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On real-analytic recurrence relations for cardinal exponential B-splines

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Abstract

Let L_{N+1} be a linear differential operator of order $N+1$ with constant coefficients and real eigenvalues $\lambda_1, \ldots, \lambda_{N+1}$, let $E(\Lambda_{N+1})$ be the space of all C^{∞}-solutions of L_{N+1} on the real line. We show that for $N \geqslant 2$ and $n = 2, \ldots, N$, there is a recurrence relation from suitable subspaces $\mathscr{E}_n$ to $\mathscr{E}_{n+1}$ involving real-analytic functions, and with $\mathscr{E}_{N+1} = E(\Lambda_{N+1})$ if and only if contiguous eigenvalues are equally spaced.

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1. Introduction

Recurrence relations for generalized splines have been discussed by several authors since the appearance of the pioneering work of De Boor and Cox in [4,5], respectively, cf. also [6,7,11–13,18,20]. In order to motivate our results, let us consider briefly the case of *cardinal*

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polynomial splines. It is well known that the cardinal B-splines M_{N+1} and M_N (of order $N+1$ and N, and support in $[0, N+1]$ and $[0, N]$, respectively) are related by the identity

$$M_{N+1}(x) = \frac{x}{N} M_N(x) + \frac{N+1-x}{N} M_N(x-1) \tag{1}$$

for all $x \in \mathbb{R}$ (see e.g. [3, p. 86]). Analogous recurrence relations were proved for trigonometric and hyperbolic B-splines in [12,18], respectively, cf. [11] for a unified proof. On the other hand, Schumaker identified the classes of generalized splines which have B-splines bases computable by recursion relations analogous to those for polynomial, trigonometric, and hyperbolic splines. He proved in [17] that, in addition to the preceding spaces, essentially the only other space of splines admitting such a basis is a certain space of Tchebycheffian splines.

Our objective is to investigate whether there exists a recurrence relation generalizing (1) to the larger class of cardinal L-splines. This question was asked independently in the Conclusion of [19, p. 1436]. Cardinal L-splines also arise in a natural way in the study of the so-called cardinal polysplines, see [1,8–10].

Polynomial and hyperbolic cardinal splines are special cases of cardinal L-splines, also known as *cardinal exponential splines*; here it is assumed that L is a linear differential operator of the form

$$L = \prod_{j=1}^{N+1} \left(\frac{d}{dx} - \lambda_j \right). \tag{2}$$

Throughout the paper we shall assume that the *eigenvalues* $\lambda_1, \ldots, \lambda_{N+1}$ are real numbers and we shall often use the notation

$$\Lambda_{N+1} := (\lambda_1, \ldots, \lambda_{N+1}). \tag{3}$$

The functions in

$$E(\Lambda_{N+1}) := E(\lambda_1, \ldots, \lambda_{N+1}) := \left\{ f \in C^\infty(\mathbb{R}) : Lf = 0 \right\} \tag{4}$$

are called *exponential polynomials*. A vector space $\mathcal{E}$ is called an *exponential space of dimension* $N+1$ if there exists $(\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$ such that

$$\mathcal{E} = E(\lambda_1, \ldots, \lambda_{N+1}). \tag{5}$$

A function $u : \mathbb{R} \to \mathbb{R}$ is a *cardinal L-spline* of order $N+1$ if u is $(N-1)$-times continuously differentiable and for every $l \in \mathbb{Z}$ there exists an $f_l \in E(\Lambda_{N+1})$ such that $u(t) = f_l(t)$ whenever $t \in (l, l+1)$. There exists (up to a nonzero scalar factor) a unique cardinal L-spline Q_{N+1} of order $N+1$ and support (equal to) $[0, N+1]$, called the B-spline of order $N+1$, see [14]. We shall also write $Q_{\Lambda_{N+1}}$ or $Q_{(\lambda_1,\ldots,\lambda_{N+1})}$ instead of Q_{N+1}.

We will study whether for a given fixed natural number N there exist "good" functions $a_N, b_N : \mathbb{R} \to \mathbb{R}$ such that the recurrence relation

$$Q_{N+1}(x) = a_N(x)\, Q_N(x) + b_N(x)\, Q_N(x-1) \tag{6}$$

holds for all $x \in \mathbb{R}$. Note that a_N necessarily coincides with Q_{N+1}/Q_N on $(0,1)$ and b_N with $Q_{N+1}/Q_N(\cdot - 1)$ on $(N, N+1)$. Moreover, if a_N is known for $x \in [1, N]$, then the function b_N must be of the form

$$b_N(x) = \frac{Q_{N+1}(x) - a_N(x)\, Q_N(x)}{Q_N(x-1)} \tag{7}$$

for $x \in (1, N)$. These arguments show that there exist many possibilities for a_N and b_N. However, if we require a_N to be real-analytic on $\mathbb{R}$ then it is uniquely determined by its values on $(0, 1)$, and then b_N is also uniquely determined on $(1, N + 1)$. If in addition b_N is real-analytic on $(-\infty, 2)$ and (N, ∞) then b_N is completely determined on $\mathbb{R}$. An analogous statement can be made by interchanging the roles of a_N and b_N. On the other hand, it is not enough to require that a_N and b_N be C^∞ to obtain uniqueness, as Example 6 shows.

