Stable recurrence relations for a class of $L-$splines and for polysplines

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Abstract. We prove stable recurrence relations for a special class of compactly supported $L-$splines. We apply them to obtain recurrence relations for compactly supported polysplines.

§1. Introduction

One of the most classical chapters of Spline theory is related to the theory of $B-$splines. A very essential part of it is the discovery of the $(2-$term) **recurrence relations** for the $B-$spline by Carl de Boor which was the start for their efficient computational methods. Many people have contributed to its development, see [6]. The main feature of the recurrence relation for the $B-$splines is the *positivity* of the coefficients; this fact together with the positivity of the $B-$splines makes the computational procedures stable and efficient. Many researchers have proved different types of recurrence relations for generalized splines, the so–called $L-$splines, [12], [15], [11], [10], [5], [16] but apparently only the references [12], [15], [10] contain relations with positive coefficients. The main purpose of the present paper is to establish new recurrence relations which are $3-$*term* and *with positive coefficients* for a special class of $L-$splines of order 4 defined by the operator (3) below. In these recurrence relations (10), the left hand side is a basic $L-$spline (compact with minimal support) for the operator (3) of order 4 and the right hand side is a sum of three terms (multiplied by some positive coefficient functions) which are basic $L-$splines (compact with minimal support) for the operator (4) which is of order 2.

The main motivation for the consideration of the above special class of $L-$splines and their recurrence relations is their relationship to the theory of polysplines which is a multivariate generalization of the one–dimensional splines, see [7], [1], [8]. Polysplines may be defined in arbitrary

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domains but they are explicitly computable only in the case of domains with special geometry, for example, on the ball or on the strip. In both cases their computation is reduced to the computation of infinitely many one–dimensional $L-$splines of a special type and the operator L for all these generalized splines has constant coefficients, see formulas (4) and (3). Polysplines are a flexible tool and may be used for surface design in CAGD, see e.g. [9].

A main problem is the efficient computation of the polysplines. This has been a motivation for our study of recurrence relations. However, our study shows that there does not exist a 3−term recurrence relation for the polysplines of a simple form; on the other hand all we need for the computation of a compactly supported polyspline is provided by means of the 3−term recurrence relation for the $L-$splines in formulas (10), (18–20).

The plan of the paper is the following: In Section 2 we prove the new 3−term recurrence relations for $L-$splines. In Section 3 we give a short introduction to polysplines with "semi–compact" support; we provide a recurrence formula for their computation.

For the sake of simplicity we consider the **cardinal** case, i.e. when the knots of the splines are integers.

§2. Recurrence relations

2.1. The classical case

Let us recall the famous recurrence formula, see [4],

$$\frac{B_{i,k}(x)}{t_{i+k}-t_i} = \frac{x-t_i}{t_{i+k}-t_i}\frac{B_{i,k-1}(x)}{t_{i+k-1}-t_i} + \frac{t_{i+k}-x}{t_{i+k}-t_i}\frac{B_{i+1,k-1}(x)}{t_{i+k}-t_{i+1}}. \qquad (1)$$

It is a remarkable fact that the coefficients are positive. Recall that the support of $B_{i,k}$ is $[t_i, t_{i+k}]$ and $B_{i,k}$ is a piecewise polynomial of degree $k-1$.

Let us consider the **cardinal cubic case,** i.e. $t_i = i$ for all $i \in \mathbb{Z}$ and $k = 4$. Then all $B-$splines are shifts of unique $B-$spline which we denote by $B_k(t)$, i.e. $B_{i,k}(x) = B_k(x-i)$.

By iterating (1) we obtain a 3−term recurrence relation

$$\frac{B_4(x)}{4} = \frac{x^2}{24}B_2(x) + \frac{8x-2x^2-4}{24}B_2(x-1) + \frac{(4-x)^2}{24}B_2(x-2). \qquad (2)$$

The coefficients in front of $B_2(x-j)$ are polynomials positive on the support of $B_2(x-j)$, for $j = 0, 1, 2$. The middle coefficient is positive on the whole interval $[1, 3]$ which is the support of $B_2(x-1)$ since we have

$$4x - x^2 - 4 + 2 = -(x-2)^2 + 2.$$

Thus we see that this is a $3-$term recurrence relation with positive coefficients.

Our **main purpose** is to generalize the above $3-$term recurrence relation (2) for a special class of $L-$splines, and to see how this may be applied to the computation of polysplines.

