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Pliska Stud. Math. Bulgar. **15** (2003), 105–130

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ABOUT STRICTLY HYPERBOLIC OPERATORS WITH NON-REGULAR COEFFICIENTS

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1. Introduction

In this note we are interested in the Cauchy problem

$$
\begin{aligned}
(1.1) \qquad & u_{tt} - \sum_{k,l=1}^{n} a_{kl}(t,x) u_{x_k x_l} = 0 \quad \text{in} \quad [0,T] \times \mathbb{R}^n, \\
& u(0,x) = \varphi(x), u_t(0,x) = \psi(x) \ .
\end{aligned}
$$

Setting $a(t,x,\xi) := \sum\limits_{k,l=1}^{n} a_{kl}(t,x)\xi_k \xi_l$ we suppose with a positive constant C the strict hyperbolicity assumption

$$
(1.2) \qquad a(t,x,\xi) \geq C|\xi|^2
$$

with $a_{kl} = a_{lk},\ k,l = 1, \cdots, n$.

As usually we will say that the Cauchy problem (1.1) is well- posed *if we can fix function spaces A_1, A_2 for the data φ, ψ in such a way that there exists a uniquely determined solution $u \in C([0,T];B_1) \cap C^1([0,T];B_2)$ possessing the finite propagation speed.*

The question we will discuss in this paper is *how the regularity of the coefficients a_{kl} is related to the well-posedness of the Cauchy problem (1.1).*

2002 *Mathematics Subject Classification:* 35L15, 35L80, 35S05, 35S30

Key words: strictly hyperbolic Cauchy problems, non-Lipschitz coefficients in time, classes of well-posedness, construction of parametrix, refined perfect diagonalization procedure, regularization techniques, sharp Gårding's inequality

Our starting point is a well-known result (see [18]) which says that if the coefficients $a_{kl} \in C([0,T];\mathcal{B}^k) \cap C^1([0,T];\mathcal{B}^0)$ and $\varphi \in H^{k+1}$, $\psi \in H^k$, then there exists a uniquely determined solution $u = u(t,x)$ belonging to $C([0,T];H^{k+1}) \cap C^1([0,T];H^k)$. By $\mathcal{B}^\infty = \mathcal{B}^\infty(\mathbb{R}^n)$ we will later denote the space of infinitely differentiable functions having bounded derivatives on $\mathbb{R}^n$. Its topology is generated by the family of norms of spaces $\mathcal{B}^k = \mathcal{B}^k(\mathbb{R}^n)$, $k \geq 0$, consisting of functions with bounded derivatives up to order k. By $H^k = H^k(\mathbb{R}^n)$, $k \geq 0$, we denote as usually the Sobolev spaces having square integrable Sobolev derivatives up to order k. In a standard way we introduce $H^\infty = H^\infty(\mathbb{R}^n)$ and $H^k = H^k(\mathbb{R}^n)$ for negative k.

Standard arguments:

• If the coefficients have more regularity $C^1([0,T],\mathcal{B}^\infty)$, and the data φ and ψ are from H^∞, then the Cauchy problem is H^∞ well-posed, that is, the uniquely determined solution is from $C^2([0,T],H^\infty)$.

• Together with the domain of dependence property this result implies C^∞ well-posedness, that is, to arbitrary data φ and ψ from C^∞ there exists a uniquely determined solution from $C^2([0,T],C^\infty)$.

Remark 1. One can *weaken* in the above statements the C^1 *property with respect to* t *to the Lip property.*

The weakest property with respect to t to get well-posedness results for (1.1) is the L_1-property. This was shown in [9]. More precisely, it is assumed in [9] for (1.1) with an elliptic operator in self-adjoint form that the coefficients satisfy the analyticity condition

$$|\sum_{k,l=1}^{n} \partial_x^\beta a_{kl}(t,x)| \leq \Lambda_K(t) A_K^{|\beta|} |\beta|!$$

for all multi-indices β and for any compact set K of an open set Ω, where $\Lambda_K \in L_1(0,T)$, and the strict hyperbolicity condition

$$C|\xi|^2 \leq a(t,x,\xi) \leq \Lambda(t)|\xi|^2,$$

with a positive C and $\sqrt{\Lambda(t)} \in L_1(0,T)$. Then to arbitrary analytic data φ and ψ on Ω, there exists a unique solution on a suitable conoid Γ_Ω^T which is C^1 in t and analytic in x.

We know from the results of [7] for the Cauchy problem

$$u_{tt} - a(t)u_{xx} = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x), \tag{1.3}$$

that assumptions like $a \in L_p(0,T), p > 1$, or even $a \in C[0,T]$, don't allow to weaken the analyticity assumption for data φ, ψ to get well-posedness results. It is proved under these conditions that there is no well-posedness result in quasi-analytic or non quasi- analytic spaces.

