NUMERICAL SOLUTIONS OF ODEs USING VOLTERRA SERIES

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Abstract

We propose a numerical approach for solving systems of nonautonomous ordinary differential equations under suitable assumptions. This approach is based on expansion of the solutions by Volterra series and allows to estimate the accuracy of the approximation. Also we can solve some ordinary differential equations for which the classical numerical methods fail.

1 Introduction

A numerical approach for solving systems of nonautonomous ordinary differential equations (ODEs) is proposed under suitable assumptions. This approach is based on expansion of the solutions of ODEs by Volterra series and allows to estimate the distance between the obtained approximation and the true trajectory.

The approximation of trajectories of nonlinear control systems can be reduced to the problem of approximation the solutions of nonautonomous systems of ODEs. As a rule, their right-hand sides depend on the time in a discontinuous way. But applying the traditional numerical approximation schemes of higher order (such as Runge-Kutta schemes) is a nontrivial task (cf. for example [2], [8], etc.). An approach for solving this problem for affinelly controlled systems is proposed in [3] and [4]. It is based on the well known expansion of the solution of ODEs systems by Volterra series (cf. [5]). Combining this approach with the ideas developed in [6], we obtain a method for approximation the trajectories of analytic control systems with guaranteed accuracy (cf. [7]).

2 Systems of ODEs and Volterra series.

For every point $y = (y^1, \dots, y^n)^T$ from R^n we define $\|y\| := \sum_{i=1}^n |y^i|$ and $B = \{y \in R^n : \|y\| \le 1\}$. For $x_0 \in R^n$, Ω is a convex compact neighbourhood of the point x_0.

If $z \in C$, the set of the complex numbers, then $|z|$, Re z and Im z denote the norm, the real and the imaginary part of z, respectively. For some $\sigma > 0$ we set

$$\Omega^\sigma := \{z = (z_1, \dots, z_n) \in C^n : \ (\mathrm{Re}\ z_1, \dots, \mathrm{Re}\ z_n) \in \Omega \ + \ \sigma B; |\mathrm{Im}\ z_i| < \sigma, \ i = 1, \dots, n\}.$$

By $\mathcal{F}_\Omega^\sigma$ we denote the set of all real analytic functions ϕ defined on Ω with bounded analytic extensions $\bar{\phi}$ on Ω^σ. We define a norm in the set $\mathcal{F}_\Omega^\sigma$ as follows:

$$\|\phi\|_\Omega^\sigma = \sup\left\{|\bar{\phi}(z)| \ : z \in \Omega^\sigma\right\}.$$

If $h(x) = (h_1(x), h_2(x), \dots, h_n(x)), x \in \Omega$ is a vector field defined on Ω, we identify h with the corresponding differential operator

$$\sum_{i=1}^{n} h_i(x) \frac{\partial}{\partial x_i}, \ x \in \Omega.$$

Let $\mathcal{V}_\Omega^\sigma$ be the set of all real analytic vector fields h defined on Ω such that every $h_i, i = 1, \dots, n$ belongs to $\mathcal{F}_\Omega^\sigma$. We define the following norm in $\mathcal{V}_\Omega^\sigma$:

$$\|h\|_\Omega^\sigma = \max\{\|h_i\|_\Omega^\sigma, \ i = 1, \dots, n\}.$$

An integrable analytic vector field $X_t(x) = (X_1(t,x), X_2(t,x), \dots, \ X_n(t,x)), \ x \in \Omega$, $t \in R$ (parameterized by t) is a map $t \to X_t \in \mathcal{V}_\Omega^\sigma$ such that:

i) for every $x \in \Omega$ the functions $X_1(.,x), X_2(\cdot,x), \dots, X_n(\cdot,x)$ are measurable;

ii) for every $t \in R$ and $x \in \Omega$, $|X_i(t,x)| \le m(t), \ i = 1, 2, \dots, n$, where m is an integrable (on every compact interval) function.