2.2. Definition of $L-$splines

We consider the $L-$splines defined for $k \geq 0$ by the operators

$$L_2 = \left(\frac{d^2}{dt^2} - k^2\right)^2 = L_1^2 \qquad \text{where} \tag{3}$$

$$L_1 = \frac{d^2}{dt^2} - k^2. \tag{4}$$

The solutions to

$$L_2 f(t) = 0 \qquad \text{for all } t \in \mathbb{R}$$

which are in $C^\infty(\mathbb{R})$ are called "generalized polynomials for L_2 ", and coincide with all linear combinations of the form

$$f(t) = Ae^{-kt} + B \cdot t \cdot e^{-kt} + Ce^{kt} + D \cdot t \cdot e^{kt}$$

with some constants A, B, C, D. In a similar way, *the generalized polynomials* for L_1 are the C^∞ solutions to $L_1 f = 0$ and are of the form

$$f(t) = Me^{-kt} + Ne^{kt}.$$

On some interval $[a, b]$ the function u is called $L-$spline for the operator L of order p if it is a piecewise solution of the equation $Lf = 0$ which is in C^{p-2}, and the points where it is not C^∞ are called break–points. The $L-$spline is called *cardinal* if it is defined on the whole real line and its breakspoints are subset of $\mathbb{Z}$.

For the theory of generalized splines, in particular for $L-$splines a basic reference is the monograph [14].

2.3. The Green functions ϕ_3 and ϕ_1

Let us recall that in the polynomial case, where the operator is $L = d^k/dt^k$, the Green function is $t_+^{k-1}/(k-1)!$. It satisfies the initial value problem (see [14]),

$$\frac{d^j}{dt^j}\left(t_+^{k-1}\right)_{|t=0} = 0 \qquad \text{for } j = 0, 1, ..., k-2,$$

$$\frac{1}{(k-1)!}\frac{d^{k-1}}{dt^{k-1}}\left(t_+^{k-1}\right)_{|t=0} = 1.$$

We use the symbol f_+ to define

$$f_+(t) = \begin{cases} f(t) & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases}$$

For the above operators L_1 and L_2 we define the Green functions $\phi_1, \phi_3 \in C^\infty(\mathbb{R})$, satisfying $L_1\phi_1 = 0$ and $L_2\phi_2 = 0$ on $\mathbb{R}$, and

$$\begin{aligned} &\phi_1(0) = 0, && \phi_1'(0) = 1; \\ &\phi_3(0) = \phi_3'(0) = \phi_3''(0) = 0, && \phi_3'''(0) = 1. \end{aligned}$$

We prove by a direct computation

$$\phi_1(t) = \phi_1^k(t) = \frac{1}{2k}\left(e^{kt} - e^{-kt}\right) \qquad \text{for } k > 0; \tag{5}$$

and

$$\phi_3(t) = \phi_3^k(t) = \frac{1}{4k^3}\left(e^{-kt} + kte^{-kt} - e^{kt} + kte^{kt}\right) \qquad \text{for } k > 0. \tag{6}$$

Let us note that the polynomial case (i.e. $k = 0$) is obtained from (5) and (6) when $k \longrightarrow 0$.

Let us note that

$$\phi_3(t) > 0 \qquad \text{and } \phi_1(t) > 0 \qquad \text{for } t > 0, \tag{7}$$

see Chapter 9 in [14].

2.4. The compactly supported $L-$splines Q_2 and Q_4

The compactly supported $L-$splines with minimal support are given by the following formulas, (see [13], or [7], Section 13.10),

$$Q_2^k(t) = \sum_{j=0}^{2} s_{1,j}\phi_{1,+}(t-j), \tag{8}$$

$$Q_4^k(t) = \sum_{j=0}^{4} s_{3,j}\phi_{3,+}(t-j), \tag{9}$$

where $s_{1,j}$ and $s_{3,j}$ are the coefficients of the polynomials $s_1(\lambda)$ and $s_3(\lambda)$, defined as follows:

$$s_1(\lambda) = \left(e^{-k} - \lambda\right)\left(e^{k} - \lambda\right) = \sum_{j=0}^{2} s_{1,j}\lambda^j;$$

$$s_{1,0} = 1, \quad s_{1,1} = -e^{-k} - e^{k}, \quad s_{1,2} = 1,$$

and

$$s_3(\lambda) = \left(e^{-k} - \lambda\right)^2 \left(e^{k} - \lambda\right)^2 = \sum_{j=0}^{4} s_{1,j}\lambda^j;$$
$$s_{3,0} = 1, \quad s_{3,1} = -2e^{-k} - 2e^{k}, \quad s_{3,2} = e^{-2k} + e^{2k} + 4,$$
$$s_{3,3} = -2e^{-k} - 2e^{k}, \quad s_{3,4} = 1.$$

Again, the polynomial case is obtained from the above for $k = 0$.

