Zeros of Non-negative Sub-biharmonic Functions and Extremal Problems in the Inverse Source Problem for the Biharmonic Potential

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1. Introduction

The notion of graviequivalent bodies or measures is fundamental in the theory of the Newton potential. Two measures μ_1, μ_2 with supports in a given domain Ω with regular boundary are called graviequivalent if and only if the following equlity holds

$$\int h(x)d\mu_1(x) = \int h(x)d\mu_2(x) \tag{1.1}$$

for every function h harmonic in Ω, i.e. staisfying there $\Delta h(x) = 0, x \in \Omega$.
Similarly, we shall say that the two measures μ_1, μ_2 are biharmonically equivalent if and only if the equality (1.1) holds for every function h which is biharmonic in the domain Ω, i.e. satisfying there $\Delta^2 h(x) = 0, x \in \Omega$.
In [Ko1] we considered the extremal problem

$$\int f(x)d\mu(x) \to \inf \tag{1.2}$$

over all non-negative measures μ graviequivalent to a given non-negative measure ν. It was proved that if $f(x)$ is continuous in $\overline{\Omega}$ and subharmonic in Ω, then problem (1.2) has unique solution. Let us remark that if f has second derivatives than its subharmonicity is equivalent to

$$\Delta f(x) \geq 0 \quad \text{for} \quad x \in \Omega. \tag{1.3}$$

In the present paper we consider problem (1.2) over non-negative measures μ biharmonically equivalent to a given measure ν. Within some natural conditions we prove

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that for continuous sub-biharmonic functions f in Ω problem (1.2) has unique solution. In the case of doubly smooth function f its sub-biharmonicity is equivalent to

$$(1.4) \qquad \Delta^2 f(x) \leq 0, \quad x \in \Omega$$

(cf. [Ni,Du]).
For convenience sake, we will introduce some notations.
Let us denote by $M^+(\Omega)$ the space of finite, non-negative regular Borel measures defined on Ω. For integers $j \geq 1$, let us introduce the spaces

$$(1.5) \qquad H_j = \{h \in C^{2j}(\Omega) : \Delta^j h(x) = 0, \quad x \in \Omega, \quad h \text{ - bounded in } \quad \overline{\Omega}\}.$$

Here we denoted by Δ^j the iterated Laplacian defined by $\Delta^{j+1} = \Delta\Delta^j$ for integers $j \geq 0$.
For each measure $\nu \in M(\Omega)$ set

$$(1.6) \qquad B_j(\nu) = \{\mu \in M^+(\Omega) : \int h(x) d\mu(x) = \int h(x) d\nu(x), \quad \text{for every } h \in H_j\}.$$

First, let us provide some historical remarks and references to the above problems.
The structure of the set $B_1(\nu)$ has been extensively studied from different points of view by many mathematicians, physicists and geophysicists. The reader may consult the papers by G. Anger [A1,A2], the book by D. Zidarov [Z], and the book by V. Isakov [Is] for the appropriate references on the history of the subject.
Our result in [Ko1] cited above may be reformulated as follows: The extremal points of the set $B_1(\nu)$ are exposed along the directions given by subharmonic functions f(considered as functionals on the space of measures).
What concerns the structure of the set $B_2(\nu)$, we refer to the book of B.-W. Schulze and G. Wildenhain [SW, §9.3, §10], and to the papers of E. Kleine [Kl1,Kl2], for some initial information. In general, the structure of the set $B_2(\nu)$ is far more complicated than that of $B_1(\nu)$.
It should be noted that the interest to the class of sub-biharmonic functions is motivated by their unusual behaviour due to the failure of the famous Hadamard conjecture on the sign of the biharmonic Green function (cf. [Du]).
In the most abstract setting the extremal points of the convex sets B_j are considered in the paper of R. Douglas [Do].

2. Motivation

The major motivation for considering extremal problems over the sets $B_j(\nu)$ of the type (1.2) is the analogy which exists with the classical moment problem.
In the interval $[a, b]$ the classical moment problem may be formulated as follows (cf. [KN,KS]):

$$(2.1) \qquad \int_a^b t^j d\mu = c_j, \quad j = 0, 1, \ldots, s,$$

where μ is a non-negative measure.
Accordingly, two measures μ_1, μ_2 are equivalent if and only if

$$\int_a^b t^j d\mu_1 = \int_a^b t^j d\mu_2, j = 0, 1, \ldots, s, \tag{2.2}$$

or which is the same

$$\int_a^b P d\mu_1 = \int_a^b P d\mu_2, \tag{2.3}$$

for every polynomial P of degree $\leq$ s.
Just like in (1.6) we have the set $B(\nu)$ of equivalent measures:

$$B(\nu) = \{\mu \in M^+([a,b]) : \int P(x) d\mu(x) = \int P(x) d\nu(x),$$

$$\text{for every polynomial } P, \quad \deg(P) \leq s\}. \tag{2.4}$$

For the purposes of the central limit theorem in probability theory P. Chebyshev and A.A. Markov considered problems of the following form:

$$\int_a^b f(x) d\mu(x) \to \inf, \tag{2.5}$$

where $\mu \in B(\nu)$, for a given non-negative measure ν. The function f is continuous. They also studied problems of the form

$$\int_c^d f(x) d\mu(x) \to \inf, \tag{2.6}$$

where the numbers c, d satisfy $a \leq c \leq d \leq$ b.
A.A. Markov created the theory of the canonical representations for describing the set of solutions of problems of the type (2.5). For the whole circle of problems see the review paper by M.G. Krein "The ideas of P.L. Chebyshev and A.A. Markov in the theory of the limiting values of the integrals and their further development", Uspekhi Mat. Nauk, v. 6, No. 4 (1951), pp. 3-120.
In particular, it was proved that if a function f satisfies

$$(-1)^s f^{(s)}(x) > 0, \quad x \in [a,b], \tag{2.7}$$

then problem (2.5) has unique solution.

The analogy which we investigate is obtained through replacing the operator of univariate differentiation d/dx with the Laplace operator in $\mathbf{R}^n$. After this we get in particular that:
(i) harmonic functions are analogues to the linear polynomials;
(ii) subharmonic functions (1.3) are analogues to convex functions;
(iii) biharmonic fucntions are analogues to cubic polynomials;
(iv) sub-biharmonic functions (1.4) correspond to (2.7) with $f^{(3)}(x) < 0$.
Finally, it is clear why problem (1.2) corresponds to problem (2.5).

3. The problem

We assume that the domain Ω has a boundary C^∞ as to guarantee the solubility of the elliptic boundary value problem as stated in [LM]. In particular, the domain Ω will be regular in the sense of solubility of the Dirichlet problem. Through $M(\Omega)$, resp. $M(\partial\Omega)$, we denote the space of finite, regular Borel measures defined on Ω, resp. $\partial\Omega$. The corresponding spaces of non-negative measures will be denoted by $M^+(\Omega)$, resp. $M^+(\partial\Omega)$.
Let f be a continuous function in $\overline{\Omega}$.
Let us introduce the following sets of functions:

$$\underline{P}_j(f) = \{h \in H_j : h(x) \leq f(x), \quad x \in \overline{\Omega}\} \tag{3.1}$$

$$\overline{P}_j(f) = \{h \in H_j : h(x) \geq f(x), \quad x \in \overline{\Omega}\}. \tag{3.2}$$

They are not empty since all constants are in H_j.
First we will prove the following result of the "infinite linear programming" which was announced in [Ko2].

THEOREM 1. Let $\nu \in M^+(\Omega)$ be given. We have the following statements:
a) the equalities hold

$$\min\{\int f(x)d\mu(x) : \mu \in B_j(\nu)\} = \max\{\int h(x)d\nu(x) : h \in \underline{P}_j(f)\} \tag{3.3}$$

$$\max\{\int f(x)d\mu(x) : \mu \in B_j(\nu)\} = \min\{\int h(x)d\nu(x) : h \in \overline{P}_j(f)\}; \tag{3.4}$$

b) the integral

$$\int f(x)d\mu(x), \quad \mu \in B_j(\nu), \tag{3.5}$$

takes its minimal (maximal) value for some $\mu = \mu_o \in B_j(\nu)$ if and only if there exists a function $h_o \in \underline{P}_j(f)$ (resp. $h_o \in \overline{P}_j(f)$) which coincides with f at those points $x \in \overline{\Omega}$ where $d\mu_o \neq 0$.