Thus it is natural to assume that the coefficients are Hölder with respect to t. That such an assumption allows to weaken the regularity of coefficients with respect to x was shown in [19]. The main result from [19] for the strictly hyperbolic case applied to (1.1) tells us that if the coefficients $a_{kl} \in C^{\kappa}([0,T], G^s)$, $s < \frac{1}{1-\kappa}$, $\kappa \in (0,1)$, and the data φ, $\psi \in G^s$, then there exists a uniquely determined solution $u \in C^2([0,T], G^s)$. Here G^s is a suitable space of Gevrey functions defined on $\mathbb{R}^n$. In the case $s = \frac{1}{1-\kappa}$, the solution exists locally in t. The counter-examples from [3] show that we have no well-posedness in Gevrey classes G^s with $s > \frac{1}{1-\kappa}$.

In the following we are interested in the question if the C^1 or Lip property of coefficients can be weakened to guarantee well-posedness in Sobolev spaces.

2. Critical conditions for C^∞ well-posedness

Let us explain for model (1.3) different strategies how to weaken the conditions $a \in C^1[0,T]$, $a \in Lip[0,T]$, respectively, to guarantee well-posedness of Sobolev solutions.

A first idea goes back to [3]. Instead of the Lip property the authors supposed the so-called LogLip property, that is, the coefficient $a = a(t)$ satisfies the property

$$|a(t_1) - a(t_2)| \leq C|t_1 - t_2||\log|t_1 - t_2|| \quad \text{for all } t_1,\ t_2 \in [0,T],\ t_1 \neq t_2.$$

Under the assumption $a \in LogLip[0,T]$ a well-posedness result in C^∞ could be proved. But what is the qualitative difference of possible results to the case $a \in Lip[0,T]$?

• This difference we feel in the energy estimate. In general, one can derive an energy estimate of the type

$$E(u)|_{H^{s-s_0}}(t) \leq C(T)E(u)|_{H^s}(0) \text{ for all } t \in (0,T], \tag{2.1}$$

where $E(u)|_{H^s}(t_0)$ denotes the strictly hyperbolic energy basing on the H^s-norm to the time $t = t_0$. The constant s_0 describes the so-called *loss of derivatives.* The loss of derivatives shows how less regular the solution is in comparison with the data. It is clear, that such an estimate yields H^∞ well-posedness and together with the domain of dependence property, immediately, a C^∞ well-posedness result.

As far as the author knows there is no classification of LogLip behaviour of the coefficient $a = a(t)$ from (1.3) with respect to the related loss of derivatives. The author expects the following classification:

Let us suppose with a nonnegative γ that

$$|a(t_1) - a(t_2)| \le C|t_1 - t_2||\log|t_1 - t_2||^\gamma \tag{2.2}$$

for all t_1, $t_2 \in [0, T]$, $t_1 \neq t_2$.

Then it holds:

- If $\gamma = 0$, then $s_0 = 0$.
- If $\gamma \in (0, 1)$, then s_0 is an arbitrary small positive constant.
- If $\gamma = 1$, then s_0 is a positive constant.
- If $\gamma > 1$, then there doesn't exist a positive constant s_0 satisfying (2.1), that is, we have an *infinite loss of derivatives*.

The statement for $\gamma = 0$ can be found in [18]. The counter-example from [8] implies the statement for $\gamma > 1$.

Open Problem 1: Prove the above statement for $\gamma \in (0, 1)$!

Open Problem 2: The results of [3] and [8] show that $\gamma = 1$ brings a finite loss of derivatives. Are these results sharp? Do we have a concrete example which shows that the solution has a finite loss of derivatives?

We already cited the paper [8]. In this paper the authors studied strictly hyperbolic Cauchy problems with coefficients in principal part depending LogLip on spatial and time variable.

- If the principal part is as in (1.1) but with an elliptic operator in divergence form, then the authors derive energy estimates basing on a suitable low energy of the data and of the right-hand side.
- If the principal part is as in (1.1) but with coefficients which are C^∞ in x and LogLip in t, then the energy estimates are basing on arbitrary high energy of the data and of the right-hand side.
- In all these energy estimates which exist for $t \in [0, T^*]$, where T^* is a suitable positive constant independent of the regularity of the data and right-hand side, the loss of derivatives depends on t.

It is clear that these energy estimates are an important tool to prove (locally in t) well-posedness results.

A second possibility to weaken the Lip behavior goes back to [5]. Under the assumptions

$$a \in C[0,T] \cap C^1(0,T], |ta'(t)| \leq C \text{ for } t \in (0,T], \tag{2.3}$$

the authors proved a C^∞ well-posedness result for (1.3) (even for more general equations), where they observed the effect of a *finite loss of derivatives*, too. The condition for the derivative of a from (2.3) is sharp in the following sense:
If we suppose

$$a \in C[0,T] \cap C^1(0,T], |t^\gamma a'(t)| \leq C \text{ for } t \in (0,T], \tag{2.4}$$

then for
- $\gamma \in [0,1)$ we have no loss of derivatives,
- $\gamma > 1$ we have an infinite loss of derivatives, the authors of [5] proved well-posedness in Gevrey classes.

Remark 2. Let us compare both strategies (2.2) and (2.4). In (2.2) the coefficient $a = a(t)$ can have an irregular behavior (in comparison with the Lip property) on the whole interval $[0,T]$. In (2.4) the coefficient has only an irregular behavior in $t = 0$, in $(0,T]$ the coefficient belongs to C^1. But the behavior of the derivative of a in $t = 0$ is more singular than the LogLip behavior in $t = 0$. Consequently, each approach to attack one of these both irregularities will have its own peculiarities.