The main purpose of the paper is to find out under which conditions both a_N and b_N can be chosen to be real-analytic on $\mathbb{R}$. Let us introduce the following terminology: we say that there exists a *real-analytic recurrence relation from* $E(\Lambda_N)$ *to* $E(\Lambda_{N+1})$ if there exist real-analytic functions a_N, b_N defined on $\mathbb{R}$ such that (6) holds for all $x \in \mathbb{R}$. The following is our main result:

Theorem 1. *Let* $(\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$. *Then there exists a sequence of exponential spaces* $\mathcal{E}_n$ *of dimension* n, $n = 1, \ldots, N + 1$,

$$\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_N \subset \mathcal{E}_{N+1} = E(\lambda_1, \ldots, \lambda_{N+1}) \tag{8}$$

with real-analytic recurrence relations from $\mathcal{E}_n$ *to* $\mathcal{E}_{n+1}$ *for* $n = 2, \ldots, N$, *if and only if there exist* $\alpha, \beta \in \mathbb{R}$ *and a permutation* σ *of* $\{1, 2, \ldots, N + 1\}$ *such that* $\lambda_{\sigma(k)} = \alpha + (k - 1)\beta$, $1 \leqslant k \leqslant N + 1$.

Let us note that the sufficiency part follows from [11,18] in the setting of L-splines with arbitrary knots; but it is also an easy byproduct of our methods of proof. This makes the paper self-contained.

Finally, we mention that recurrence relations of a different nature were obtained by Dyn and Ron in [6,7]. When specialized to cardinal L-splines, and under the assumption $\lambda_1 \neq \lambda_{N+1}$, their results yield the following four-term recurrence relation (see e.g. [8]):

$$\begin{aligned} Q_{(\lambda_1,\lambda_2,\ldots,\lambda_{N+1})}(x) = {} & \frac{e^{-\lambda_{N+1}} Q_{(\lambda_2,\ldots,\lambda_{N+1})}(x)}{\lambda_1 - \lambda_{N+1}} - \frac{e^{-\lambda_1} Q_{(\lambda_1,\ldots,\lambda_N)}(x)}{\lambda_1 - \lambda_{N+1}} \\ & - \frac{Q_{(\lambda_2,\ldots,\lambda_{N+1})}(x-1)}{\lambda_1 - \lambda_{N+1}} + \frac{Q_{(\lambda_1,\ldots,\lambda_N)}(x-1)}{\lambda_1 - \lambda_{N+1}}. \end{aligned}$$

2. Preliminaries

The general theory of cardinal L-splines was developed by Micchelli [14], cf. also [8, Chapter 13]. Let $(\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{C}^{N+1}$. We define the function φ_{N+1} for the operator L given in (2) as the unique function in the space $E(\Lambda_{N+1})$ such that

$$\frac{d^m}{dx^m} \varphi_{N+1}(0) = 0 \text{ for } m = 0, \ldots, N - 1 \quad \text{and} \quad \frac{d^N}{dx^N} \varphi_{N+1}(0) = 1. \tag{9}$$

We shall also write $\varphi_{(\lambda_1,\ldots,\lambda_{N+1})}$ instead of φ_{N+1}. Another useful way to explain properties of φ_{N+1} is the identity

$$\varphi_{(\lambda_1,\ldots,\lambda_{N+1})}(x) := \left[\lambda_1, \ldots, \lambda_{N+1}\right] h_x, \tag{10}$$

where h_x is the function defined by $h_x(t) = e^{xt}$ and $\left[\lambda_1, \ldots, \lambda_{N+1}\right]$ is the divided difference operator with respect to the variable t, see [16]. Recall that for pairwise distinct $\lambda_1, \ldots, \lambda_{N+1}$ and for any suitable function f

$$\left[\lambda_1, \ldots, \lambda_{N+1}\right] f = \sum_{j=1}^{N+1} d_j f\left(\lambda_j\right), \quad d_j := \prod_{k=1, k\neq j}^{N+1} \left(\lambda_j - \lambda_k\right)^{-1}. \tag{11}$$

Note that $\varphi_{(\lambda_1)}(x) = e^{\lambda_1 x}$; furthermore $\varphi_{(\lambda_1,\lambda_1)}(x) = x e^{\lambda_1 x}$ for $\lambda_1 = \lambda_2$ and

$$\varphi_{(\lambda_1,\lambda_2)}(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2} \quad \text{for } \lambda_1 \neq \lambda_2. \tag{12}$$

From identity (11) one obtains the following simple consequence:

Lemma 2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_{N+1}$ be pairwise distinct complex numbers and let $N \geqslant 1$. Then there exist nonzero constants c_j, $j = 2, \ldots, N+1$, such that*

$$\varphi_{N+1}(x) = \sum_{j=2}^{N+1} c_j \varphi_{(\lambda_1,\lambda_j)}(x). \tag{13}$$

The last lemma can be generalized to the case of arbitrary $\lambda_1, \ldots, \lambda_{N+1}$: for $0 \leqslant k \leqslant N$ one has the identity

$$\left[\lambda_1, \ldots, \lambda_{N+1}\right] f = \left[\lambda_{k+1}, \ldots, \lambda_{N+1}\right] F_k \text{ with } F_k(y) := \left[\lambda_1, \ldots, \lambda_k, y\right] f. \tag{14}$$

This is easy to check for pairwise distinct $\lambda_{k+1}, \ldots, \lambda_{N+1}$, using the classical recurrence relation for divided differences. The continuity of divided differences gives then the general case. Using this the following is easily established:

Lemma 3. *Let $\lambda_1, \ldots, \lambda_{N+1}$ be complex numbers, define $F_x(\lambda) = \varphi_{(\lambda_1,\lambda)}(x)$ and denote by $F_x^{(l)}$ its lth derivative with respect to the variable λ. Suppose that, up to a permutation, $(\lambda_2, \ldots, \lambda_{N+1})$ is equal to $(\mu_1, \ldots, \mu_1, \ldots, \mu_r, \ldots, \mu_r)$ where $\mu_1, \ldots, \mu_r$ are pairwise distinct and μ_j has multiplicity $\alpha_j > 0$ for $j = 1, \ldots, r$. Then there exist nonzero constants $c_{j,l}$, $j = 1, \ldots, r$; $l = 1, \ldots, \alpha_j - 1$, such that*

$$\varphi_{N+1}(x) = \sum_{j=1}^{r} \sum_{l=0}^{\alpha_j - 1} c_{j,l} F_x^{(l)}\left(\mu_j\right). \tag{15}$$

Set $\varphi_{N+1}^{+}(x) := \varphi_{N+1}(x)$ for $x \geqslant 0$ and $\varphi_{N+1}^{+}(x) := 0$ for $x < 0$. The basic cardinal L-spline Q_{N+1} is defined (up to a factor) as the unique cardinal L-spline of order $N+1$ with support in $[0, N+1]$. The basic spline Q_{N+1} can be introduced via divided differences, see [11,14]. We use the formula

$$Q_{N+1}(x) = \sum_{j=0}^{N+1} s_{N+1,j} \varphi_{N+1}^{+}(x - j), \tag{16}$$

where the coefficients $s_{N+1,j}$ are defined by the equation

$$\prod_{j=1}^{N+1}\left(e^{-\lambda_j}-z\right)=\sum_{j=0}^{N+1}s_{N+1,j}z^j. \tag{17}$$

Later we shall use the identity

$$\sum_{j=0}^{N+1}s_{N+1,j}\varphi_{N+1}(x-j)=0 \tag{18}$$

which implies that $Q_{N+1}(x)=0$ for all $x\geqslant N+1$. Further we need the formulas $s_{N+1,N+1}=(-1)^{N+1}$, and

$$s_{N+1,0}=e^{-(\lambda_1+\cdots+\lambda_{N+1})},\quad s_{N+1,1}=-e^{-(\lambda_1+\cdots+\lambda_{N+1})}\sum_{i=1}^{N+1}e^{\lambda_i}. \tag{19}$$

3. Real-analytic recurrence relations: necessary conditions

First, note that for $N=1$ there exists always a real-analytic recurrence relation from $E(\lambda_1)$ to $E(\lambda_1,\lambda_2)$. Indeed, $Q_{(\lambda_1)}$ is given by $Q_{(\lambda_1)}(x)=e^{\lambda_1 x}1_{[0,1]}$, where $1_{[0,1]}$ denotes the characteristic function of the interval [0, 1]. Then

$$Q_{(\lambda_1,\lambda_2)}(x)=a_1(x)\,Q_{(\lambda_1)}(x)+b_1(x)\,Q_{(\lambda_1)}(x-1), \tag{20}$$

where a_1 and b_1 are defined by real-analytic continuation of the functions $Q_{(\lambda_1,\lambda_2)}/Q_{(\lambda_1)}$ on (0, 1) and $Q_{(\lambda_1,\lambda_2)}/Q_{(\lambda_1)}(\cdot-1)$ on (1, 2), respectively.

3.1. Uniqueness

We shall assume that L is of the form (2), *where all λ_j are real* if not otherwise stated. Then $\varphi_{N+1}(x)\neq 0$ for all $x\in\mathbb{R}\setminus\{0\}$ since φ_{N+1} has at most N real zeros on $\mathbb{R}$. Further we know that $Q_{N+1}(x)>0$ for all $x\in(0,N+1)$.

Proposition 4. *For any $N\geqslant 2$, uniqueness of the functions a_N and b_N satisfying* (6) *is guaranteed by requiring either a_N to be real-analytic on $\mathbb{R}$ and b_N to be real-analytic on* $(-\infty,2)$ *and* (N,∞) *or b_N to be real-analytic on $\mathbb{R}$ and a_N to be real-analytic on* $(-\infty,1)$ *and* $(N-1,\infty)$.

Proof. By (6), (16), (19), we have for all $x\in(0,1)$

$$a_N(x)=\frac{Q_{N+1}(x)}{Q_N(x)}=\frac{s_{N+1,0}\varphi_{N+1}(x)}{s_{N,0}\varphi_N(x)}=e^{-\lambda_{N+1}}\frac{\varphi_{N+1}(x)}{\varphi_N(x)}, \tag{21}$$

and for all $x\in(N,N+1)$ using (18) and $s_{N+1,N+1}=(-1)^{N+1}$

$$b_N(x)=\frac{Q_{N+1}(x)}{Q_N(x-1)}=-\frac{\varphi_{N+1}(x-N-1)}{\varphi_N(x-N-1)}. \tag{22}$$

Since φ_n vanishes only at 0, with multiplicity n, the function φ_{N+1}/φ_N has a real-analytic extension to all $\mathbb{R}$. Thus, if we require a_N to be real-analytic on $\mathbb{R}$, then a_N is uniquely defined by (21) on $\mathbb{R}$. Since (6) implies (7) for all $x \in (1, N+1)$, the function b_N is uniquely defined on $(1, N+1)$. If we want b_N to be real-analytic on (N, ∞) we have to define $b_N(x)$ on (N, ∞) by (22). If we want it to be real-analytic on $(-\infty, 2)$, we have to define b_N as the real-analytic extension of b_N restricted to $(1, 2)$. Using (7), (16) for $x \in (1, 2)$, and (21) it is simple to see that for $x \in (1, 2)$

$$b_N(x) = \frac{s_{N+1,1}\varphi_{N+1}(x-1)}{s_{N,0}\varphi_N(x-1)} - \frac{s_{N,1}}{s_{N,0}} \frac{s_{N+1,0}\varphi_{N+1}(x)}{s_{N,0}\varphi_N(x)}. \tag{23}$$

An entirely analogous argument works in the second case of the proposition. □

3.2. Nonanalytic recurrence relations

The preceding proof also yields the following result.

