On real-analytic recurrence relations for cardinal exponential B-splines

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Abstract

Let \(L_{N+1}\) be a linear differential operator of order \(N+1\) with constant coefficients and real eigenvalues \(\lambda_1, \ldots, \lambda_{N+1}\), let \(E(A_{N+1})\) be the space of all \(C^\infty\)-solutions of \(L_{N+1}\) on the real line. We show that for \(N \geq 2\) and \(n = 2, \ldots, N\), there is a recurrence relation from suitable subspaces \(E_n\) to \(E_{n+1}\) involving real-analytic functions, and with \(E_{N+1} = E(A_{N+1})\) if and only if contiguous eigenvalues are equally spaced.

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1. Introduction

Recurrence relations for generalized splines have been discussed by several authors since the appearance of the pioneering work of De Boor and Cox in [4,5], respectively, cf. also [6,7,11–13,18,20]. In order to motivate our results, let us consider briefly the case of cardinal...
 polynomial splines. It is well known that the cardinal B-splines \( M_{N+1} \) and \( M_N \) (of order \( N+1 \) and \( N \), and support in \([0, N+1]\) and \([0, N]\), respectively) are related by the identity

\[
M_{N+1}(x) = \frac{x}{N} M_N(x) + \frac{N+1-x}{N} M_N(x-1)
\]

for all \( x \in \mathbb{R} \) (see e.g. [3, p. 86]). Analogous recurrence relations were proved for trigonometric and hyperbolic B-splines in [12,18], respectively, cf. [11] for a unified proof. On the other hand, Schumaker identified the classes of generalized splines which have B-splines bases computable by recursion relations analogous to those for polynomial, trigonometric, and hyperbolic splines. He proved in [17] that, in addition to the preceding spaces, essentially the only other space of splines admitting such a basis is a certain space of Tchebycheffian splines.

Our objective is to investigate whether there exists a recurrence relation generalizing (1) to the larger class of cardinal L-splines. This question was asked independently in the Conclusion of [19, p. 1436]. Cardinal L-splines also arise in a natural way in the study of the so-called cardinal polysplines, see [1,8–10].

Polynomial and hyperbolic cardinal splines are special cases of cardinal L-splines, also known as cardinal exponential splines; here it is assumed that \( L \) is a linear differential operator of the form

\[
L = \prod_{j=1}^{N+1} \left( \frac{d}{dx} - \lambda_j \right).
\]

Throughout the paper we shall assume that the \( \lambda_1, \ldots, \lambda_{N+1} \) are real numbers and we shall often use the notation

\[
\Lambda_{N+1} := (\lambda_1, \ldots, \lambda_{N+1}).
\]

The functions in

\[
E(\Lambda_{N+1}) := E(\lambda_1, \ldots, \lambda_{N+1}) := \{ f \in C^\infty(\mathbb{R}) : Lf = 0 \}
\]

are called exponential polynomials. A vector space \( E \) is called an exponential space of dimension \( N+1 \) if there exists \( (\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1} \) such that

\[
E = E(\lambda_1, \ldots, \lambda_{N+1}).
\]

A function \( u : \mathbb{R} \to \mathbb{R} \) is a cardinal L-spline of order \( N+1 \) if \( u \) is \((N-1)\)-times continuously differentiable and for every \( l \in \mathbb{Z} \) there exists an \( f_l \in E(\Lambda_{N+1}) \) such that \( u(t) = f_l(t) \) whenever \( t \in (l, l+1) \). There exists (up to a nonzero scalar factor) a unique cardinal L-spline \( Q_{N+1} \) of order \( N+1 \) and support (equal to) \([0, N+1]\), called the B-spline of order \( N+1 \), see [14]. We shall also write \( Q_{\Lambda_{N+1}} \) or \( Q(\lambda_1,\ldots,\lambda_{N+1}) \) instead of \( Q_{N+1} \).

We will study whether for a given fixed natural number \( N \) there exist “good” functions \( a_N, b_N : \mathbb{R} \to \mathbb{R} \) such that the recurrence relation

\[
Q_{N+1}(x) = a_N(x) Q_N(x) + b_N(x) Q_N(x-1)
\]

holds for all \( x \in \mathbb{R} \). Note that \( a_N \) necessarily coincides with \( Q_{N+1}/Q_N \) on \((0, 1)\) and \( b_N \) with \( Q_{N+1}/Q_N \cdot (-1) \) on \((N, N+1)\). Moreover, if \( a_N \) is known for \( x \in [1, N] \), then the function \( b_N \) must be of the form

\[
b_N(x) = \frac{Q_{N+1}(x) - a_N(x) Q_N(x)}{Q_N(x-1)}
\]
for $x \in (1, N)$. These arguments show that there exist many possibilities for $a_N$ and $b_N$. However, if we require $a_N$ to be real-analytic on $\mathbb{R}$ then it is uniquely determined by its values on $(0, 1)$, and then $b_N$ is also uniquely determined on $(1, N+1)$. If in addition $b_N$ is real-analytic on $(-\infty, 2)$ and $(N, \infty)$ then $b_N$ is completely determined on $\mathbb{R}$. An analogous statement can be made by interchanging the roles of $a_N$ and $b_N$. On the other hand, it is not enough to require that $a_N$ and $b_N$ be $C^\infty$ to obtain uniqueness, as Example 6 shows.