The following considerations are devoted to the second strategy. In the next section we will show how more regularity for $a = a(t)$ brings a more refined classification of possible oscillating behavior. In Section 4 we will study our starting model problem (1.1). Finally, we will present recent developments in the theory of strictly hyperbolic equations with non-Lipschitz coefficients in Section 5.

3. A refined classification of oscillating behavior

The considerations of this section are basing on the papers [6] and [11]. Let us suppose, that the coefficient $a = a(t)$ satisfies

$$a \in L_\infty[0,T] \cap C^2(0,T], \quad |a^{(k)}(t)| \leq C_k \left(\frac{1}{t}\left(\log\frac{1}{t}\right)^\gamma\right)^k, \; k = 0,1,2,$$

where $\gamma \geq 0$.

Definition 1. *We say, that the oscillating behavior of a is*
- *very slow if* $\gamma = 0$,
- *slow if* $\gamma \in (0,1)$,
- *fast if* $\gamma = 1$,
- *very fast if condition (3.1) is not satisfied for* $\gamma = 1$.

Then we are going to prove the next statement yielding a connection between the type of oscillations and the loss of derivatives which appears.

Theorem 1. *Let us consider*

$$u_{tt} - a(t)u_{xx} = 0, \quad u(0,x) = \varphi(x),\, u_t(0,x) = \psi(x),$$

where $a = a(t)$ *satisfies the condition (3.1) and the data* φ, ψ *belong to* H^{s+1}, H^s, *respectively. Then the following energy inequality holds:*

$$E(u)|_{H^{s-s_0}}(t) \leq C(T)E(u)|_{H^s}(0) \text{ for all } t \in (0,T], \tag{3.1}$$

where
- $s_0 = 0$ *if* $\gamma = 0$,
- s_0 *is an arbitrary small positive constant if* $\gamma \in (0,1)$,
- s_0 *is a positive constant if* $\gamma = 1$,
- *there doesn't exist a positive constant* s_0 *satisfying (3.1) if* $\gamma > 1$, *that is, we have an infinite loss of derivatives.*

Proof. Without loss of generality we can suppose that T is small enough. Otherwise, we derive the energy inequality (3.1) for $t \in (0,T_1]$ and use $C^1[T_1,T]$ property of $a = a(t)$ to get the estimate for $t \in (0,T]$. The proof will be divided into several steps.

1.step: Tools

We divide the phase space $\{(t,\xi) \in [0,T] \times \mathbb{R}\}$ into two zones by using the function $t = t_\xi$ which solves

$$t_\xi \langle \xi \rangle = N(\ln\langle \xi \rangle)^\gamma. \tag{3.2}$$

The constant N will be determined later. Then the *pseudo-differential zone* $Z_{pd}(N)$, *hyperbolic zone* $Z_{hyp}(N)$, respectively, is defined by

$$Z_{pd}(N) = \{(t,\xi) \in [0,T] \times \mathbb{R} : \; t \leq t_\xi\}\;, \tag{3.3}$$

$$Z_{hyp}(N) = \{(t,\xi) \in [0,T] \times \mathbb{R} : \; t \geq t_\xi\}\;. \tag{3.4}$$

Moreover, we need a symbol class in $Z_{hyp}(N)$.

Definition 2. *To given real numbers* m_1, m_2, $r \leq 2$; *we denote by*

$$S_r\{m_1, m_2\} = \Big\{ d = d(t,\xi) \in L_\infty([0,T] \times \mathbb{R}) :$$
$$|D_t^k D_\xi^\alpha d(t,\xi)| \leq C_{k,\alpha} \langle \xi \rangle^{m_1 - |\alpha|} \big(\tfrac{1}{t}(\log \tfrac{1}{t})^\gamma\big)^{m_2+k},\ k \leq r,\ (t,\xi) \in Z_{hyp}(N) \Big\}.$$

2.step: Considerations in $Z_{pd}(N)$

We apply partial Fourier transformation with respect to x and get with $v = \hat{u}$

$$v_{tt} + a(t)\xi^2 v = 0\ , \quad v(0,\xi) = \hat{\varphi}(\xi),\ v_t(0,\xi) = \hat{\psi}(\xi).$$

Setting $V = (\xi v, D_t v)^T$ this equation can be transformed to the system of first order

$$D_t V = \begin{pmatrix} 0 & \xi \\ a(t)\xi & 0 \end{pmatrix} V =: A(t,\xi) V\ . \tag{3.5}$$

We are interested in the fundamental solution to the Cauchy problem, this is the matrix-valued solution $\mathcal{V} = \mathcal{V}(t,r,\xi)$ of this system with initial condition $\mathcal{V}(r,r,\xi) = I$ (identity matrix). Using the matrizant we can write $\mathcal{V}$ in an explicit way by

$$\mathcal{V}(t,r,\xi) = I + \sum_{k=1}^{\infty} i^k \int_r^t A(t_1,\xi) \int_r^{t_1} A(t_2,\xi) ... \int_r^{t_{k-1}} A(t_k,\xi) dt_k ... dt_1.$$