Theorem 5. *Let $N \geqslant 2$, be a natural number. Then there exist a real-analytic function $a_N : \mathbb{R} \to \mathbb{R}$ and a function $b_N \in C^{N-2}(\mathbb{R})$, real-analytic on $\mathbb{R} \setminus \{2, \ldots, N\}$, such that for all $x \in \mathbb{R}$*

$$Q_{N+1}(x) = a_N(x)\, Q_N(x) + b_N(x)\, Q_N(x-1). \tag{24}$$

Positivity over the interval $(0, N+1)$ of the functions a_N and b_N appearing in the recurrence relation is always desirable from the viewpoint of stability, cf. also the polynomial case in (1). From (21) it is clear that a_N in Theorem 5 is always positive on the half line $(0, \infty)$. Moreover (23) implies that $b_N(1) = -\frac{s_{N,1}}{s_{N,0}} a_N(1) > 0$ since $a_N(1) > 0$, $s_{N,0} > 0$, and $s_{N,1} < 0$, cf. (19). However, in general b_N is not positive on $(1, N+1)$, cf. Example 12.

The following example shows that the functions a_N and b_N are not unique if they are only required to be C^∞, even in the polynomial case.

Example 6. Let $\Lambda = (0, 0, 0)$ and take $N = 2$ in (1), i.e. $M_3(x) = \frac{x}{2} M_2(x) + \frac{3-x}{2} M_2(x-1)$. Then there exist $c, d \in C^\infty(\mathbb{R})$, $c \not\equiv 0$, $d \not\equiv 0$ such that $0 = c(x)\, M_2(x) + d(x)\, M_2(x-1)$. Thus, $M_3(x) = \left(\frac{x}{2} + c(x)\right) M_2(x) + \left(\frac{3-x}{2} + d(x)\right) M_2(x-1)$ is a different decomposition with C^∞-coefficients.

To see why such c and d exist, one may simply take $c \neq 0$ to be a C^∞-function with support contained in the open interval $(1, 2)$. Define d just by the equation $d(x) = -c(x)\, M_2(x) / M_2(x-1)$ for $x \in (1, 2)$ and 0 otherwise. Then d is a C^∞-function.

3.3. Necessary conditions

Lemma 7. *Let $L_{\Lambda_{N+1}} = \prod_{j=1}^{N+1} \left(\frac{d}{dx} - \lambda_j\right)$ and $(\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$. If $\varphi \neq 0$ is a solution of $L_{\Lambda_{N+1}} \varphi = 0$, then there exists an $M > 0$ such that φ has zeros only in a strip $|\operatorname{Re} z| \leqslant M$.*

Proof. This follows from the asymptotics of φ, since it is a sum of exponentials and all λ_j are real. □

Theorem 8. *Let $N \geqslant 2$ and $F_N := \varphi_{N+1}/\varphi_N$. Then each property below implies the next one:*

(i) *there exists a real-analytic recurrence relation from $E(\Lambda_N)$ to $E(\Lambda_{N+1})$;*

(ii) *there exist nonzero constants* A_N, B_N *such that for all* $x \in \mathbb{R}$

$$A_N F_N(x) + B_N F_N(x-1) + F_N(x-N-1) = 0; \tag{25}$$

(iii) *the function* F_N *has an entire extension.*

Proof. For (i) $\Rightarrow$ (ii) suppose that there exist real-analytic functions a_N and b_N on the real line satisfying the recurrence relation (6). Comparing (22) with (23) one obtains (25) where

$$A_N = -\frac{s_{N+1,0}s_{N,1}}{s_{N,0}^2}, \quad B_N := \frac{s_{N+1,1}}{s_{N,0}}. \tag{26}$$

It is clear from (19) that A_N and B_N are nonzero.

Let us prove now (ii) $\Rightarrow$ (iii). Clearly φ_{N+1}/φ_N is a meromorphic function. Hence we can write $\varphi_{N+1}/\varphi_N = \psi_{N+1}/\psi_N$, where ψ_{N+1} and ψ_N are entire functions without any common zero, and for $j = N, N+1$, each zero of ψ_j is a zero of φ_j. Now (25) implies that for each $z \in \mathbb{C}$

$$\begin{aligned} 0 = {} & A_N\psi_{N+1}(z)\,\psi_N(z-1)\,\psi_N(z-N-1) \\ & + B_N\psi_{N+1}(z-1)\,\psi_N(z)\,\psi_N(z-N-1) \\ & + \psi_{N+1}(z-N-1)\,\psi_N(z)\,\psi_N(z-1). \end{aligned}$$