Let us remind that $\operatorname{supp}(Q_2) = [0,2]$ and $\operatorname{supp}(Q_4) = [0,4]$, as well as $Q_2, Q_4 \geq 0$, (see [7], Section 1.10).

Now we present our first main result.

Theorem 1. *For every k the compactly supported $L-$splines Q_4^k and Q_2^k for the operators (3) and (4) satisfy the $3-$term recurrence relation*

$$Q_4^k(t) = c_0^k(t)\, Q_2^k(t) + c_1^k(t)\, Q_2^k(t-1) + c_0^k(4-t)\, Q_2^k(t-2) \qquad (10)$$

where the coefficients c_0 is given by formula (12) on the interval $[0,4]$, and c_1 is given by (13) on the interval $[1,2]$ and by formula (14) on $[2,3]$. The coefficients c_0 and c_1 are ***positive*** *on their definition intervals.*

We split the proof in several steps.

In general, we follow the idea of the proof of Carl de Boor in the classical case. We compare the two expressions for finding the coefficients c_0, c_1, by substituting in (10) the expressions (8) and (9),

$$\begin{aligned} Q_4^k(t) &= s_{3,0}\phi_{3,+}(t) + s_{3,1}\phi_{3,+}(t-1) + s_{3,2}\phi_{3,+}(t-2) \\ &\quad + s_{3,3}\phi_{3,+}(t-3) + s_{3,4}\phi_{3,+}(t-4) \\ &= c_0 s_{1,0}\phi_{1,+}(t) + (c_0 s_{1,1} + c_1 s_{1,0})\,\phi_{1,+}(t-1) \\ &\quad + (c_0 s_{1,2} + c_1 s_{1,1} + c_2 s_{1,0})\,\phi_{1,+}(t-2) \\ &\quad + (c_1 s_{1,2} + c_2 s_{1,1})\,\phi_{1,+}(t-3) + c_2 s_{1,2}\phi_{1,+}(t-4). \end{aligned} \qquad (11)$$

We will use the last equality for finding the coefficients c_0 and c_1 by simply comparing the terms.

2.5. The coefficient c_0 and its positivity

In the interval $[0,1]$ we compare only the non–zero terms in (11) and we find the coefficient

$$c_0(t) := c_0^k(t) = \frac{Q_4^k(t)}{Q_2^k(t)} = \frac{s_{3,0}\phi_{3,+}(t)}{s_{1,0}\phi_{1,+}(t)} = \frac{\phi_{3,+}(t)}{\phi_{1,+}(t)} > 0 \qquad (12)$$

for all $t > 0$, the last inequality following trivially from (7). Note that we need c_0 only for $t \in [0,2]$ since the support of $Q_2(t)$ is $[0,2]$. Simple calculus shows that $\lim c_0(t) \overset{t \longrightarrow 0}{\longrightarrow} 0$ for $t \to 0$, hence we may put $c_0(0) = 0$.

From (12) we obtain the explicit formula for c_0 in the interval $t \in [0,1]$,

$$c_0(t) = \begin{cases} -\frac{1}{2k^2}(1 - kt \coth kt) & \text{for } k > 0 \\ \frac{t^2}{6} & \text{for } k = 0, \end{cases}$$

where $\coth x = \cosh x / \sinh x$ and $\cosh x = \frac{e^{-k}+e^{k}}{2}$, $\sinh x = \frac{e^{k}-e^{-k}}{2}$. We define further $c_0(t)$ on the interval $[1,2]$ by the same formula as above.