The norm $\|A(t,\xi)\|$ can be estimated by $C\langle\xi\rangle$. Consequently,

$$\int_0^{t_\xi} \|A(s,\xi)\| ds \leq C t_\xi \langle \xi \rangle\ .$$

We deduce for $t \in [0, t_\xi]$ the estimate

$$\|\mathcal{V}(t,0,\xi)\| \leq \exp\Big(\int_0^t \|A(s,\xi)\| ds\Big) \leq \exp(CN(\ln\langle\xi\rangle)^\gamma).$$

Lemma 1. *The solution to (3.5) satisfies in* $Z_{pd}(N)$ *for large* ξ *the energy estimate*

$$|V(t,\xi)| \leq \exp(CN(\ln\langle\xi\rangle)^\gamma)|V(0,\xi)|. \tag{3.6}$$

3.step: Considerations in $Z_{hyp}(N)$

Substituting $V = (\sqrt{a(t)}\xi\, v,\ D_t v)^T$ brings the system of first order

$$D_t V - \begin{pmatrix} \frac{D_t a}{2a} & \sqrt{a}\xi \\ \sqrt{a}\xi & 0 \end{pmatrix} V = 0\,. \tag{3.7}$$

To derive an energy estimate we carry out the first two steps of a diagonalization procedure. Here we use the $C^2(0,T]$ property of $a = a(t)$.

• *Diagonalization of the principal part*
Let us define the matrices

$$M = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then the system (3.7) can be transformed to a first order system for a new function $V_0 := M\,V$ in the following way:

$$D_t V_0 - M\begin{pmatrix} 0 & \sqrt{a}\xi \\ \sqrt{a}\xi & 0 \end{pmatrix} M^{-1} V_0 - M\begin{pmatrix} \frac{D_t a}{2a} & 0 \\ 0 & 0 \end{pmatrix} M^{-1} V_0 = 0\,,$$

$$D_t V_0 - \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} V_0 - \frac{1}{2}\begin{pmatrix} 0 & \frac{D_t a}{2a} \\ \frac{D_t a}{2a} & 0 \end{pmatrix} V_0 = 0\,,$$

where

$$\tau_1(t,\xi) := -\sqrt{a(t)}\xi + \tfrac{1}{2}\,\tfrac{D_t a}{2a}\ ,\ \tau_2(t,\xi) := \sqrt{a(t)}\xi + \tfrac{1}{2}\tfrac{D_t a}{2a}\ .$$

We deduce $\sqrt{a}\xi \in S_2\{1,0\}$ and $\frac{D_t a}{2a} \in S_1\{0,1\}$. In $Z_{hyp}(N)$ we have $S_{k+1}\{1,0\} \subset S_k\{1,0\}$ and $S_k\{0,1\} \subset S_k\{1,0\}$. Consequently, substituting $V_0 := MV$ in (3.7) we obtain the following system of first order:

$$D_t V_0 - \mathcal{D} V_0 - R_0 V_0 = 0\,, \tag{3.8}$$

where

$$\mathcal{D} := \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \in S_1\{1,0\}\ ;\ R_0 := \frac{1}{2}\begin{pmatrix} 0 & \frac{D_t a}{2a} \\ \frac{D_t a}{2a} & 0 \end{pmatrix} \in S_1\{0,1\}\,.$$

• *Diagonalization of the remainder R_0 modulo $S_0\{-1,2\}$*
Let us set

$$\mathcal{N}^{(1)} := -\frac{1}{2}\begin{pmatrix} 0 & \frac{\frac{D_t a}{2a}}{\tau_1 - \tau_2} \\ \frac{\frac{D_t a}{2a}}{\tau_2 - \tau_1} & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & \frac{\frac{D_t a}{2a}}{\sqrt{a}\xi} \\ \frac{\frac{D_t a}{2a}}{-\sqrt{a}\xi} & 0 \end{pmatrix}.$$

Then the matrix $N_1 := I + \mathcal{N}^{(1)}$ is invertible in $Z_{hyp}(N)$ for sufficiently large N. We observe that on the one hand $\mathcal{D}N_1 - N_1\mathcal{D} = R_0$ and on the other hand

$$(D_t - \mathcal{D} - R_0)N_1 = N_1(D_t - \mathcal{D} - R_1),$$

where

$$R_1 := -N_1^{-1}(D_t\mathcal{N}^{(1)} - R_0\mathcal{N}^{(1)}).$$

Taking account of Definition 2 we have $\mathcal{N}^{(1)} \in S_1\{-1,1\}$, $N_1 \in S_1\{0,0\}$ and $R_1 \in S_0\{-1,2\}$. Setting $V_1 := N_1^{-1}V_0$ in (3.8) gives the following first order system:

$$D_tV_1 - \mathcal{D}V_1 - R_1V_1 = 0.$$

- *Representation of solution*

Now let us devote to the Cauchy problem

$$D_tV_1 - \mathcal{D}V_1 - R_1V_1 = 0,\ V_1(t_\xi, \xi) = V_{1,0}(\xi)\ . \tag{3.9}$$

The datum $V_{1,0}$ arises from the solution $V = V(t,\xi)$ in $Z_{pd}(N)$ on $t = t_\xi$ and the above transformations, that is,

$$V_{1,0}(\xi) = N_1^{-1}(t_\xi, \xi)MV(t_\xi, \xi). \tag{3.10}$$

Conversely, if we have a solution $V_1 = V_1(t,\xi)$ in $Z_{hyp}(N)$, then $V = V(t,\xi)$ which is defined by

$$V(t,\xi) = M^{-1}N_1(t,\xi)V_1(t,\xi) \tag{3.11}$$

solves (3.7) with given $V(t_\xi,\xi)$ on $t = t_\xi$.