The main purpose of the paper is to find out under which conditions both $a_N$ and $b_N$ can be chosen to be real-analytic on $\mathbb{R}$. Let us introduce the following terminology: we say that there exists a \textit{real-analytic recurrence relation from} $E(A_N)$ to $E(A_{N+1})$ if there exist real-analytic functions $a_N, b_N$ defined on $\mathbb{R}$ such that (6) holds for all $x \in \mathbb{R}$. The following is our main result:

**Theorem 1.** Let $(\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$. Then there exists a sequence of exponential spaces $E_n$ of dimension $n$, $n = 1, \ldots, N+1$,

$$E_1 \subset E_2 \subset \cdots \subset E_N \subset E_{N+1} = E(\lambda_1, \ldots, \lambda_{N+1})$$

with real-analytic recurrence relations from $E_n$ to $E_{n+1}$ for $n = 2, \ldots, N$, if and only if there exist $\alpha, \beta \in \mathbb{R}$ and a permutation $\sigma$ of $\{1, 2, \ldots, N+1\}$ such that $\lambda_{\sigma(k)} = \alpha + (k-1) \beta$, $1 \leq k \leq N+1$.

Let us note that the sufficiency part follows from $[11,18]$ in the setting of L-splines with arbitrary knots; but it is also an easy byproduct of our methods of proof. This makes the paper self-contained.

Finally, we mention that recurrence relations of a different nature were obtained by Dyn and Ron in $[6,7]$. When specialized to cardinal L-splines, and under the assumption $\lambda_1 \neq \lambda_{N+1}$, their results yield the following four-term recurrence relation (see e.g. $[8]$): $${Q(\lambda_1, \ldots, \lambda_{N+1}) (x) = \frac{e^{-\lambda_{N+1}} Q(\lambda_2, \ldots, \lambda_{N+1}) (x)}{\lambda_1 - \lambda_{N+1}} - \frac{e^{-\lambda_1} Q(\lambda_1, \ldots, \lambda_{N}) (x)}{\lambda_1 - \lambda_{N+1}} - \frac{Q(\lambda_2, \ldots, \lambda_{N+1}) (x-1)}{\lambda_1 - \lambda_{N+1}} + \frac{Q(\lambda_1, \ldots, \lambda_{N}) (x-1)}{\lambda_1 - \lambda_{N+1}}.}$$

2. Preliminaries

The general theory of cardinal $L$-splines was developed by Micchelli $[14]$, cf. also $[8$, Chapter 13]. Let $(\lambda_1, \ldots, \lambda_{N+1}) \in C^{N+1}$. We define the function $\varphi_{N+1}$ for the operator $L$ given in (2) as the unique function in the space $E(A_{N+1})$ such that

$$\frac{d^m}{dx^m} \varphi_{N+1} (0) = 0 \text{ for } m = 0, \ldots, N-1 \quad \text{and} \quad \frac{d^N}{dx^N} \varphi_{N+1} (0) = 1. \quad (9)$$

We shall also write $\varphi(\lambda_1, \ldots, \lambda_{N+1})$ instead of $\varphi_{N+1}$. Another useful way to explain properties of $\varphi_{N+1}$ is the identity

$$\varphi(\lambda_1, \ldots, \lambda_{N+1}) (x) := [\lambda_1, \ldots, \lambda_{N+1}] h_x,$$

where $h_x$ is the function defined by $h_x (t) = e^{xt}$ and $[\lambda_1, \ldots, \lambda_{N+1}]$ is the divided difference operator with respect to the variable $t$, see $[16]$. Recall that for pairwise distinct $\lambda_1, \ldots, \lambda_{N+1}$ and
for any suitable function $f$

$$[\lambda_1, \ldots, \lambda_{N+1}] f = \sum_{j=1}^{N+1} d_j f (\lambda_j), \quad d_j := \prod_{k=1, k \neq j}^{N+1} (\lambda_j - \lambda_k)^{-1}. \quad (11)$$

Note that $\varphi(\lambda_1)(x) = e^{\lambda_1 x}$; furthermore $\varphi(\lambda_1, \lambda_2)(x) = x e^{\lambda_1 x}$ for $\lambda_1 = \lambda_2$ and

$$\varphi(\lambda_1, \lambda_2)(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2} \quad \text{for } \lambda_1 \neq \lambda_2. \quad (12)$$

From identity (11) one obtains the following simple consequence:

**Lemma 2.** Let $\lambda_1, \lambda_2, \ldots, \lambda_{N+1}$ be pairwise distinct complex numbers and let $N \geq 1$. Then there exist nonzero constants $c_j$, $j = 2, \ldots, N + 1$, such that

$$\varphi_{N+1}(x) = \sum_{j=2}^{N+1} c_j \varphi(\lambda_1, \lambda_j)(x). \quad (13)$$

The last lemma can be generalized to the case of arbitrary $\lambda_1, \ldots, \lambda_{N+1}$: for $0 \leq k \leq N$ one has the identity

$$[\lambda_1, \ldots, \lambda_{N+1}] f = [\lambda_{k+1}, \ldots, \lambda_{N+1}] F_k \quad \text{with } F_k(y) := [\lambda_1, \ldots, \lambda_k, y] f. \quad (14)$$

This is easy to check for pairwise distinct $\lambda_{k+1}, \ldots, \lambda_{N+1}$, using the classical recurrence relation for divided differences. The continuity of divided differences gives then the general case. Using this the following is easily established:

**Lemma 3.** Let $\lambda_1, \ldots, \lambda_{N+1}$ be complex numbers, define $F_x(\lambda) = \varphi(\lambda_1, \lambda)(x)$ and denote by $F_x^{(l)}$ its $l$th derivative with respect to the variable $\lambda$. Suppose that, up to a permutation, $(\lambda_2, \ldots, \lambda_{N+1})$ is equal to $(\mu_1, \ldots, \mu_1, \ldots, \mu_r, \ldots, \mu_r)$ where $\mu_1, \ldots, \mu_r$ are pairwise distinct and $\mu_j$ has multiplicity $\alpha_j > 0$ for $j = 1, \ldots, r$. Then there exist nonzero constants $c_{j,l}$, $j = 1, \ldots, r$; $l = 1, \ldots, \alpha_j - 1$, such that

$$\varphi_{N+1}(x) = \sum_{j=1}^{r} \sum_{l=0}^{\alpha_j - 1} c_{j,l} F_x^{(l)}(\mu_j). \quad (15)$$

Set $\varphi_{N+1}^+(x) := \varphi_{N+1}(x)$ for $x \geq 0$ and $\varphi_{N+1}^+(x) := 0$ for $x < 0$. The basic cardinal $L$-spline $Q_{N+1}$ is defined (up to a factor) as the unique cardinal $L$-spline of order $N + 1$ with support in $[0, N + 1]$. The basic spline $Q_{N+1}$ can be introduced via divided differences, see [11,14]. We use the formula

$$Q_{N+1}(x) = \sum_{j=0}^{N+1} s_{N+1,j} \varphi_{N+1}^+(x - j). \quad (16)$$
where the coefficients \( s_{N+1, j} \) are defined by the equation
\[
\prod_{j=1}^{N+1} \left( e^{-i j} - z \right) = \sum_{j=0}^{N+1} s_{N+1, j} z^j. \tag{17}
\]
Later we shall use the identity
\[
\sum_{j=0}^{N+1} s_{N+1, j} \varphi_{N+1}(x - j) = 0 \tag{18}
\]
which implies that \( Q_{N+1}(x) = 0 \) for all \( x \geq N + 1 \). Further we need the formulas \( s_{N+1, N+1} = (-1)^{N+1} \), and
\[
s_{N+1, 0} = e^{-(\lambda_1 + \cdots + \lambda_{N+1})}, \quad s_{N+1, 1} = -e^{-(\lambda_1 + \cdots + \lambda_{N+1})} \sum_{i=1}^{N+1} e^{i \lambda_i}. \tag{19}
\]

3. Real-analytic recurrence relations: necessary conditions

First, note that for \( N = 1 \) there exists always a real-analytic recurrence relation from \( E(\lambda_1) \) to \( E(\lambda_1, \lambda_2) \). Indeed, \( Q(\lambda_1) \) is given by \( Q(\lambda_1)(x) = e^{\lambda_1 x} 1_{[0,1]} \), where \( 1_{[0,1]} \) denotes the characteristic function of the interval \([0,1]\). Then
\[
Q(\lambda_1, \lambda_2)(x) = a_1(x) Q(\lambda_1)(x) + b_1(x) Q(\lambda_1)(x - 1), \tag{20}
\]
where \( a_1 \) and \( b_1 \) are defined by real-analytic continuation of the functions \( Q(\lambda_1, \lambda_2)/Q(\lambda_1) \) on \((0, 1)\) and \( Q(\lambda_1, \lambda_2)/Q(\lambda_1) (-1) \) on \((1, 2)\), respectively.

3.1. Uniqueness

We shall assume that \( L \) is of the form (2), where all \( \lambda_j \) are real if not otherwise stated. Then \( \varphi_{N+1}(x) \neq 0 \) for all \( x \in \mathbb{R} \setminus \{0\} \) since \( \varphi_{N+1} \) has at most \( N \) real zeros on \( \mathbb{R} \). Further we know that \( Q_{N+1}(x) > 0 \) for all \( x \in (0, N + 1) \).

**Proposition 4.** For any \( N \geq 2 \), uniqueness of the functions \( a_N \) and \( b_N \) satisfying (6) is guaranteed by requiring either \( a_N \) to be real-analytic on \( \mathbb{R} \) and \( b_N \) to be real-analytic on \((-\infty, 2)\) and \((N, \infty)\) or \( b_N \) to be real-analytic on \( \mathbb{R} \) and \( a_N \) to be real-analytic on \((-\infty, 1)\) and \((N - 1, \infty)\).

**Proof.** By (6), (16), (19), we have for all \( x \in (0, 1) \)
\[
a_N(x) = \frac{Q_{N+1}(x)}{Q_N(x)} = \frac{s_{N+1, 0} \varphi_{N+1}(x)}{s_{N, 0} \varphi_N(x)} = e^{-\lambda_{N+1}} \frac{\varphi_{N+1}(x)}{\varphi_N(x)}, \tag{21}
\]
and for all \( x \in (N, N + 1) \) using (18) and \( s_{N+1, N+1} = (-1)^{N+1} \)
\[
b_N(x) = \frac{Q_{N+1}(x)}{Q_N(x - 1)} = -\frac{\varphi_{N+1}(x - N - 1)}{\varphi_N(x - N - 1)}. \tag{22}
\]
Since \( \varphi_n \) vanishes only at 0, with multiplicity \( n \), the function \( \varphi_{N+1}/\varphi_N \) has a real-analytic extension to all \( \mathbb{R} \). Thus, if we require \( a_N \) to be real-analytic on \( \mathbb{R} \), then \( a_N \) is uniquely defined.
by (21) on \( \mathbb{R} \). Since (6) implies (7) for all \( x \in (1, N + 1) \), the function \( b_N \) is uniquely defined on \( (1, N + 1) \). If we want \( b_N \) to be real-analytic on \( (N, \infty) \) we have to define \( b_N (x) \) on \( (N, \infty) \) by (22). If we want it to be real-analytic on \( (-\infty, 2) \), we have to define \( b_N \) as the real-analytic extension of \( b_N \) restricted to \( (1, 2) \). Using (7), (16) for \( x \in (1, 2) \), and (21) it is simple to see that for \( x \in (1, 2) \)