We show that ψ_N has no zero in the complex plane, so ψ_{N+1}/ψ_N is entire. Suppose there exists a zero of ψ_N. By Lemma 7 there exists an $K \in \mathbb{R}$ such that all zeros of φ_N (and hence of ψ_N) satisfy $\operatorname{Re} z \geqslant K$. Let K_0 be the infimum of $\{\operatorname{Re} z : \psi_N(z) = 0\}$. Then there exists a zero z_0 of ψ_N with $\operatorname{Re} z_0 < K_0 + \frac{1}{2}$. It follows that $\psi_N(z_0 - 1) \neq 0$ and $\psi_N(z_0 - N - 1) \neq 0$. Then the equation above shows that $0 = A_N\psi_{N+1}(z_0)\,\psi_N(z_0-1)\,\psi_N(z_0-N-1)$. By (ii), $A_N \neq 0$, so we conclude that $\psi_{N+1}(z_0) = 0$. This contradicts the fact that ψ_{N+1} and ψ_N have no common zeros. $\square$

Theorem 9. *Assume that* $\lambda_1, \ldots, \lambda_{N+1}$ *are given with* $\lambda_1 \neq \lambda_2$. *Suppose that for each* $n = 2, \ldots, N$ *there exists a real-analytic recurrence relation from* $E(\lambda_1, \ldots, \lambda_n)$ *to* $E(\lambda_1, \ldots, \lambda_{n+1})$. *Then there exist pairwise distinct nonzero integers* $m_3, \ldots, m_{N+1}$ *such that*

$$\lambda_j - \lambda_1 = m_j(\lambda_2 - \lambda_1) \quad \textit{for } j = 3, \ldots, N+1. \tag{27}$$

We first prove the following two lemmas:

Lemma 10. *With the notations of Lemma* 3, *given* $(\lambda_1, \ldots, \lambda_{N+1})$, *the following holds*: *All functions* φ_n/φ_2, $2 \leqslant n \leqslant N+1$, *have entire extensions if and only if so do all functions* $F_x^{(l)}(\mu_j)/\varphi_2$ *for* $0 \leqslant l \leqslant \alpha_j - 1$ *and* $j = 1, \ldots, r$.

Proof. Sufficiency is clear since by Lemma 3, φ_n is a linear combination of the functions $F_x^{(l)}(\mu_j)$. For the necessity, use induction over N. For $N = 1$ the statement is trivial. Suppose now that φ_n/φ_2, $2 \leqslant n \leqslant N+1$, have entire extensions, so they have entire extensions for $2 \leqslant n \leqslant N$. By the induction hypothesis each summand (necessarily nonzero) of φ_N/φ_2 in the corresponding sum arising from (15) has an entire extension. By Lemma 3, φ_{N+1}/φ_2 is a linear combination of multiples of the same summands and one more term with a *nonzero* coefficient, either the value $F_x(\mu_j)/\varphi_2$ for a new μ_j or of the type $F_x^{(l)}(\mu_j)/\varphi_2$ at an old one. Since the other summands and φ_{N+1}/φ_2 have entire extensions it follows that the new term also has an entire extension. $\square$

Lemma 11. *Suppose that* $\lambda_1 \neq \lambda_2$. *Given* $\lambda \in \mathbb{C}$, *the function* $x \longmapsto \varphi_{(\lambda_1,\lambda)}(x) / \varphi_{(\lambda_1,\lambda_2)}(x)$ *has an entire extension if and only if there exists a nonzero* $m \in \mathbb{Z}$ *such that*

$$\lambda - \lambda_1 = m(\lambda_2 - \lambda_1). \tag{28}$$

Moreover, if $x \longmapsto \varphi_{(\lambda_1,\lambda)}(x) / \varphi_{(\lambda_1,\lambda_2)}(x)$ *has an entire extension, it cannot be so for* $x \longmapsto \frac{d}{d\lambda}\varphi_{(\lambda_1,\lambda)}(x) / \varphi_{(\lambda_1,\lambda_2)}(x)$.

Proof. Suppose that $x \longmapsto \varphi_{(\lambda_1,\lambda)}(x) / \varphi_{(\lambda_1,\lambda_2)}(x)$ has an entire extension. Then by (12), any nonzero complex zero z_0 of $e^{\lambda_1 z} - e^{\lambda_2 z}$ must be a zero of $z \longmapsto \varphi_{(\lambda_1,\lambda)}(z)$. Since $z_0 := 2\pi i/(\lambda_2 - \lambda_1)$ is a zero of $e^{\lambda_1 z} - e^{\lambda_2 z}$ we conclude that $0 = \varphi_{(\lambda_1,\lambda)}(z_0)$. This implies that $\lambda \neq \lambda_1$, and $e^{\lambda z_0} - e^{\lambda_1 z_0} = 0$. The existence of some nonzero integer m satisfying (28) follows immediately.

Conversely, from (28) and (12), one may derive that

$$\varphi_{(\lambda_1,\lambda)}(x) / \varphi_{(\lambda_1,\lambda_2)}(x) = \frac{1}{m} \frac{e^{m(\lambda_2-\lambda_1)x} - 1}{e^{(\lambda_2-\lambda_1)x} - 1}. \tag{29}$$

Since $\frac{X^m - 1}{X - 1} = 1 + X + \cdots + X^{m-1}$ we conclude that $x \longmapsto \varphi_{(\lambda_1,\lambda)}(x) / \varphi_{(\lambda_1,\lambda_2)}(x)$ has an entire extension.