The matrix-valued function

$$E_2(t,r,\xi) := \begin{pmatrix} \exp\Big(i\int_r^t \Big(-\sqrt{a(s)}\xi + \frac{1}{4}\frac{D_sa(s)}{a(s)}\Big)ds\Big) & 0 \\ 0 & \exp\Big(i\int_r^t \Big(\sqrt{a(s)}\xi + \frac{1}{4}\frac{D_sa(s)}{a(s)}\Big)ds\Big) \end{pmatrix}$$

solves the Cauchy problem

$$(D_t - \mathcal{D})E(t,r,\xi) = 0\ ,\ E(r,r,\xi) = I\ .$$

The matrix-valued function $H = H(t,r,\xi)$ is defined by

$$H(t,r,\xi) := E_2(r,t,\xi)R_1(t,\xi)E_2(t,r,\xi)\ ,\ t,r \geq t_\xi\ ,$$

satisfies in $Z_{hyp}(N)$ the estimate

$$\|H(t,r,\xi)\| \leq C\langle\xi\rangle^{-1}\Big(\tfrac{1}{t}(\ln\tfrac{1}{t})^\gamma\Big)^2. \tag{3.12}$$

By the aid of H we define the matrix-valued function $Q = Q(t,r,\xi)$ by

$$Q(t,r,\xi) := \sum_{k=1}^{\infty} i^k \int_r^t H(t,r,\xi)dt_1 \int_r^{t_1} H(t_2,r,\xi)dt_2 ... \int_r^{t_{k-1}} H(t_k,r,\xi)dt_k\ .$$

The reason for introducing the function Q is that

$$V_1 = V_1(t,\xi) := E_2(t,t_\xi,\xi)(I + Q(t,t_\xi,\xi))V_{1,0}(\xi) \tag{3.13}$$

represents a solution to (3.9).

• *Basic estimate in $Z_{hyp}(N)$*
Using (3.12) and the estimate

$$\int_{t_\xi}^t \|H(s,t_\xi,\xi)\|ds \leq C_N(\ln\langle\xi\rangle)^\gamma$$

we get from the representation for Q immediately

$$\|Q(t,t_\xi,\xi)\| \leq \exp(\int_{t_\xi}^t \|H(s,t_\xi,\xi)\|ds) \leq \exp(C_N(\ln\langle\xi\rangle)^\gamma).$$

Now we are in position to conclude a basic estimate for V taking account of (3.10), (3.11) and (3.13).

Lemma 2. *The solution to (3.7) satisfies in $Z_{hyp}(N)$ for large ξ the energy estimate*

$$|V(t,\xi)| \leq C\exp(C_N(\ln\langle\xi\rangle)^\gamma)|V(t_\xi,\xi)|. \tag{3.14}$$

4.step: Conclusion

From Lemmas 1 and 2 we conclude

Lemma 3. *The solution $v = v(t,\xi)$ to*

$$v_{tt} + a(t)\xi^2 v = 0, \quad v(0,\xi) = \hat{\varphi}(\xi),\ v_t(0,\xi) = \hat{\psi}(\xi)$$

satisfies the a-priori estimate

$$\Big|\begin{pmatrix} \xi v \\ v_t \end{pmatrix}\Big| \le C \exp(C_N (\ln\langle\xi\rangle)^\gamma) \Big|\begin{pmatrix} \xi\, \hat{\varphi}(\xi) \\ \hat{\psi}(\xi) \end{pmatrix}\Big|$$

for all $(t,\xi) \in [0,T] \times \mathbb{R}$.

This estimate proves the statements for $\gamma \in [0,1]$. The statement for $\gamma > 1$ follows from Theorem 2 which we formulate after this proof if we choose $\omega(t) = \ln^q \frac{C(q)}{t}$ with $q \ge 2$. $\square$

Theorem 2 (see [6]) *Let $\omega : (0,1/2] \to (0,\infty)$ be a continuous, decreasing function satisfying* $\lim \omega(s) = \infty$ *for $s \to +0$ and $\omega(s/2) \le c\omega(s)$ for all $s \in (0,1/2]$. Then there exists a function $a \in C^\infty(\mathbb{R} \setminus \{0\}) \cap C^0(\mathbb{R})$ with the following properties:*
- *$1/2 \le a(t) \le 3/2$ for all $t \in \mathbb{R}$;*
- *there exists a suitable positive t_0 and to every p a positive constant C_p such that*

$$|a^{(p)}(t)| \le C_p \omega(t) \Big(\frac{1}{t} \ln \frac{1}{t}\Big)^p \ \textit{for all } 0 < t < t_0;$$

- *there exist two functions φ and ψ from $C^\infty(\mathbb{R})$ such that the Cauchy problem*

$$u_{tt} - a(t)u_{xx} = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x)$$

has no solution in $C^0([0,r]; \mathcal{D}'(\mathbb{R}))$ for all $r > 0$.