\[
b_N (x) = \frac{s_{N+1,1} \varphi_{N+1} (x)}{s_{N,0} \varphi_N (x)} (x - 1) - \frac{s_{N,1} s_{N+1,0} \varphi_{N+1} (x)}{s_{N,0} s_{N,0} \varphi_N (x)}.
\]  

(23)

An entirely analogous argument works in the second case of the proposition. \( \square \)

### 3.2. Nonanalytic recurrence relations

The preceding proof also yields the following result.

**Theorem 5.** Let \( N \geq 2 \), be a natural number. Then there exist a real-analytic function \( a_N : \mathbb{R} \rightarrow \mathbb{R} \) and a function \( b_N \in C^{N-2} (\mathbb{R}) \), real-analytic on \( \mathbb{R} \setminus \{2, \ldots, N\} \), such that for all \( x \in \mathbb{R} \)

\[
Q_{N+1} (x) = a_N (x) Q_N (x) + b_N (x) Q_N (x - 1).
\]  

(24)

Positivity over the interval \((0, N + 1)\) of the functions \( a_N \) and \( b_N \) appearing in the recurrence relation is always desirable from the viewpoint of stability, cf. also the polynomial case in (1). From (21) it is clear that \( a_N \) in Theorem 5 is always positive on the half line \((0, \infty)\). Moreover (23) implies that \( b_N (1) = -\frac{s_{N,1}}{s_{N,0}} a_N (1) > 0 \) since \( a_N (1) > 0 \), \( s_{N,0} > 0 \), and \( s_{N,1} < 0 \), cf. (19). However, in general \( b_N \) is not positive on \((1, N + 1)\), cf. Example 12.

The following example shows that the functions \( a_N \) and \( b_N \) are not unique if they are only required to be \( C^\infty \), even in the polynomial case.

**Example 6.** Let \( A = (0, 0, 0) \) and take \( N = 2 \) in (1), i.e. \( M_3 (x) = \frac{x}{2} M_2 (x) + \frac{3-x}{2} M_2 (x - 1) \). Then there exist \( c, d \in C^\infty (\mathbb{R}) \), \( c \neq 0 \), \( d \neq 0 \) such that \( 0 = c (x) M_2 (x) + d (x) M_2 (x - 1) \). Thus, \( M_3 (x) = \left( \frac{x}{2} + c (x) \right) M_2 (x) + \left( \frac{3-x}{2} + d (x) \right) M_2 (x - 1) \) is a different decomposition with \( C^\infty \)-coefficients.

To see why such \( c \) and \( d \) exist, one may simply take \( c = 0 \) to be a \( C^\infty \)-function with support contained in the open interval \((1, 2)\). Define \( d \) just by the equation \( d (x) = -c (x) M_2 (x) / M_2 (x - 1) \) for \( x \in (1, 2) \) and 0 otherwise. Then \( d \) is a \( C^\infty \)-function.

### 3.3. Necessary conditions

**Lemma 7.** Let \( L_{A_N+1} = \prod_{j=1}^{N+1} \left( \frac{d}{dx} - \lambda_j \right) \) and \( (\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1} \). If \( \varphi \neq 0 \) is a solution of \( L_{A_N+1} \varphi = 0 \), then there exists an \( M > 0 \) such that \( \varphi \) has zeros only in a strip \(| \text{Re} \, z | \leq M \).

**Proof.** This follows from the asymptotics of \( \varphi \), since it is a sum of exponentials and all \( \lambda_j \) are real. \( \square \)

**Theorem 8.** Let \( N \geq 2 \) and \( F_N := \varphi_{N+1} / \varphi_N \). Then each property below implies the next one:

(i) there exists a real-analytic recurrence relation from \( E (A_N) \) to \( E (A_{N+1}) \);
(ii) there exist nonzero constants $A_N$, $B_N$ such that for all $x \in \mathbb{R}$
\[ A_N F_N(x) + B_N F_N(x - 1) + F_N(x - N - 1) = 0; \]

(iii) the function $F_N$ has an entire extension.

**Proof.** For (i) $\Rightarrow$ (ii) suppose that there exist real-analytic functions $a_N$ and $b_N$ on the real line satisfying the recurrence relation (6). Comparing (22) with (23) one obtains (25) where
\[ A_N = -\frac{s_{N+1,0}s_{N,1}}{s_{N,0}^2}, \quad B_N := \frac{s_{N+1,1}}{s_{N,0}}. \]

It is clear from (19) that $0$ is a zero of $\varphi_{N+1}/\varphi_N$ is a meromorphic function. Hence we can write $\varphi_{N+1}/\varphi_N = \psi_{N+1}/\psi_N$, where $\psi_{N+1}$ and $\psi_N$ are entire functions without any common zero, and for $j = N, N + 1$, each zero of $\psi_j$ is a zero of $\varphi_j$. Now (25) implies that for each $z \in \mathbb{C}$
\[ 0 = A_N \psi_{N+1}(z) \psi_N(z - 1) \psi_N(z - N - 1) + B_N \psi_{N+1}(z - 1) \psi_N(z) \psi_N(z - N - 1) + \psi_{N+1}(z - N - 1) \psi_N(z) \psi_N(z - 1). \]