Let t_0 be a real number, M be a compact set contained in the interior of Ω and $x_0 \in M$, and let X_t be an integrable analytic vector field defined on Ω. Then there exists $T(M, X_t) > t_0$ such that for every point x of M the solution $y(.,x)$ of the differential equation

$$\dot{y}(t,x) = X_t(y(t,x)), \quad y(t_0,x) = x \tag{1}$$

is defined on the interval $[t_0, T(M, X_t)]$ and $y(T,x) \in \Omega$ for every T from $[t_0, T(M, X_t)]$. In this case we denote by $\exp \int_{t_0}^{T} X_t \, dt : \ M \to \Omega$ the diffeomorphism defined by

$$\exp \int_{t_0}^{T} X_t \ dt \ (x) := y(T,x).$$

According to Proposition 2.1 from [?], the positive number $T(M, X_t)$ can be chosen in such a way that $T(M, X_t) > t_0$ and for every T from the open interval $(t_0, T(M, X_t))$, for every point x from M and for every function ϕ from $\mathcal{F}_\Omega^\sigma$, the following expansion of $\phi\left(\exp \int_{t_0}^{T} X_t dt(x)\right)$ in Volterra series holds true:

$$\phi\left(\exp \int_{t_0}^{T} X_t dt(x)\right) = \phi(x) + \sum_{N=1}^{\infty} \int_{t_0}^{T} \int_{t_0}^{\tau_1} \dots \int_{t_0}^{\tau_{N-1}} X_{\tau_N} X_{\tau_{N-1}} \dots X_{\tau_1} \phi(x) d\tau_N d\tau_{N-1} \dots d\tau_1 \tag{2}$$

and the series is absolutely convergent.

3 A computational procedure

We consider the following system of ODEs:

$$\frac{d}{dt} x(t) = f(t, x(t)), \ x(t_0) \in M_0, \tag{3}$$

where M_0 is a convex and compact subset of R^n. We assume that there exist $\sigma > 0$ and a convex and compact neighbourhood Ω of the set M_0 such that $f(t,\cdot)$ belongs to $\mathcal{V}_\Omega^\sigma$ for each t from $[t_0, \hat{T}]$. Trajectories are the absolutely continuous functions $x(t)$, $t \in [0, \hat{T}]$ satisfying (3) for almost every t in this interval.

Let $\varepsilon > 0$. First, using [7], Proposition 1 we can find a positive real T, $t_0 < T \leq \hat{T}$, such that for every point x_0 from M_0 the corresponding trajectory $x(\cdot, x_0) : [t_0, T] \to R^n$ starting from x_0 is well defined on $[t_0, T]$ and $x(t) \in \Omega$ for every t from $[t_0, T]$. Next, we show how to calculate a point $y \in \Omega$ such that $|x(T, x_0) - y| < \varepsilon$. We choose the points $t_0 < t_1 < \ldots < t_k = T$ such that

$$\frac{2n}{\sigma} \int_{t_i}^{t_{i+1}} \|f(s, \cdot)\|_\Omega^\sigma \, ds < \frac{1}{2}$$

for each $i = 0, \ldots, k-1$. Let $\alpha > 1$ be so large that

$$\frac{1}{k^\alpha} \exp\left(\frac{4n^2}{\sigma} \int_{t_0}^{T} \|f(s, \cdot)\|_\Omega^\sigma \, ds \right) < \varepsilon.$$

At the end we determine the accuracy of the local approximation by choosing the positive integer ω to be so large that the following inequality holds true

$$\frac{n}{\omega} \left(\frac{2n}{\sigma} \right)^{\omega - 1} \left(\int_{t_i}^{t_{i+1}} \|f(s, \cdot)\|_\Omega^\sigma \, ds \right)^\omega < \frac{1}{3k^{\alpha+1}}.$$

We set $y_0 := x_0$ and

$$y_{i+1} := y_i \ + \sum_{N=1}^{\omega-1} \int_{t_i}^{t_{i+1}} \int_{t_i}^{\tau_1} \ldots \int_{t_i}^{\tau_{N-1}} f(\tau_N, \cdot) f(\tau_{N-1}, \cdot) \ldots f(\tau_1, \cdot) E(y_i) d\tau_N d\tau_{N-1} \ldots d\tau_1$$

for every $i = 0, 1, \ldots, k-1$. Since the trajectory $x(\cdot, x_0) : [t_0, T] \to R^n$ is well defined on $[0, T]$ and belongs to Ω, we obtain according to [7], Proposition 2 that $|x(T, x_0) - y_k| < \varepsilon$.