Remark 3. If we would stop the diagonalization procedure after diagonalization of principal part, then we have to assume in Theorem 1 only $a \in C[0,T] \cap C^1(0,T]$ and $|ta'(t)| \le C$ in correspondence to [5].

Remark 4. The counter-examples from [6] possess the regularity $a \in C^\infty(\mathbb{R} \setminus \{0\})$. Thus it was valuable to have a counter-example from [11] with regularity $a \in C^2(\mathbb{R} \setminus \{0\}$. This counterexample was obtained by the application of Floquet theorem influenced the research of [12] (see Section 4.2 and the following open problems).

Open Problem 3: In the moment it seems to be not clear what kind of oscillations do we have if $a \in C[0,T] \cap C^1(0,T]$ and $|ta'(t)| \le C$ as in [5]. The problem is to prove that under the assumptions

$$a \in C[0,T] \cap C^1(0,T],\ |a'(t)| \le C\frac{1}{t}\Big(\ln\frac{1}{t}\Big)^{\gamma},\ \gamma > 0,$$

we have very fast oscillations, this means, we cannot expect C^∞ well-posedness. To study this problem we have to use in a right way the low regularity $C^1(0,T]$.

Remark 5. Let us consider the Cauchy problem

$$u_{tt} + b(t)u_{xt} - a(t)u_{xx} = 0, \quad u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x).$$

Does the existence of a mixed derivative of second order change the classification of oscillations from Definition 1? In a very recent paper [13] the authors gave a positive answer. The method how to get this answer is connected with the proof of necessity of Levi conditions for lower order terms. Its answer is important for quasi-linear equations of second (or even higher order) with non-Lipschitz behavior. From the results of [1] we know that $a, b \in \mathit{LogLip}[0,T]$ implies C^∞ well-posedness of the above Cauchy problem, that is, in this case the mixed derivatives have no deteriorating influence.

4. Solvability in Sobolev spaces for the general case

4.1. Construction of parametrix

In this section we come back to our general Cauchy problem (1.1) taking account of the classification of oscillations was supposed in Definition 1 and (3.1). We assume

$$a_{kl} \in C([0,T], \mathcal{B}^\infty(\mathbb{R}^n)) \cap C^\infty((0,T], \mathcal{B}^\infty(\mathbb{R}^n)). \tag{4.1}$$

The non-Lipschitz behavior of coefficients is characterized by

$$|D_t^k D_x^\beta a_{kl}(t,x)| \le C_{k,\beta}\left(\tfrac{1}{t}\left(\ln\ \tfrac{1}{t}\right)^{\gamma}\right)^k \tag{4.2}$$

for all k, β and $(t,x) \in (0,T] \times \mathbb{R}^n$, where T is sufficiently small and $\gamma \ge 0$. The transformation $U = (\langle D_x\rangle u, D_t u)^T$ transfers our starting Cauchy problem (1.1) to a Cauchy problem for $D_t U - AU = F$, where $A = A(t,x,D_x)$ is a matrix-valued pseudo-differential operator. The goal of this section is the construction of parametrix to $D_t - A$.

Definition 3. *An operator $E = E(t,s)$, $0 \le s \le t \le T_0$, is said to be a parametrix to the operator $D_t - A$ if $D_tE - AE \in L_\infty([0,T_0]^2, \Psi^{-\infty}(\mathbb{R}^n))$. Here $\Psi^{-\infty}$ denotes the classical pseudo-differential operators with symbols from $S^{-\infty}$ (see [17]).*

We will prove that E is a matrix Fourier integral operator. The considerations of this section are basing on [15], where the case $\gamma = 1$ was studied, and on [20]. We will sketch this construction of parametrix and show how the different loss of derivatives appears. It is more or less standard to get from the parametrix the existence of C^1 solutions in t of (1.1) with values in Sobolev spaces.

1.step: Tools

With the function $t = t_\xi$ from (3.2) we define for $\langle \xi \rangle \ge M$ the *pseudo-differential zone* $Z_{pd}(N)$, *hyperbolic zone* $Z_{hyp}(N)$, respectively, by

$$Z_{pd}(N) = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n} : \ t \le t_\xi\} \ , \tag{4.3}$$

$$Z_{hyp}(N) = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n} : \ t \ge t_\xi\} \ . \tag{4.4}$$

Moreover, we divide $Z_{hyp}(N)$ into the so-called *oscillations subzone* $Z_{osc}(N)$ and the *regular subzone* $Z_{reg}(N)$. These subzones are defined by

$$Z_{osc}(N) = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n} : \ t_\xi \le t \le \tilde{t}_\xi\} \ , \tag{4.5}$$

$$Z_{reg}(N) = \{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2n} : \ \tilde{t}_\xi \le t\} \ , \tag{4.6}$$

where $t = \tilde{t}_\xi$ solves

$$\tilde{t}_\xi \langle \xi \rangle = 2N(\ln\langle \xi \rangle)^{2\gamma}. \tag{4.7}$$

In each of these zones we define its own class of symbols.