2.6. The coefficient c_1 on the interval $[1,2]$

We proceed in a similar way for finding the coefficient c_1 from the second equality in (11), by comparing the coefficients of the non–zero quantities. In the interval $[1,2]$ we obtain the equality

$$s_{3,0}\phi_3^+(t) + s_{3,1}\phi_3^+(t-1) = c_0 s_{1,0}\phi_1^+(t) + (c_0 s_{1,1} + c_1 s_{1,0})\,\phi_1^+(t-1);$$

Since $s_{3,0} = s_{1,0} = 1$ and by using equation (12) for c_0 in $1 \leq t \leq 2$, we obtain after some straightforward calculations, for all $k \neq 0$ the equality

$$c_1(t) = \frac{\cosh k}{k^2} \cdot \{1 - 2k(t-1)\coth k(t-1) + kt \coth kt\}. \qquad (13)$$

2.7. The symmetry of the coefficients c_j

We may apply a symmetry argument for finding the coefficient of the recurrence relation in the interval $[2,4]$ from those in the interval $[0,2]$. We shall use the fact that the basic splines Q_4 and Q_2 are symmetric about the point $t = 2$ and $t = 1$ respectively, i.e.

$$Q_4(4-t) = Q_4(t) \qquad \text{for } 0 \leq t \leq 4,$$
$$Q_2(2-t) = Q_2(t) \qquad \text{for } 0 \leq t \leq 2,$$

which is Theorem 13.51 in [7], see also [13]. From the relation

$$Q_4(t) = c_0(t)\,Q_2(t) + c_1(t)\,Q_2(t-1) + c_2(t)\,Q_2(t-2) = Q_4(4-t)$$

follows that $Q_4(4-t)$ is equal to

$$c_0(4-t)\,Q_2(4-t) + c_1(4-t)\,Q_2(3-t) + c_2(4-t)\,Q_2(2-t)$$

hence,

$$Q_4(4-t) = c_0(4-t)\,Q_2(t-2) + c_1(4-t)\,Q_2(t-1) + c_2(4-t)\,Q_2(t).$$

Comparing the coefficients shows that

$$c_2(4-t) = c_0(t) \qquad \text{for } 0 \leq t \leq 2, \qquad (14)$$
$$c_1(4-t) = c_1(t) \qquad \text{for } 1 \leq t \leq 3,$$

and we use the last to define the coefficient c_1 on the interval $[2,3]$ by means of $c_1(2+\tau) = c_1(2-\tau)$ for $0 \leq \tau \leq 1$.

2.8. Positivity of $c_1(t)$ on $(1,3)$

The positivity of the coefficient $c_1(t)$ is not evident. Due to (14) it suffices to prove the positivity of $c_1(t)$ only in the interval $(1,2)$.

Theorem 2. *The function $c_1(t)$ is positive on the interval $(1,2)$. More precisely, for all $t \in (1,2)$ we have*

$$c_1(t) = \frac{\cosh k}{\sinh kt \sinh k(t-1)} \sum_{m=0}^{\infty} \frac{k^{2m}}{(2m+2)!} b_m[t]$$

with coefficients $b_m[t]$ given by

$$b_m[t] = (2m+2)\left[(2t-1)^{2m+1}(2-t) - (3t-2)\right] + (2t-1)^{2m+2} - 1,$$

which are positive.

The proof is rather technical and we will omit it. It is based on direct calculations.

This completes the **proof** of Theorem 1.

§3. Recurrence relation for compactly supported polysplines

The main motivation for considering the $L-$splines with operator L given by (3) is their application to the theory of polysplines. Here we will outline the way to obtain $3-$term recurrence relations for the compactly supported polysplines. For simplicity sake we consider the $2\pi-$*periodic cardinal* case.

Let us recall that the Laplace operator Δ in the space $\mathbb{R}^n$ with variables $(x_1, x_2, ..., x_n)$ is defined as

$$\Delta = \sum\nolimits_{j=1}^{n} \partial^2/\partial x_j^2$$

and the biharmonic operator is defined as an iteration of Δ, i.e. as $\Delta^2 = \Delta\Delta$.

Let us recall the definition of cardinal biharmonic polyspline, [1].

Definition 1. *The function $u(t,y)$ is called **cardinal biharmonic polyspline** in the space $\mathbb{R}^n$ where $t \in \mathbb{R}$ and $y = (y_1, ..., y_{n-1}) \in \mathbb{R}^{n-1}$, with $x = (t,y)$, if it satisfies the following:*

1. *For all $j \in \mathbb{Z}$ the function $u(t,y)$ is biharmonic in the strip $(j, j+1) \times \mathbb{R}^{n-1}$ with $t \in (j, j+1)$ and $y \in \mathbb{R}^{n-1}$, i.e. satisfies*

$$\Delta^2 u(t,y) = 0.$$

2. *The function $u(t,y)$ is C^2 in the whole space $\mathbb{R}^n$ (including the "interfaces" $\{(t,y) \in \mathbb{R}^n : t = j\}$).*

Now we consider the following Definition.