Proof. If we take $h = 1, x \in \Omega$, we obtain that

$$\| \mu \| = \int d\mu(x) = \int d\nu(x) = \| \nu \|$$

for every $\mu \in B_j(\nu)$. Hence, through the Banach-Alaouglou theorem $B_j(\nu)$ is a weak*-compact set (cf. [An1,Ho]). It is evidently also convex. Consequently, the max and min on the left-hand sides of statement a) exist. What concerns the min and max on the right-hand sides of a), their existence follows through the compactness of the families of functions

$$\{h \in H_j : \min_{\overline{\Omega}} f(.) \leq h(x) \leq f(x), \quad x \in \overline{\Omega}\} \qquad \text{and}$$

$$\{h \in H_j : f(x) \leq h(x) \leq \max_{\overline{\Omega}} f(.), \quad x \in \overline{\Omega}\}$$

(cf. [Ni] and [PP]).
By the definitions, we have

$$\int h(x)d\nu(x) = \int h(x)d\mu(x) \leq \int f(x)d\mu(x)$$

for every $h \in \underline{P}_j$ and every $\mu \in B_j(\nu)$, and similarly

$$\int f(x)d\mu(x) \leq \int h(x)d\mu(x) = \int h(x)d\nu(x)$$

for every $h \in \overline{P}_j$ and every $\mu \in B_j(\nu)$.
Hence, we obtain that

$$\underline{\gamma} = \max\{\int h(x)d\nu(x) : h \in \underline{P}_j(f)\} \leq \int f(x)d\mu(x) \leq$$

$$\leq \min\{\int h(x)d\nu(x) : \nu \in \overline{P}_j(f)\} = \overline{\gamma}; \tag{3.6}$$

for every $\mu \in B_j(\nu)$.
Let us prove the first inequality in a).
We will use the following version of the Hahn-Banach theorem which is due to M.G. Krein and M.A. Rutman (cf. [Ho]):

LEMMA. *Let K be a cone, F a subspace of E, and I , a linear functional on F such that $I(K \cap F) \geq 0$. If $F \cap$ Int(K) $\neq \emptyset$ then there exists an extension $\overline{I}$ of I to E such that $\overline{I}(K) \geq 0$.*

It is clear that the measure $d\nu$ defines a positive functional on the space H_j. Take an arbitrary $\gamma \in [\underline{\gamma}, \overline{\gamma}]$.
Let us define the functional ψ on the linear space $L = [H_j + \{f\}]^{\text{lin}}$ spanned by H_j and the function f, by putting $\psi(h) = \int h(x)d\nu(x)$ for $h \in H_j$, and $\psi(f) = \gamma$.
Let us see that the functional ψ is positive on the subspace L. Indeed, let $h_1(x) = h(x) + \alpha f(x) \geq 0, x \in \overline{\Omega}$. If $\alpha > 0$ then $(-1/\alpha)h(x) \in \underline{P}_j(f)$ which implies $(-1/\alpha)\psi(h) = (-1/\alpha)\int h(x)d\nu(x) \leq \underline{\gamma} \leq \gamma = \psi(f)$. In a similar way, if $\alpha < 0$, then $(-1/\alpha)h(x) \in \overline{P}_j(f)$ which implies $\psi(f) = \gamma \leq \overline{\gamma} \leq (-1/\alpha)\psi(h)$. In both cases we obtain $\psi(h_1) \geq 0$. The case of $\alpha = 0$ is trivially included.
According to the above-cited theorem of Krein-Rutman it follows that ψ admits a positive extension as a functional in $C(\overline{\Omega})$. The Riesz representation theorem provides that there exists a non-negative measure $d\mu, \mu \in B_j(\nu)$ such that $\int f(x)d\mu(x) = \gamma$.
The proof of *a*) is finished, since the second equality is proved in the same way.
The statement *b*) is obtained by taking $\gamma = \underline{\gamma}$ and $\gamma = \overline{\gamma}$. Indeed, for $\gamma = \underline{\gamma}$ we have

$$\int \{f(x) - h_o(x)\}d\mu_o(x) = 0 \tag{3.7}$$

for some $h_o \in \underline{P}_j(f)$, and some $\mu_o \in B_j(\nu)$. Since $f(x) \geq h_o(x)$, for $x \in \overline{\Omega}$, it follows that $\text{supp}(\mu_o) \subseteq \{x : f(x) = h(x), x \in \overline{\Omega}\}$.
The inverse is also true. Indeed, if $h_o \in \underline{P}_j(f)$, and for some $\mu_o \in B_j(\nu)$ we have $f(x) = h_o(x)$, when $x \in supp(\mu_o)$, then we obtain

$$\int f(x)d\mu_o(x) \leq \int h_o(x)d\mu_o(x) = \int h_o(x)d\mu(x) \leq \int f(x)d\mu(x),$$

i.e. μ_o provides the minimum in (3.5). Q.E.D.

