$$

$$r=|x-y|,\qquad R=r\sqrt{\frac{\mu}{k}}$$

- Numerical approximation:

$$\tfrac{1}{2}v_i\left(\xi^{(m)}\right)=v_i^{\infty}\left(\xi^{(m)}\right)+\sum_{l=1}^{N}v_j\left(\xi^{(m)}\right)\int_{\Delta S_m}\mathbf{K}_{ij}\left(\xi^{(m)},y\right)dS_y-\sum_{l=1}^{N}f_j\left(\xi^{(l)}\right)\int_{\Delta S_m}\mathbf{L}_{ij}\left(\xi^{(m)},y\right)dS_y$$
$$\tfrac{1}{2}v_i\left(\xi^{(m)}\right)=-\sum_{l=1}^{N}v_j\left(\xi^{(l)}\right)\int_{\Delta S_m}\mathbf{K}\mathbf{B}_{ij}\left(\xi^{(m)},y\right)dS_y+\sum_{l=1}^{N}f_j\left(\xi^{(l)}\right)\int_{\Delta S_m}\mathbf{L}\mathbf{B}_{ij}\left(\xi^{(m)},y\right)dS_y$$
$$m=1,N$$

To avoid difficulties with the apparent singularities of the Brinkman kernels for small values of $R$, the kernel decompositions as described by Richardson and Power (1997) have been used:

$$\mathbf{K}\mathbf{B}_{ij}(x,y)=\mathbf{K}_{ij}(x,y)+\mathbf{D}_{ij}(x,y),\qquad \mathbf{L}\mathbf{B}_{ij}(x,y)=\mathbf{L}_{ij}(x,y)+\mathbf{C}_{ij}(x,y)$$

Rigid body analogy has been used to avoid any numerical integration of the double kernel potential at the singular element. Youngren and Acrivos approach has been used for the numerical integration of the weak singular kernel of the single-layer potential at the singular point.

Fig. 7 shows excellent agreement between the numerically and analytically calculated drag force acting on a porous prolate spheroid in uniform far-field flow.