We show that $\psi_N$ has no zero in the complex plane, so $\psi_{N+1}/\psi_N$ is entire. Suppose there exists a zero of $\psi_N$. By Lemma 7 there exists an $K \in \mathbb{R}$ such that all zeros of $\varphi_N$ (and hence of $\psi_N$) satisfy $\text{Re} z \geq K$. Let $K_0$ be the infimum of $\{\text{Re} z : \psi_N(z) = 0\}$. Then there exists a zero $z_0$ of $\psi_N$ with $\text{Re} z_0 < K_0 + \frac{1}{2}$. It follows that $\psi_N(z_0 - 1) \neq 0$ and $\psi_N(z_0 - N - 1) \neq 0$. Then the equation above shows that $0 = A_N \psi_{N+1}(z_0) \psi_N(z_0 - 1) \psi_N(z_0 - N - 1)$. By (ii), $A_N \neq 0$, so we conclude that $\psi_{N+1}(z_0) = 0$. This contradicts the fact that $\psi_{N+1}$ and $\psi_N$ have no common zeros. \( \square \)

**Theorem 9.** Assume that $\lambda_1, \ldots, \lambda_{N+1}$ are given with $\lambda_1 \neq \lambda_2$. Suppose that for each $n = 2, \ldots, N$ there exists a real-analytic recurrence relation from $E(\lambda_1, \ldots, \lambda_n)$ to $E(\lambda_1, \ldots, \lambda_{n+1})$. Then there exist pairwise distinct nonzero integers $m_3, \ldots, m_{N+1}$ such that
\[ \lambda_j - \lambda_1 = m_j (\lambda_2 - \lambda_1) \quad \text{for } j = 3, \ldots, N + 1. \]

We first prove the following two lemmas:

**Lemma 10.** With the notations of Lemma 3, given $(\lambda_1, \ldots, \lambda_{N+1})$, the following holds: All functions $\varphi_n/\varphi_2$, $2 \leq n \leq N + 1$, have entire extensions if and only if so do all functions $F^{(l)}_x(\mu_j)/\varphi_2$ for $0 \leq l < \alpha_j - 1$ and $j = 1, \ldots, r$.

**Proof.** Sufficiency is clear since by Lemma 3, $\varphi_n$ is a linear combination of the functions $F^{(l)}_x(\mu_j)$. For the necessity, use induction over $N$. For $N = 1$ the statement is trivial. Suppose now that $\varphi_n/\varphi_2$, $2 \leq n \leq N + 1$, have entire extensions, so they have entire extensions for $2 \leq n \leq N$. By the induction hypothesis each summand (necessarily nonzero) of $\varphi_N/\varphi_2$ in the corresponding sum arising from (15) has an entire extension. By Lemma 3, $\varphi_{N+1}/\varphi_2$ is a linear combination of multiples of the same summands and one more term with a nonzero coefficient, either the value $F^{(l)}_x(\mu_j)/\varphi_2$ for a new $\mu_j$ or of the type $F^{(l)}_x(\mu_j)/\varphi_2$ at an old one. Since the other summands and $\varphi_{N+1}/\varphi_2$ have entire extensions it follows that the new term also has an entire extension. \( \square \)
Lemma 11. Suppose that $\lambda_1 \neq \lambda_2$. Given $\lambda \in \mathbb{C}$, the function $x \mapsto \varphi(\lambda_1, x) / \varphi(\lambda_1, \lambda_2) (x)$ has an entire extension if and only if there exists a nonzero $m \in \mathbb{Z}$ such that

$$\lambda - \lambda_1 = m (\lambda_2 - \lambda_1).$$

Moreover, if $x \mapsto \varphi(\lambda_1, x) / \varphi(\lambda_1, \lambda_2) (x)$ has an entire extension, it cannot be so for $x \mapsto \frac{d}{dx} \varphi(\lambda_1, x) / \varphi(\lambda_1, \lambda_2) (x)$.

Proof. Suppose that $x \mapsto \varphi(\lambda_1, x) / \varphi(\lambda_1, \lambda_2) (x)$ has an entire extension. Then by (12), any nonzero complex zero $z_0$ of $e^{\lambda_1 z} - e^{\lambda_2 z}$ must be a zero of $z \mapsto \varphi(\lambda_1, z)$ (z). Since $z_0 := 2\pi i / (\lambda_2 - \lambda_1)$ is a zero of $e^{\lambda_1 z} - e^{\lambda_2 z}$ we conclude that $0 = \varphi(\lambda_1, z_0)$. This implies that $\lambda \neq \lambda_1$, and $e^{\lambda z_0} - e^{\lambda_1 z_0} = 0$. The existence of some nonzero integer $m$ satisfying (28) follows immediately.

Conversely, from (28) and (12), one may derive that

$$\varphi(\lambda_1, x) / \varphi(\lambda_1, \lambda_2) (x) = \frac{1}{m} \frac{e^{m(\lambda_2 - \lambda_1)x} - 1}{e^{(\lambda_2 - \lambda_1)x} - 1}.$$  

Since $\frac{e^m - 1}{e^m - 1} = 1 + X + \cdots + X^{m-1}$ we conclude that $x \mapsto \varphi(\lambda_1, x) / \varphi(\lambda_1, \lambda_2) (x)$ has an entire extension.