Our choice of the points $t_0, t_1, \ldots, t_k$, of $\alpha > 0$ and of $\omega > 0$ guarantee the needed accuracy of the numerical approximation. But, the used estimates are not precise enough and in practice the obtained accuracy is much better than the expected one (cf. the examples from the next section).

4 Examples

We consider some examples illustrating the applicability of the proposed numerical approach [7]. First, this approach is suitable for solving ODEs and systems of ODEs.

Example 1. Let us consider the following ODE:

$$\dot{x} = \gamma t^{\gamma-1} x^2, \quad x(0) = x_0 > 0,$$

where $\gamma \in (0, 1]$. For $T \geq x_1 = x_0^{-\frac{1}{\gamma}}$ it can be seen that there does not exist a compact set M_1 satisfying the assumptions of [7], Proposition 1. Hence, we can not conclude that the solution of this ODE exists on $[0, T]$. In fact, this ODE has no solution in this interval. But, if we choose an arbitrary T, such that $0 \leq T < x_1$, the solution exists. By setting $f(t, x) := \gamma t^{\gamma-1} x^2$, it can be directly calculated that

$$\int_0^t \int_0^{\tau_1} \ldots \int_0^{\tau_{k-1}} f(\tau_k, \cdot) f(\tau_{k-1}, \cdot) \ldots f(\tau_1, \cdot) E(x_0) d\tau_k d\tau_{k-1} \ldots d\tau_1 = x_0^{k+1} t^{k\gamma}.$$

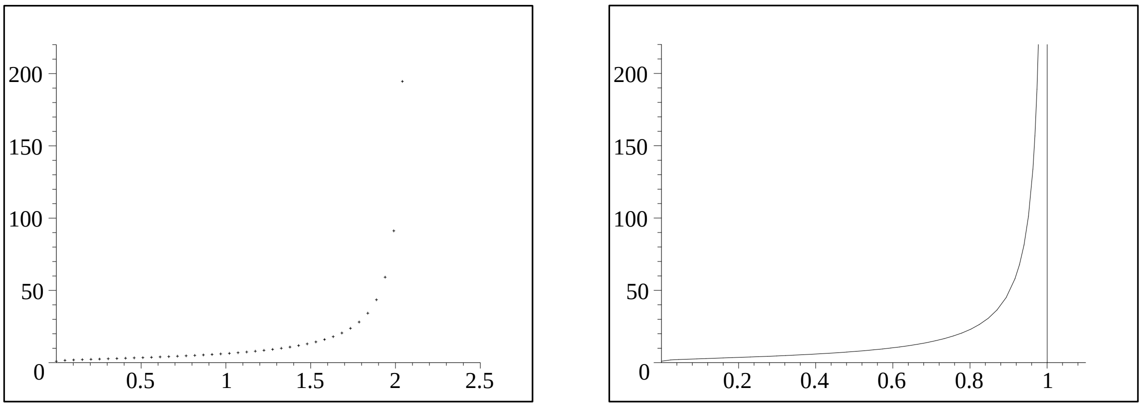


Figure 1: A Runge-Kutta approximation and the true solution

Hence, the series (2) for this ODE is:

$$x_0 + x_0^2 t^\gamma + x_0^3 t^{2\gamma} + \cdots + x_0^{k+1} t^{k\gamma} + \cdots,$$

which is convergent and tends to its analytic solution

$$x(t) = \frac{x_0}{1 - x_0 t^\gamma}, \quad t \in [0, x_1).$$

We made some numerical experiments with MAPLE for $\gamma = 0.2$ and $x_0 = 1$. We used `dsolve` command with options numeric and `method=dverk78` that finds a numerical solution using a seventh-eighth order continuous Runge-Kutta method (Fig. 1, left). This method calculates a solution in the interval $[0, T)$ for $T > 2$ while the analytic solution $x(t)$ tends to infinity for t tends to $T = 1$ (Fig. 1, right).