Now we are going to apply Theorem 1 to the case $j = 2$. The case $j = 1$, of gravitational equivalence, was treated in [Ko1].
The following theorem is an analogue to a Chebyshev-Markov theorem in the univariate moment problem:

THEOREM 2. *Let the domain* $\Omega \subseteq \mathbf{R}^2$. *Assume that* f *is a real-analytic function in* Ω *which is continuous in* $\overline{\Omega}$ *, and satisfies* $\Delta^2 f(x) < 0, x \in \Omega$ *, i.e. is sub-biharmonic. Let us suppose that for a given measure* $\nu \in M^+(\Omega)$ *the following extremal problem*

$$\int f(x)d\mu(x) \to \min, \quad \mu \in B_2(\nu), \tag{3.8}$$

has a solution μ_o *satisfying* $\text{supp}(\mu_o) \subseteq \Omega$. *Such a solution is unique.*

Proof. 1.) According to Theorem 1 there exists a function $h_o \in H_2$ such that $h_o(x) \leq f(x), x \in \overline{\Omega}$, and satisfying (3.7). Hence $\text{supp}(\mu_o) \subseteq N = \{x \in \overline{\Omega} : f(x) = h_o(x)\}$.
2.) Let us put $F(x) = f(x) - h_o(x)$. Clearly, N is the set of zeroes of F. According to the condition of the theorem $\text{supp}(\mu_o) \subseteq \Omega$, so we will be interested only in the set

$N_1 = N \cap \Omega$. We will prove that the complement $\mathbf{C}N_1$ of N_1 in $\mathbf{R}^2$ has no compact components in $\mathbf{R}^2$.
3.) Let us notice that $F(x) \geq 0$ in $\overline{\Omega}$, and if a point $x_o \in \Omega$ satisfies $F(x_o) = 0$, then like a point of interior local minimum we have $\partial F(x_o)/\partial x_j = 0$, for $j = 1, 2$ where $x = (x_1, x_2) \in \mathbf{R}^2$, and $\partial^2 F(x_o)/\partial^2 x_j \geq 0$, for $j = 1, 2$. Hence $\Delta F(x_o) \geq 0$.
On the other hand the function $\Delta F(x)$ is superharmonic since $\Delta^2 F(x) = \Delta^2 f(x) < 0$.
4.) Now let us assume that $\mathbf{C}N_1$ has a compact component, i.e. a domain $G \subseteq \Omega$ such that $\partial G \subseteq N_1$. In 3.) we obtained that $\Delta F(x) \geq 0$ on $N_1 \supseteq \partial G$, and $\Delta^2 F(x) < 0$ in G. According to the well-known properties of the superharmonic functions (cf. [HK]), it follows that $\Delta F(x) \geq 0, x \in$ G. The last means that the function $F(x)$ is subharmonic in G. But we have $F(x) = 0, x \in \partial$G. Now the same property for subharmonic functions in [HK] implies that $F(x) \leq 0$ for every $x \in$ G. Since $F(x) \geq 0$ in $\overline{\Omega}$, we obtain $F(x) = f(x) - h_o(x) = 0$ in G which implies $\Delta^2 f(x) = 0$ in G. The last contradicts our basic assumption that $\Delta^2 f(x) < 0, x \in \Omega$. Hence $G = \emptyset$, and $\mathbf{C}N_1$ is a connected set.
5.) Up till now we did not use the fact that $\Omega \subseteq \mathbf{R}^2$.
Let us suppose that there exist two measures $\mu_1, \mu_2 \in B_2(\nu)$ solving problem (3.8) with $\text{supp}(\mu_j) \subseteq \Omega, j = 1, 2$. It follows that

$$\int h(x) d\mu_1(x) = \int h(x) d\mu_2(x), \quad h \in H_2.$$

We saw in 1.) that $\text{supp}(\mu_j) \subseteq N_1, j = 1, 2$. It follows that $\text{supp}(\mu_1 - \mu_2) \subseteq N_1$, and

$$\int h(x) d(\mu_1 - \mu_2) = 0, \quad h \in H_2.$$

According to the theorem of Mergelyan (cf. [Ga]), such equality for harmonic functions only, and a set N_1 with a connected complement in $\mathbf{R}^2$ implies $\mu_1 - \mu_2 = 0$.
This proves the theorem. Q.E.D.

REMARKS. 1. The case when $\text{supp}(\mu_j) \cap \partial\Omega \neq \emptyset$ is more complicated and needs more considerations. It is very plausible that Theorem 2 holds without such severe restriction on the supports of the measures.
2. Clearly, if $\Omega \subseteq \mathbf{R}^n, n \geq 3$, we have to require regularity in Wiener sense of the points of the set N in order to get density result (cf. [La]).

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