Finally, suppose that (28) holds for some nonzero $m$. Then, with $z_0$ as above, we get

$$\frac{d}{dx} \varphi(\lambda_1, z_0) \neq 0.$$  

Since $\varphi(\lambda_1, \lambda_2) (z_0) = 0$ it follows that $\frac{d}{dx} \varphi(\lambda_1, x) / \varphi(\lambda_1, \lambda_2) (x)$ is not entire. □

Proof of Theorem 9. Suppose that for each $n = 2, \ldots, N$ there exists a real-analytic recurrence relation from $E(\lambda_1, \ldots, \lambda_n)$ to $E(\lambda_1, \ldots, \lambda_{n+1})$. By Theorem 8 all functions $\varphi_{n+1} / \varphi_n$, $2 \leq n \leq N$ have entire extensions. Thus, so do all functions $\varphi_n / \varphi_2$, $2 \leq n \leq N$. The previous two lemmas prove that, if $\lambda_1 \neq \lambda_2$, (27) holds for nonzero integers. Furthermore $\lambda_1, \ldots, \lambda_N$ are pairwise distinct by the second statement of Lemma 11. □

We have already seen that the coefficient function $a_N$ in (24) is positive on $(0, \infty)$. It is a natural question whether the coefficient function $b_N$ is also positive on $[1, N + 1]$. By example we show that $b_N (x)$ can be negative on the interval $(1, 2)$.

Example 12. Let $A = (0, 1, \lambda_3)$ with $\lambda_3 > 1$, and set

$$C_{\lambda_3} (x) := \lambda_3 (\lambda_3 - 1) b_2 (x) \varphi_2 (x) \varphi_2 (x - 1).$$  

Then $C_{\lambda_3}$ and $b_2$ have the same sign on $(1, 2)$, and a computation shows that

$$C_{\lambda_3} (x) = (1 + e) \left( e^x - e^{x-1} \right) + (1 - e^x) (1 + e) e^{(x-1)\lambda_3} + (e^{x-1} - e) e^{x\lambda_3} - (1 - e^x) e^{\lambda_3} + \lambda_3 \left( 1 - e^{x-1} \right) (1 - e^x) e^{\lambda_3}.$$  

Take $x = \frac{3}{2}$. Then, since $e^{1.5\lambda_3}$ is the dominating term and the coefficient $(e^{0.5} - e)$ is negative, $C_{\lambda_3} \left( \frac{3}{2} \right) < 0$ whenever $\lambda_3$ is large enough. So $b_2 \left( \frac{3}{2} \right)$ is also negative.
4. Existence of real-analytic recurrence relations: a characterization

At first we notice the following simple observation:

**Proposition 13.** If there is a real-analytic recurrence relation from the exponential space $E(\lambda_1, \ldots, \lambda_N)$ to $E(c + \lambda_1, \ldots, c + \lambda_{N+1})$, then there is also one from $E(c + \lambda_1, \ldots, c + \lambda_N)$ to $E(c + \lambda_1, \ldots, c + \lambda_{N+1})$ for any $c \in \mathbb{R}$.

**Proof.** For simplicity sake put $c + A_N = (c + \lambda_1, \ldots, c + \lambda_N)$. Using the fact that $\varphi_{c + A_N}(x) = e^{cx} \varphi_{A_N}(x)$ it is not difficult to see that $Q_{c + A_N}(x) = c_N e^{cx} Q_{A_N}(x)$ for some nonzero constant $c_N$. Assuming the recurrence relation $Q_{A_{N+1}}(x) = a_N(x) Q_{A_N}(x) + b_N(x) Q_{A_N}(x - 1)$, it is obvious that

$$c_{N+1}^{-1} c_N Q_{c + A_{N+1}}(x) = a_N(x) Q_{c + A_N}(x) + e^c b_N(x) Q_{c + A_N}(x - 1).$$

(32)

In the following we shall make use of a general remark: let $U_{N+1}$ be the linear space of functions over an open interval $I$, spanned by the functions $1, X, \ldots, X^{N-1}$ and a real-analytic function $u(X)$ over $I$. Then, given $a \in I$, one can define an element $\Phi_u$ in $U_{N+1}$ which satisfies $\Phi_u(a) = \cdots = \Phi_u^{(N-1)}(a) = 0$ by

$$\Phi_u(X) = u(X) - \sum_{k=0}^{N-1} \frac{u^{(k)}(a)}{k!} (X - a)^k.$$  

(33)

By expanding $u(X)$ in a Taylor series about $a$ this implies

$$\Phi_u(X) = (X - a)^N \sum_{k=0}^{\infty} \frac{u^{(k+N)}(a)}{(k+N)!} (X - a)^k.$$  

(34)

**Lemma 14.** Suppose $A_N = (0, \lambda_2, \ldots, (N - 1) \lambda_2)$ and $A_{N+1} = (A_N, M \lambda_2)$, with a natural number $M \geq N \geq 1$, and let $\varphi_N, \varphi_{N+1}$ be defined by (9). Then $\varphi_{N+1}/\varphi_N$ is an entire function of the form

$$\frac{\varphi_{N+1}(x)}{\varphi_N(x)} = cR(e^{\lambda_2 x})$$

(35)

for some non-zero constant $c$ and a polynomial $R$ defined by

$$R(X) = (X - 1) \sum_{k=0}^{M-N} \binom{M}{k+N} (X - 1)^k.$$  

(36)