The proposed numerical approach is used for approximations of reachable sets of a nonlinear control system.

Example 2. [1] Let us consider the following system of ODEs:

$$\left| \begin{array}{lll} \dot{x}_1 = u, & |u| \le 1, & x_1(0) = 0, \\ \dot{x}_2 = x_1 + \dfrac{1}{2} x_1^2, & & x_2(0) = 0. \end{array} \right.$$

The Volterra series of this system is finite, it has four non zero terms. It follows that for each integrable control function u with values from the interval $[-1, 1]$, we can compute the exact single trajectory. The most complicated task is the choice of control sets such that the trajectory endpoints could outline the reachable set. The reachable set $R(t)$ is called the set of all point which can be steered from the origin within the fixed time t.

The boundary of $R(t)$ can be derived (cf. [1]) by using switching curves, corresponding to the bang-bang values of the control ($u = \pm 1$) and singular arcs (Fig. 2).

We obtained the boundary of $R(6)$ as follows. The points from the upper boundary, and the points from the lower boundary in the interval $[4, 6]$, correspond to piecewise constant control functions with only one jump and values ± 1, i.e.

$$u_p(t) := \begin{cases} e, & t \in [0, p), \\ -e, & t \in [p, 6), \end{cases} \tag{4}$$

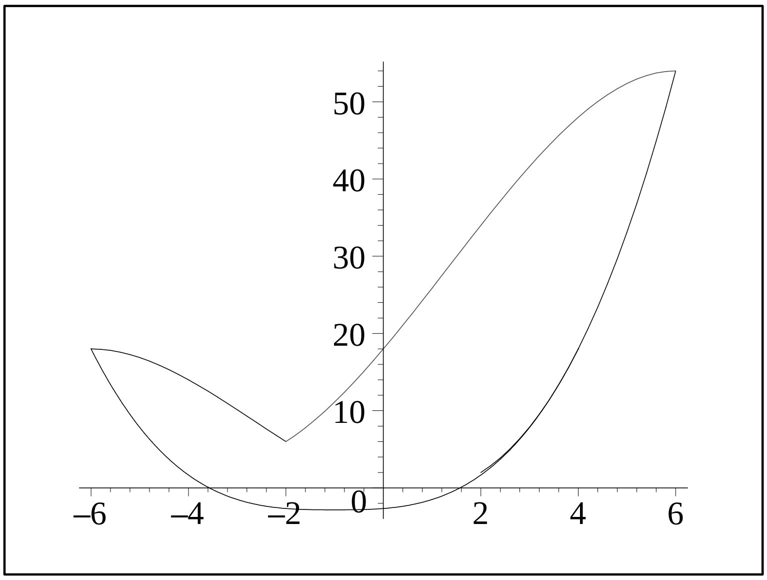


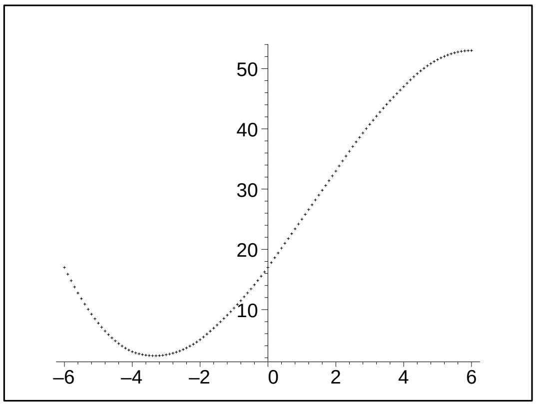
Figure 2: The boundary of the set $R(6)$

where $e \in \{-1, 1\}$ (Fig. 3).

The most points from the lower boundary of the reachable set are computed by the following family of control functions:

$$u_{p,e}(t) := \begin{cases} -1, & t \in [0, 1), \\ 0, & t \in [1, p), \\ e, & t \in [p, 6), \end{cases} \tag{5}$$

where $e \in \{-1, 1\}$ and $p \in (1, 6]$ (Fig. 4). The true reachable set $R(6)$ and its approximation of second order, with time-step 1 and step 0.05 for p, are shown on Fig. 5.