Definition 4. *By T_{2N} we denote the class of all amplitudes $a = a(t,x,\xi) \in L_\infty([0,T], C^\infty(\mathbb{R}^{2n}))$ satisfying for $(t,x,\xi) \in Z_{pd}(2N)$ and all α, β the estimates*

$$\operatorname*{ess\,sup}_{(t,x)\in[0,t_\xi]\times\mathbb{R}^n} |\partial_x^\beta \partial_\xi^\alpha a(t,x,\xi)| \le C_{\beta\alpha} \langle \xi \rangle^{1-|\alpha|}. \tag{4.8}$$

By $S^m_{\rho,\delta}(\mathbb{R}^n)$ we will denote the classical symbol spaces (see [17]).
To describe the behavior in oscillations subzone $Z_{osc}(N)$ we need the next class of symbols.

Definition 5. *By $S_N\{m_1, m_2\}$, $m_2 \geq 0$, we denote the class of all amplitudes $a = a(t, x, \xi) \in C^\infty((0, T] \times \mathbb{R}^{2n})$ satisfying*

$$|\partial_t^k \partial_x^\beta \partial_\xi^\alpha a(t, x, \xi)| \leq C_{k\beta\alpha} \langle\xi\rangle^{m_1 - |\alpha|} \left(\tfrac{1}{t}(\ln \tfrac{1}{t})^\gamma\right)^{m_2 + k} \tag{4.9}$$

for all k, α, β and $(t, x, \xi) \in Z_{hyp}(N)$.

Finally, we use symbols describing the behavior of solution in the regular part $Z_{reg}(N)$ of $Z_{hyp}(N)$.

Definition 6. *By $S_N^\star\{m_1, m_2\}$, $m_2 \geq 0$, we denote the class of all amplitudes $a = a(t, x, \xi) \in C^\infty((0, T] \times \mathbb{R}^{2n})$ satisfying*

$$|\partial_t^k \partial_x^\beta \partial_\xi^\alpha a(t, x, \xi)| \leq C_{k\beta\alpha} \langle\xi\rangle^{m_1 - |\alpha|} \left(\tfrac{1}{t}(\ln \tfrac{1}{t})^\gamma\right)^{m_2 + k} \tag{4.10}$$

for all k, α, β and $(t, x, \xi) \in Z_{reg}(N)$.

To all these symbol classes one can define corresponding pseudo-differential operators. To get a calculus for these symbol classes it is useful to know that under assumptions to the behavior of the symbols in $Z_{pd}(N)$ we have relations to classical parameter dependent symbol classes.

Lemma 4. *Assume that the symbol $a \in S_N\{m_1, m_2\}$ is constant in $Z_{pd}(N)$. Then*

$$\begin{aligned} &a \in L_\infty([0, T],\ S_{1,0}^{\max(0, m_1 + m_2)}(\mathbb{R}^n)) \\ &\partial_t^k a \in L_\infty([0, T],\ S_{1,0}^{m_1 + m_2 + k}(\mathbb{R}^n)) \end{aligned} \tag{4.11}$$

for all $k \geq 1$.

The statements (4.11) allow to apply the standard rules of classical symbolic calculus. One can show

- a hierarchy of symbol classes $S_N\{m_{1,k}, m_2\}$ for $m_{1,k} \to -\infty$;
- a hierarchy of symbol classes $S_N\{m_1 - k, m_2 + k\}$ for $k \geq 0$;
- asymptotic representations of symbols vanishing in $Z_{pd}(N)$ by using these hierarchies;
- a composition formula of pseudo-differential operators which symbols are constant in $Z_{pd}(N)$;
- the existence of parametrix to elliptic matrix pseudo- differential operators belonging to $S_N\{0, 0\}$ and which are constant in $Z_{pd}(N)$.

2.step: Diagonalization procedure

We have to carry out perfect diagonalization. The main problem is to understand what does the perfect diagonalization procedure mean. Here we follow the next strategy:

- The first step we carry out in all zones.
- The second step we carry out only in $Z_{hyp}(N)$.
- The perfect diagonalization we carry out in $Z_{reg}(N)$.

Perfect diagonalization means diagonalization modulo $T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\}) \cap \Big\{ \bigcap_{r\geq 0} S^\star_{2N}\{-r, r+1\}\Big\}$.