**Proof.** By the assumptions of the lemma, the space $E(A_{N+1})$ is generated by $1, e^{\lambda_2 x}, \ldots, e^{e^{M \lambda_2} x}$ and $e^{M \lambda_2 x}$. So we are working, up to a change of variable $X := e^{\lambda_2 x}$, in the space $1, X, \ldots, X^{N-1}, X^M$ over the interval $I = (0, \infty)$. Use now the above notations $\Phi_u$ for $u(X) = X^M$ and $a = 1$ in (33). Then there exists a nonzero constant $d_{N+1}$ with

$$\varphi_{N+1}(x) = d_{N+1} \Phi_u(e^{\lambda_2 x}).$$

(37)
Similarly, for the system 1, $X$, $X^{N-2}$, $v(X)$ with $v(X) = X^{N-1}$, one has that $\Phi_v(X) = (X - 1)^{N-1}$ and $\phi_N(x) = d_N \Phi_v \left( e^{\lambda_2 x} \right)$ for some $d_N \neq 0$. An immediate consequence of (34) is that $\frac{\phi_N(x)}{\Phi_v(x)}$ is equal to $R(X)$ defined in (36). This completes the proof of the lemma. □

The following proposition provides the central step in the proof of our main theorem. In particular it shows that for the exponential space $E(0, \ldots, N)$ there exist two different exponential spaces $E_1$ and $E_2$ admitting a real-analytic recurrence relation from $E(0, \ldots, N)$ to $E_j$ for $j = 1, 2$, namely $E_1 = E(0, \ldots, N+1)$ and $E_2 = E(-1, 0, \ldots, N)$.

**Proposition 15.** Given two real numbers $\alpha$, $\beta$ with $\beta \neq 0$, an integer $N \geq 1$, and an integer $M$, $M \not\in \{0, \ldots, N-1\}$, let us set

$$A_N := (\alpha, \alpha + \beta, \ldots, \alpha + (N-1)\beta), \ A_{N+1} := (\alpha, \alpha + \beta, \ldots, \alpha + (N-1)\beta, \alpha + \beta M).$$

Then, the following assertions are equivalent:

(i) There exists a real-analytic recurrence relation from $E(A_N)$ to $E(A_{N+1})$;

(ii) $M = N$ or $M = -1$.

**Proof.** Assume that (i) holds. Due to Proposition 13, we may assume that $\alpha = 0$. We will show that $M = N$ if $M > 0$ and $M = -1$ if $M < 0$.

First, assume that $M > 0$. Then $M > N$ by our assumptions and we can use the last lemma: if $M > N$, then the polynomial $R$ defined in (36) has degree $M - N + 1 \geq 2$. Now (35) and Theorem 8 yield

$$A_N R \left( e^{\lambda_2 x} \right) + B_N R \left( e^{\lambda_2 (x-1)} \right) + C_N R \left( e^{\lambda_2 (x-N-1)} \right) = 0. \tag{38}$$

Putting $\gamma = e^{-\lambda_2}$ and $X = e^{\lambda_2 x}$ one arrives at

$$A_N R (X) + B_N R (\gamma X) + C_N R (\gamma^{N+1} X) = 0 \tag{39}$$

for all $X > 0$, hence for all $X \in \mathbb{R}$. Then $(A_N + B_N + C_N) R(0) = 0$, and differentiation gives the following two relations:

$$(A_N + \gamma B_N + C_N \gamma^{N+1}) R'(0) = (A_N + \gamma^2 B_N + C_N \gamma^{2N+2}) R''(0) = 0. \tag{40}$$

Since $R(0)$, $R'(0)$ and $R''(0)$ are nonzero and $\lambda_2 \neq 0$, this implies $A_N = B_N = C_N = 0$, a contradiction. Hence $M = N$.

Now assume that $M < 0$. We will see that this is reduced to the previous case. We apply Proposition 13 with $c := -(N-1)\beta$: so assumption (i) with $\alpha = 0$ implies that there exists a real-analytic recurrence relation from $E(c + A_N)$ to $E(c + A_{N+1})$. Now $c + A_N$ consists of the values

$$-(N-1)\beta + j\beta = (-\beta)(N-1-j) \tag{41}$$

for $j = 0, \ldots, N-1$ and

$$c + \lambda_{N+1} = -(N-1)\beta + M\beta = (-\beta)(N-1-M). \tag{42}$$

Since $M < 0$ we know that $\tilde{M} := N-1-M > 0$. By the first case applied to $c + A_N$ and $c + A_{N+1}$ we conclude that $\tilde{M} = N$ which clearly implies that $M = -1$. 

□
For (ii) ⇒ (i) we assume at first that $M = N$. Then the real change of variable $X = e^{\lambda x}$ transforms the cardinal spline spaces based on $E(A_N)$ and $E(A_{N+1})$ into the polynomial splines of degree $N$ and $N + 1$ on $(0, \infty)$ relative to the simple knots $t_j := e^{\lambda j}$. Recurrence relations are known in such spaces, and their coefficients are real-analytic. This implies the statement by taking the inverse transform $x = \lambda^{-1} \ln X$. The case $M = -1$ is handled in a similar way. □

**Proposition 16.** Let $\alpha$ be a real number. Suppose $A_N = (x, \ldots, x)$ and $A_{N+1} = (A_N, \lambda)$ for $\lambda \in \mathbb{R}$. Then the following assertions are equivalent:

(i) There exists a real-analytic recurrence relation from $E(A_N)$ to $E(A_{N+1})$;

(ii) $\lambda = \alpha$.