Definition 2. *A cardinal biharmonic polyspline is called* ***semi–periodic*** *if $u(t,y)$ is a $2\pi-$periodic function in every variable y_j for $j = 1, 2, ..., n-1$. We say that the biharmonic polyspline $u(t,y)$ is* ***semi–compactly supported*** *if its support lies in some compact cylinder $[M,N] \times \mathbb{R}^{n-1}$ for some integers M and N with $M \leq N$.*

Semi–periodic cardinal polysplines have been called "periodic polysplines on strips" in [7].

A basic result in [7], Theorem 9.3, says that by considering the Fourier series in the variables y we reduce the computation of a semi–periodic cardinal polyspline u to a computation of infinitely many one–dimensional $L-$splines. Let us outline shortly their computation. Let us assume that $u(t,y)$ is a semi–compactly supported polyspline in some set $(0,M) \times [0,2\pi]^{n-1}$. For simplicity we consider the case $n = 2$. The Fourier series expansion of $u(t,y)$ is

$$u(t,y) = \frac{u_0(t)}{2} + \sum_{k=1}^{\infty} u_k(t)\cos ky + \sum_{k=1}^{\infty} d_k(t)\sin ky,$$

where the coefficients are defined as

$$u_k(t) = \frac{1}{\pi}\int_0^{2\pi} u(t,y)\cos ky dy \qquad \text{for all } k \geq 0,$$
$$d_k(t) = \frac{1}{\pi}\int_0^{2\pi} u(t,y)\sin ky dy \qquad \text{for all } k \geq 0.$$

Again by Theorem 9.3 in [7], the coefficients $u_k(t)$, $d_k(t)$ are compactly supported $L-$splines of the variable t with support in the interval $[0,M]$, for the operator $L = \left(\frac{d^2}{dt^2} - k^2\right)^2$ defined in (3).

Of special interest are those cardinal semi–periodic polysplines u with semi–compact support such that their support is minimal. From considerations analogous to those in Chapter 15.8 of [7] we see that all such "minimally supported polysplines" have their supports in $[N, N+4] \times [0,2\pi]^{n-1}$ for some integer N.

We argument as in Chapter 13 of [7] to see that the following Theorem holds.

Theorem 3. *Suppose that $u(t,y)$ is a semi–periodic biharmonic cardinal polyspline with support in the set $[0,4] \times [0,2\pi]^{n-1}$. Then it is represented* *by the following formula*

$$u(t,y) = \frac{u_0(2)}{2}\frac{Q_4^0(t)}{Q_4^0(2)} + \sum_{k=1}^{\infty} u_k(2)\frac{Q_4^k(t)}{Q_4^k(2)}\cos ky \tag{15}$$
$$+\sum_{k=1}^{\infty} d_k(2)\frac{Q_4^k(t)}{Q_4^k(2)}\sin ky.$$

Indeed, it is clear that all $L-$splines $u_k(t)$, $d_k(t)$ are proportional to the compactly supported $L-$splines with minimal support, i.e.

$$\frac{u_k(t)}{u_k(2)} = \frac{d_k(t)}{d_k(2)} = \frac{Q_4^k(t)}{Q_4^k(2)} \qquad \text{for } t\in\mathbb{R}, \quad \text{for } k\geq 0;$$

here $Q_4^k(t)$ is the minimal compactly supported $L-$spline defined in (9). Hence, we obtain (15).

In a similar way we prove as in [7] the following

Theorem 4. *Suppose that $u(t,y)$ is a semi–periodic biharmonic cardinal polyspline which has support in the set $[0,N+4]\times\mathbb{R}^{n-1}$ for some integer N. Then it is uniquely representable in the form*

$$u(t,y) = \sum_{k=0}^{\infty}\left[\sum_{j=0}^{N} c_j^k Q_4^k(t-j)\right]\cos ky + \sum_{k=1}^{\infty}\left[\sum_{j=0}^{N} d_j^k Q_4^k(t-j)\right]\sin ky, \tag{16}$$

for some constants c_j^k and d_j^k.