Finally, suppose that (28) holds for some nonzero m. Then, with z_0 as above, we get

$$\frac{d}{d\lambda}\varphi_{(\lambda_1,\lambda)}(z_0) \neq 0. \tag{30}$$

Since $\varphi_{(\lambda_1,\lambda_2)}(z_0) = 0$ it follows that $\frac{d}{d\lambda}\varphi_{(\lambda_1,\lambda)}(x) / \varphi_{(\lambda_1,\lambda_2)}(x)$ is not entire. □

Proof of Theorem 9. Suppose that for each $n = 2, \ldots, N$ there exists a real-analytic recurrence relation from $E(\lambda_1, \ldots, \lambda_n)$ to $E(\lambda_1, \ldots, \lambda_{n+1})$. By Theorem 8 all functions φ_{n+1}/φ_n, $2 \leqslant n \leqslant N$ have entire extensions. Thus, so do all functions φ_n/φ_2, $2 \leqslant n \leqslant N$. The previous two lemmas prove that, if $\lambda_1 \neq \lambda_2$, (27) holds for nonzero integers. Furthermore $\lambda_1, \ldots, \lambda_{N+1}$ are pairwise distinct by the second statement of Lemma 11. □

We have already seen that the coefficient function a_N in (24) is positive on $(0, \infty)$. It is a natural question whether the coefficient function b_N is also positive on $[1, N+1]$. By example we show that $b_N(x)$ can be negative on the interval $(1, 2)$.

Example 12. Let $\Lambda = (0, 1, \lambda_3)$ with $\lambda_3 > 1$, and set

$$C_{\lambda_3}(x) := \lambda_3(\lambda_3 - 1)\, b_2(x)\, \varphi_2(x)\, \varphi_2(x-1). \tag{31}$$

Then C_{λ_3} and b_2 have the same sign on $(1, 2)$, and a computation shows that

$$\begin{aligned} C_{\lambda_3}(x) = {} & (1+e)\left(e^x - e^{x-1}\right) + \left(1 - e^x\right)(1+e)\, e^{(x-1)\lambda_3} \\ & + \left(e^{x-1} - e\right) e^{x\lambda_3} - \left(1 - e^x\right) e^{\lambda_3} + \lambda_3 \left(1 - e^{x-1}\right)\left(1 - e^x\right) e^{\lambda_3}. \end{aligned}$$

Take $x = \frac{3}{2}$. Then, since $e^{1.5\lambda_3}$ is the dominating term and the coefficient $\left(e^{0.5} - e\right)$ is negative, $C_{\lambda_3}\left(\frac{3}{2}\right) < 0$ whenever λ_3 is large enough. So $b_2\left(\frac{3}{2}\right)$ is also negative.

4. Existence of real-analytic recurrence relations: a characterization

At first we notice the following simple observation:

Proposition 13. *If there is a real-analytic recurrence relation from the exponential space* $E(\lambda_1, \ldots, \lambda_N)$ *to* $E(\lambda_1, \ldots, \lambda_{N+1})$, *then there is also one from* $E(c+\lambda_1, \ldots, c+\lambda_N)$ *to* $E(c+\lambda_1, \ldots, c+\lambda_{N+1})$ *for any* $c \in \mathbb{R}$.

Proof. For simplicity sake put $c + \Lambda_N = (c+\lambda_1, \ldots, c+\lambda_N)$. Using the fact that $\varphi_{c+\Lambda_N}(x) = e^{cx}\varphi_{\Lambda_N}(x)$ it is not difficult to see that $Q_{c+\Lambda_N}(x) = c_N e^{cx} Q_{\Lambda_N}(x)$ for some nonzero constant c_N. Assuming the recurrence relation $Q_{\Lambda_{N+1}}(x) = a_N(x)\, Q_{\Lambda_N}(x) + b_N(x)\, Q_{\Lambda_N}(x-1)$, it is obvious that

$$c_{N+1}^{-1} c_N Q_{c+\Lambda_{N+1}}(x) = a_N(x)\, Q_{c+\Lambda_N}(x) + e^c b_N(x)\, Q_{c+\Lambda_N}(x-1). \qquad \square \tag{32}$$

In the following we shall make use of a general remark: let U_{N+1} be the linear space of functions over an open interval I, spanned by the functions $1, X, \ldots, X^{N-1}$ and a real-analytic function $u(X)$ over I. Then, given $a \in I$, one can define an element Φ_u in U_{N+1} which satisfies $\Phi_u(a) = \cdots = \Phi_u^{(N-1)}(a) = 0$ by

$$\Phi_u(X) = u(X) - \sum_{k=0}^{N-1} \frac{u^{(k)}(a)}{k!}(X-a)^k. \tag{33}$$

By expanding $u(X)$ in a Taylor series about a this implies

$$\Phi_u(X) = (X-a)^N \sum_{k=0}^{\infty} \frac{u^{(k+N)}(a)}{(k+N)!}(X-a)^k. \tag{34}$$

Lemma 14. *Suppose* $\Lambda_N = (0, \lambda_2, \ldots, (N-1)\lambda_2)$ *and* $\Lambda_{N+1} = (\Lambda_N, M\lambda_2)$, *with a natural number* $M \geqslant N \geqslant 1$, *and let* φ_N, φ_{N+1} *be defined by* (9). *Then* φ_{N+1}/φ_N *is an entire function of the form*

$$\frac{\varphi_{N+1}(x)}{\varphi_N(x)} = cR\left(e^{\lambda_2 x}\right) \tag{35}$$