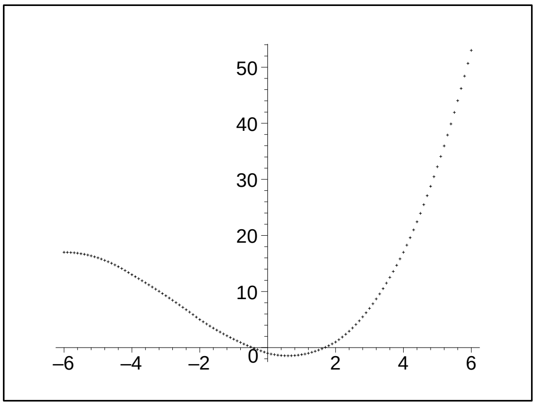


Figure 3: Trajectory endpoints for (4) and $e \equiv 1$ (left) and $e \equiv -1$ (right)

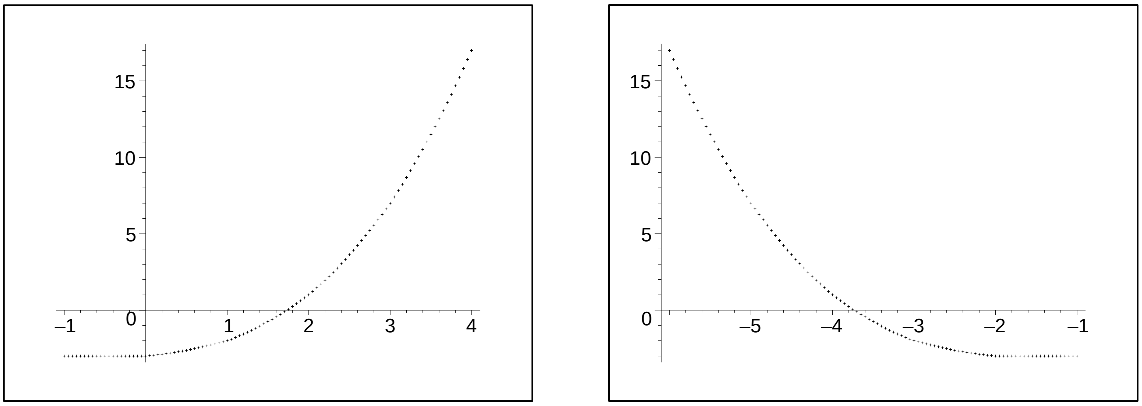


Figure 4: Trajectory endpoints for (5) and $e \equiv 1$ (left) and $e \equiv -1$ (right)

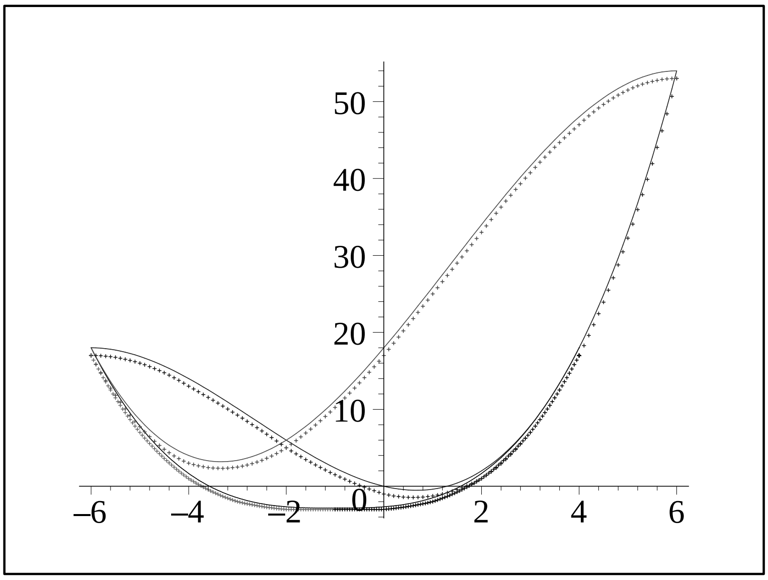


Figure 5: The true reachable set (—) and second order approximation $(\cdots)$

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