**Proof.** Due to Proposition 13, we may assume that $\alpha = 0$. By Theorem 8 and assumption (i) there exist nonzero constants $A_N, B_N, C_N$ such that for all $x \in \mathbb{R}$

$$A_N \frac{\phi_{N+1}}{\phi_N}(x) + B_N \frac{\phi_{N+1}}{\phi_N}(x - 1) + C_N \frac{\phi_{N+1}}{\phi_N}(x - N - 1) = 0. \quad (43)$$

Suppose that $\lambda \neq 0$. Note that $\phi_N(x) = x^{N-1}/(N-1)!$ and, according to (33) and Lemma 14, there exists a nonzero constant $d_{N+1}$ such that

$$\phi_{N+1}(x) = d_{N+1}(e^{\lambda x} - R(x)), \quad R(x) = \sum_{k=0}^{N-1} \frac{(\lambda x)^k}{k!}. \quad (44)$$

Multiply (43) with $[(x-1)(x-N-1)]^{N-1}$. It follows that there exists a polynomial $Q$ such that

$$e^{\lambda x} P(x) - Q(x) = 0 \quad \text{for all } x \in \mathbb{R}, \quad (45)$$

where the polynomial $P$ is defined by

$$P(x) = A_N[(x-1)(x-N-1)]^{N-1} + B_N e^{-\lambda}[(x-N-1)]^{N-1} + C_N e^{-\lambda(N+1)}[(x-1)]^{N-1}.$$ 

This is impossible unless $P = Q = 0$. But $P = 0$ implies $A_N = B_N = C_N = 0$. Thus we cannot have $\lambda \neq 0$.

For (ii) ⇒ (i) note that $E(A_{N+1})$ is the classical polynomial spline space. □

Now we are going to prove our main result stated as Theorem 1.

**Proof of Theorem 1.** Proof of the necessity by induction. For $N = 1$ there is nothing to prove. Suppose that there exists a sequence of exponential spaces $E_1 \subset E_2 \subset \cdots \subset E_N \subset E_{N+1} = E(A_{N+1})$ with real-analytic recurrence relations from $E_n$ to $E_{n+1}$ for $n = 2, \ldots, N$. The recursive assumption enables us to assume, without loss of generality, that $\lambda_j = \alpha + (j-1)\beta$ for $1 \leq j \leq N$.

Suppose that $\beta \neq 0$. From Theorem 9 we can deduce that $\lambda_{N+1} = \alpha + \beta M$ for some integer $M$ different from $0, \ldots, N - 1$. Proposition 15 ensures that either $M = N$ or $M = -1$. If $M = N$, then the equality $\lambda_j = \alpha + (j-1)\beta$ is valid for $j = N + 1$ too. If $M = -1$, then $\lambda_{\sigma(j)} = \tilde{\alpha} + (j-1)\beta$ for $1 \leq j \leq N + 1$, with $\tilde{\alpha} := \alpha - \beta$, and with $\sigma(1) = N + 1, \sigma(j) := j - 1$ for $j = 2, \ldots, N + 1$.

The case $\beta = 0$ follows from Proposition 16.

Sufficiency follows from Propositions 15 and 16. □
Consider the exponential space $E(\lambda_1, \lambda_2)$. For simplicity assume that $\lambda_1 = 0$, and put $\beta = \lambda_2 - \lambda_1$. Then the proof of our main theorem shows how to construct all increasing sequences of exponential spaces admitting analytic relations, starting from $E(\lambda_1, \lambda_2) = E(0, \beta)$ in the following (uncomplete) scheme:

\[
\begin{array}{c}
E(0, \beta) \\
E(-\beta, 0, \beta) \quad \quad \quad \quad E(0, \beta, 2\beta) \\
E(-2\beta, -\beta, 0, \beta) \quad \quad \quad E(-\beta, 0, \beta, 2\beta) \quad \quad \quad \quad E(0, \beta, 2\beta, 3\beta)
\end{array}
\] (46)

Let us look at the particular case that $A_{N+1} = (\lambda_1, \ldots, \lambda_{N+1})$ is ordered, so $\lambda_1 \leq \cdots \leq \lambda_{N+1}$. Then there exists a real-analytic recurrence relation from $E(\lambda_1, \ldots, \lambda_n)$ to $E(\lambda_1, \ldots, \lambda_{n+1})$ for $n = 2, \ldots, N$, if and only if

$$\lambda_n = \lambda_1 + (n-1)(\lambda_2 - \lambda_1).$$

(47)

The following description is obvious from the above scheme:

**Theorem 17.** Let $(\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$. Then there exist real-analytic recurrence relations from $E(\lambda_1, \ldots, \lambda_n)$ to $E(\lambda_1, \ldots, \lambda_{n+1})$ for $n = 1, 2, \ldots, N$ if and only if for $3 \leq j \leq N + 1$

$$\lambda_j = \lambda_1 + m_j(\lambda_2 - \lambda_1)$$

(48)

with either $m_{j+1} = \min\{m_1, \ldots, m_j\} - 1$ or $m_{j+1} = \max\{m_1, \ldots, m_j\} + 1$, and with $m_1 = 0, m_2 = 1$.

It follows from our results that the only exponential spaces admitting real-analytic recurrence relations are either the classical polynomial spaces, or transformations of polynomial spaces via an exponential map, cf. the discussion in Section 6 in [17].

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**References**