for some non-zero constant c *and a polynomial* R *defined by*

$$R(X) = (X-1)\sum_{k=0}^{M-N} \binom{M}{k+N}(X-1)^k. \tag{36}$$

Proof. By the assumptions of the lemma, the space $E(\Lambda_{N+1})$ is generated by $1, e^{\lambda_2 x}, \ldots, e^{\lambda_2(N-1)x}$ and $e^{M\lambda_2 x}$. So we are working, up to a change of variable $X := e^{\lambda_2 x}$, in the space $1, X, \ldots, X^{N-1}, X^M$ over the interval $I = (0, \infty)$. Use now the above notations Φ_u for $u(X) = X^M$ and $a = 1$ in (33). Then there exists a nonzero constant d_{N+1} with

$$\varphi_{N+1}(x) = d_{N+1}\Phi_u\left(e^{\lambda_2 x}\right). \tag{37}$$

Similarly, for the system $1, X, \ldots, X^{N-2}, v(X)$ with $v(X) = X^{N-1}$, one has that $\Phi_v(X) = (X-1)^{N-1}$ and $\varphi_N(x) = d_N \Phi_v\left(e^{\lambda_2 x}\right)$ for some $d_N \neq 0$. An immediate consequence of (34) is that $\frac{\Phi_u(X)}{\Phi_v(X)}$ is equal to $R(X)$ defined in (36). This completes the proof of the lemma. □

The following proposition provides the central step in the proof of our main theorem. In particular it shows that for the exponential space $E(0, \ldots, N)$ there exist two different exponential spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ admitting a real-analytic recurrence relation from $E(0, \ldots, N)$ to $\mathcal{E}_j$ for $j = 1, 2$, namely $\mathcal{E}_1 = E(0, \ldots, N+1)$ and $\mathcal{E}_2 = E(-1, 0, \ldots, N)$.

Proposition 15. *Given two real numbers α, β with $\beta \neq 0$, an integer $N \geqslant 1$, and an integer M, $M \notin \{0, \ldots, N-1\}$, let us set*

$$\Lambda_N := (\alpha, \alpha+\beta, \ldots, \alpha+(N-1)\beta),\ \Lambda_{N+1} := (\alpha, \alpha+\beta, \ldots, \alpha+(N-1)\beta, \alpha+\beta M).$$

Then, the following assertions are equivalent:
(i) *There exists a real-analytic recurrence relation from $E(\Lambda_N)$ to $E(\Lambda_{N+1})$*;
(ii) *$M = N$ or $M = -1$.*

Proof. Assume that (i) holds. Due to Proposition 13, we may assume that $\alpha = 0$. We will show that $M = N$ if $M > 0$ and $M = -1$ if $M < 0$.

First, assume that $M > 0$. Then $M > N$ by our assumptions and we can use the last lemma: if $M > N$, then the polynomial R defined in (36) has degree $M - N + 1 \geqslant 2$. Now (35) and Theorem 8 yield

$$A_N R\left(e^{\lambda_2 x}\right) + B_N R\left(e^{\lambda_2 (x-1)}\right) + C_N R\left(e^{\lambda_2 (x-N-1)}\right) = 0. \tag{38}$$

Putting $\gamma = e^{-\lambda_2}$ and $X = e^{\lambda_2 x}$ one arrives at

$$A_N R(X) + B_N R(\gamma X) + C_N R\left(\gamma^{N+1} X\right) = 0 \tag{39}$$

for all $X > 0$, hence for all $X \in \mathbb{R}$. Then $(A_N + B_N + C_N) R(0) = 0$, and differentiation gives the following two relations:

$$(A_N + \gamma B_N + C_N \gamma^{N+1}) R'(0) = (A_N + \gamma^2 B_N + C_N \gamma^{2N+2}) R''(0) = 0. \tag{40}$$

Since $R(0)$, $R'(0)$ and $R''(0)$ are nonzero and $\lambda_2 \neq 0$, this implies $A_N = B_N = C_N = 0$, a contradiction. Hence $M = N$.

Now assume that $M < 0$. We will see that this is reduced to the previous case. We apply Proposition 13 with $c := -(N-1)\beta$: so assumption (i) with $\alpha = 0$ implies that there exists a real-analytic recurrence relation from $E(c + \Lambda_N)$ to $E(c + \Lambda_{N+1})$. Now $c + \Lambda_N$ consists of the values

$$-(N-1)\beta + j\beta = (-\beta)(N-1-j) \tag{41}$$

for $j = 0, \ldots, N-1$ and

$$c + \lambda_{N+1} = -(N-1)\beta + M\beta = (-\beta)(N-1-M). \tag{42}$$

Since $M < 0$ we know that $\tilde{M} := N - 1 - M > 0$. By the first case applied to $c + \Lambda_N$ and $c + \Lambda_{N+1}$ we conclude that $\tilde{M} = N$ which clearly implies that $M = -1$.

For (ii) $\Rightarrow$ (i) we assume at first that $M = N$. Then the real change of variable $X = e^{\lambda_2 x}$ transforms the cardinal spline spaces based on $E(\Lambda_N)$ and $E(\Lambda_{N+1})$ into the polynomial splines of degree N and $N+1$ on $(0,\infty)$ relative to the simple knots $t_j := e^{\lambda_2 j}$. Recurrence relations are known in such spaces, and their coefficients are real-analytic. This implies the statement by taking the inverse transform $x = \lambda_2^{-1} \ln X$. The case $M = -1$ is handled in a similar way. $\square$

Proposition 16. *Let α be a real number. Suppose $\Lambda_N = (\alpha, \dots, \alpha)$ and $\Lambda_{N+1} = (\Lambda_N, \lambda)$ for $\lambda \in \mathbb{R}$. Then the following assertions are equivalent*:
(i) *There exists a real-analytic recurrence relation from $E(\Lambda_N)$ to $E(\Lambda_{N+1})$*;
(ii) $\lambda = \alpha$.