Now we may apply the recurrence relation for the $L-$splines from Theorem 1 in the form

$$\frac{Q_4^k(t)}{Q_4^k(2)} = \frac{c_0^k(t)}{Q_4^k(2)}Q_2^k(t) + \frac{c_1^k(t)}{Q_4^k(2)}Q_2^k(t-1) + \frac{c_0^k(4-t)}{Q_4^k(2)}Q_2^k(t-2).$$

In fact, we obtain a 3−term recurrence relation for the polyspline, namely $u(t,y)$ is equal to

$$\frac{u_0(2)}{2}\left(\frac{c_0^0(t)}{Q_4^0(2)}Q_2^0(t) + \frac{c_1^0(t)}{Q_4^0(2)}Q_2^0(t-1) + \frac{c_0^0(4-t)}{Q_4^0(2)}Q_2^0(t-2)\right) +$$
$$+\sum_{k=1}^{\infty} u_k(2)\left(\frac{c_0^k(t)}{Q_4^k(2)}Q_2^k(t) + \frac{c_1^k(t)}{Q_4^k(2)}Q_2^k(t-1) + \frac{c_0^k(4-t)}{Q_4^k(2)}Q_2^k(t-2)\right)\cos ky$$
$$+\sum_{k=1}^{\infty} d_k(2)\left(\frac{c_0^k(t)}{Q_4^k(2)}Q_2^k(t) + \frac{c_1^k(t)}{Q_4^k(2)}Q_2^k(t-1) + \frac{c_0^k(4-t)}{Q_4^k(2)}Q_2^k(t-2)\right)\sin ky,$$

which implies

$$u\left(t,y\right)=v_1\left(t,y\right)+v_2\left(t,y\right)+v_3\left(t,y\right). \tag{17}$$

Here we have put

$$v_1\left(t,y\right)=\frac{u_0\left(2\right)}{2}\frac{c_0^0\left(t\right)}{Q_4^0\left(2\right)}Q_2^0\left(t\right)+\sum_{k=1}^{\infty}u_k\left(2\right)\frac{c_0^k\left(t\right)}{Q_4^k\left(2\right)}Q_2^k\left(t\right)\cos ky \tag{18}$$
$$+\sum_{k=1}^{\infty}d_k\left(2\right)\frac{c_0^k\left(t\right)}{Q_4^k\left(2\right)}Q_2^k\left(t\right)\sin ky,$$

and

$$v_2\left(t,y\right)=\frac{u_0\left(2\right)}{2}\frac{c_1^0\left(t\right)}{Q_4^0\left(2\right)}Q_2^0\left(t-1\right)+\sum_{k=1}^{\infty}u_k\left(2\right)\frac{c_1^k\left(t\right)}{Q_4^k\left(2\right)}Q_2^k\left(t-1\right)\cos ky \tag{19}$$
$$+\sum_{k=1}^{\infty}d_k\left(2\right)\frac{c_1^k\left(t\right)}{Q_4^k\left(2\right)}Q_2^k\left(t-1\right)\sin ky,$$

and

$$v_3\left(t,y\right)=\frac{u_0\left(2\right)}{2}\frac{c_0^0\left(4-t\right)}{Q_4^0\left(2\right)}Q_2^0\left(t-2\right)+\sum_{k=1}^{\infty}u_k\left(2\right)\frac{c_0^k\left(4-t\right)}{Q_4^k\left(2\right)}Q_2^k\left(t-2\right)\cos ky \tag{20}$$
$$+\sum_{k=1}^{\infty}d_k\left(2\right)\frac{c_0^k\left(4-t\right)}{Q_4^k\left(2\right)}Q_2^k\left(t-2\right)\sin ky,$$

and we see that the functions $v_j\left(t,y\right)$ have supports in $[0,2]$, $[1,3]$, and $[2,4]$ respectively. Let us remark that they are in general **not** harmonic polysplines as one might will. The main reason is the impossibility to separate the dependence on t and k in the coefficients c_0 and c_1. Anyway, formulas (17), (18–20) provide us with an efficient and stable way for the computation of the compactly supported polyspline u.

Finally, let us remark that an important motivation for the consideration of the above recurrence relations for polysplines is their application to the theory of "polyspline surfaces", which are a generalization of the "biharmonic surfaces" introduced by M. Bloor and M. Wilson for the purposes of CAGD in 1988, see [2]. For space limitations the polyspline surfaces will be introduced and studied in a further research.

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§4. References

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