Proof. Due to Proposition 13, we may assume that $\alpha = 0$. By Theorem 8 and assumption (i) there exist nonzero constants A_N, B_N, C_N such that for all $x \in \mathbb{R}$

$$A_N \frac{\varphi_{N+1}}{\varphi_N}(x) + B_N \frac{\varphi_{N+1}}{\varphi_N}(x-1) + C_N \frac{\varphi_{N+1}}{\varphi_N}(x-N-1) = 0. \tag{43}$$

Suppose that $\lambda \neq 0$. Note that $\varphi_N(x) = x^{N-1}/(N-1)!$ and, according to (33) and Lemma 14, there exists a nonzero constant d_{N+1} such that

$$\varphi_{N+1}(x) = d_{N+1}(e^{\lambda x} - R(x)), \quad R(x) = \sum_{k=0}^{N-1} \frac{(\lambda x)^k}{k!}. \tag{44}$$

Multiply (43) with $[x(x-1)(x-N-1)]^{N-1}$. It follows that there exists a polynomial Q such that

$$e^{\lambda x} P(x) - Q(x) = 0 \quad \text{for all } x \in \mathbb{R}, \tag{45}$$

where the polynomial P is defined by

$$\begin{aligned} P(x) = A_N[(x-1)(x-N-1)]^{N-1} + B_N e^{-\lambda}[x(x-N-1)]^{N-1} \\ + C_N e^{-\lambda(N+1)}[x(x-1)]^{N-1}. \end{aligned}$$

This is impossible unless $P = Q = 0$. But $P = 0$ implies $A_N = B_N = C_N = 0$. Thus we cannot have $\lambda \neq 0$.

For (ii) $\Rightarrow$ (i) note that $E(\Lambda_{N+1})$ is the classical polynomial spline space. $\square$

Now we are going to prove our main result stated as Theorem 1.

Proof of Theorem 1. Proof of the necessity by induction. For $N = 1$ there is nothing to prove. Suppose that there exists a sequence of exponential spaces $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_N \subset \mathcal{E}_{N+1} = E(\Lambda_{N+1})$ with real-analytic recurrence relations from $\mathcal{E}_n$ to $\mathcal{E}_{n+1}$ for $n = 2, \dots, N$. The recursive assumption enables us to assume, without loss of generality, that $\lambda_j = \alpha + (j-1)\beta$ for $1 \leqslant j \leqslant N$.

Suppose that $\beta \neq 0$. From Theorem 9 we can deduce that $\lambda_{N+1} = \alpha + \beta M$ for some integer M different from $0, \dots, N-1$. Proposition 15 ensures that either $M = N$ or $M = -1$. If $M = N$, then the equality $\lambda_j = \alpha + (j-1)\beta$ is valid for $j = N+1$ too. If $M = -1$, then $\lambda_{\sigma(j)} = \tilde{\alpha} + (j-1)\beta$ for $1 \leqslant j \leqslant N+1$, with $\tilde{\alpha} := \alpha - \beta$, and with $\sigma(1) = N+1$, $\sigma(j) := j-1$ for $j = 2, \dots, N+1$.

The case $\beta = 0$ follows from Proposition 16.

Sufficiency follows from Propositions 15 and 16. $\square$

Consider the exponential space $E(\lambda_1, \lambda_2)$. For simplicity assume that $\lambda_1 = 0$, and put $\beta = \lambda_2 - \lambda_1$. Then the proof of our main theorem shows how to construct all increasing sequences of exponential spaces admitting analytic relations, starting from $E(\lambda_1, \lambda_2) = E(0, \beta)$ in the following (uncomplete) scheme:

$$\begin{array}{ccccccc} & & & E(0,\beta) & & & \\ & & \swarrow & & \searrow & & \\ & E(-\beta,0,\beta) & & & & E(0,\beta,2\beta) & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ E(-2\beta,-\beta,0,\beta) & & & E(-\beta,0,\beta,2\beta) & & & E(0,\beta,2\beta,3\beta) \end{array} \tag{46}$$

Let us look at the particular case that $\Lambda_{N+1} = (\lambda_1, \ldots, \lambda_{N+1})$ is ordered, so $\lambda_1 \leqslant \cdots \leqslant \lambda_{N+1}$. Then there exists a real-analytic recurrence relation from $E(\lambda_1, \ldots, \lambda_n)$ to $E(\lambda_1, \ldots, \lambda_{n+1})$ for $n = 2, \ldots, N$, if and only if

$$\lambda_n = \lambda_1 + (n-1)(\lambda_2 - \lambda_1). \tag{47}$$

The following description is obvious from the above scheme:

Theorem 17. *Let $(\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$. Then there exist real-analytic recurrence relations from $E(\lambda_1, \ldots, \lambda_n)$ to $E(\lambda_1, \ldots, \lambda_{n+1})$ for $n = 1, 2, \ldots, N$ if and only if for $3 \leqslant j \leqslant N+1$*

$$\lambda_j = \lambda_1 + m_j(\lambda_2 - \lambda_1) \tag{48}$$

with either $m_{j+1} = \min\{m_1, \ldots, m_j\} - 1$ or $m_{j+1} = \max\{m_1, \ldots, m_j\} + 1$, and with $m_1 = 0, m_2 = 1$.

It follows from our results that the only exponential spaces admitting real-analytic recurrence relations are either the classical polynomial spaces, or transformations of polynomial spaces via an exponential map, cf. the discussion in Section 6 in [17].

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