Lemma 5. *The determination of parametrix to the matrix pseudo-differential operator $D_t - A$ can be reduced after transformations by elliptic matrix pseudo-differential operators (corresponds to perfect diagonalization) to the determination of parametrix to the matrix pseudo-differential operator $D_t - \mathcal{D} + F_2 + P_\infty$, where the matrix pseudo-differential operators $\mathcal{D}$, F_2, P_∞, possess the following properties:*

- $\mathcal{D}$: $\sigma(\mathcal{D}) \in T_{2N} \cap S_N\{1,0\}$,

$$\sigma(\mathcal{D}) = \begin{pmatrix} \varphi_1 + \frac{\langle\xi\rangle}{2\varphi_2} D_t \frac{\varphi_2}{\langle\xi\rangle} & 0 \\ 0 & \varphi_2 + \frac{\langle\xi\rangle}{2\varphi_2} D_t \frac{\varphi_2}{\langle\xi\rangle} \end{pmatrix};$$

- F_2: *diagonal,* $\sigma(F_2) \in (S^\star_N\{0,0\} + S^\star_N\{-1,2\})$, $\sigma(F_2) \equiv 0$ in $Z_{pd}(N) \cup Z_{osc}(N)$*;*

- P_∞: $\sigma(P_\infty) \in T_{2N} \cap (S_{2N}\{0,0\} + S_{2N}\{-1,2\}) \cap \Big\{ \bigcap_{p\geq 0} S^\star_{2N}\{-p, p+1\}\Big\}$.

Here we use

$$\varphi_k(t,x,\xi) = d_k\langle\xi\rangle\, \chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big) + \tau_k(t,x,\xi)\Big(1 - \chi\Big(\frac{t\langle\xi\rangle}{N(\ln\langle\xi\rangle)^\gamma}\Big)\Big),$$

where $d_2 = -d_1$ is a positive constant and

$$\tau_k(t,x,\xi) = (-1)^k\sqrt{a(t,x,\xi)}\,,\ a(t,x,\xi) := \sum_{k,l=1}^{n} a_{k,l}(t,x)\xi_k\xi_l.$$

The function $\chi = \chi(s)$ is from $C_0^\infty(\mathbb{R})$, $\chi(s) \equiv 1$ for $|s| \leq 1$, $\chi(s) \equiv 0$ for $|s| \geq 2$ and $0 \leq \chi(s) \leq 1$.

3.step: Construction of parametrix

We need four steps for the construction of parametrix.

• *Transformation by an elliptic pseudo-differential operator*
Let K be the diagonal elliptic pseudo-differential operator with symbol

$$\sigma(K) = \begin{pmatrix} \sqrt{\frac{\varphi_2}{\langle\xi\rangle}} & 0 \\ 0 & \sqrt{\frac{\varphi_2}{\langle\xi\rangle}} \end{pmatrix} .$$

This symbol is constant in $Z_{pd}(N)$, $\sigma(K) \in S_N\{0,0\}$. Then the following operator-valued identity holds modulo regularizing operator:

$$(D_t - \mathcal{D} + F_2)K = K(D_t - \mathcal{D}_1 + F_3), \tag{4.12}$$

where

$$\sigma(\mathcal{D}_1) := \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} , \; \sigma(F_3) \equiv 0 \;\text{ in }\; Z_{pd}(N),$$
$$\sigma(F_3) \in T_{2N} \cap (S_N\{0,0\} + S_N^\star\{-1,2\}).$$

Remark 6. This transformation corresponds to the special structure of our starting operator and explains that we have no contribution to the loss of derivatives from $\mathcal{D}$. This we already observed in Section 3 during the proof of Theorem 1. In the representation of V_1 from (3.13) there appears $E_2 = E_2(t, t_\xi, \xi)$. Although in E_2 there appears the term $\frac{1}{2}\frac{D_s a}{a}$ which belongs to $S_1\{0,1\}$ (see Definition 2), this term has no contribution to the loss of derivatives.

• *Parametrix to* $D_t - \mathcal{D}_1$
Lemma 6. *The parametrix* $E_2(t,s) = \operatorname{diag}(E_2^-(t,s), E_2^+(t,s))$ *to* $D_t - \mathcal{D}_1$ *is a diagonal Fourier integral operator with*

$$E_2^\mp(t,s)w(x) = \int\limits_{\mathbb{R}^n} e^{i\phi^\mp(t,s,x,\xi)} e_2^\mp(t,s,x,\xi)\hat{w}(\xi)d\xi \;,$$
$$\phi^\mp(s,s,x,\xi) = x\cdot\xi \;,\; e_2^\mp(s,s,x,\xi) = 1 \;.$$

The phase functions $\phi^\mp$ *satisfy*

- $\phi^\mp(t,s,x,\xi) = x\cdot\xi + d_k\langle\xi\rangle(t-s)$, $k = 1$ *for* ϕ^-, $k = 2$ *for* ϕ^+ *if* $0 \le s,t \le t_\xi$;
- $|\partial_\xi^\alpha \partial_x^\beta(\phi^\mp(t,s,x,\xi) - x\cdot\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{1-|\alpha|}\max(s,t)$ *if* $\max(s,t) \ge t_\xi$.

The amplitude functions $e_2^{\mp}$ satisfy

- $e_2^{\mp}(t,s,x,\xi) = 1$ *if* $0 \le s,t \le t_\xi$;
- $e_2^{\mp} \in C([0,T_0]^2, S^0_{1,0}(\mathbb{R}^n))$.

To prove this result we follow the next steps:
• Study of the Hamiltonian flow generated by $\varphi_1 = \varphi_1(t,x,\xi)$ and $\varphi_2 = \varphi_2(t,x,\xi)$.
• Construction of phase functions $\phi^{\mp}$ solving the eikonal equations.
• Construction of amplitudes $e_2^{\mp}$ by solving the transport equations and by using the asymptotic representation theorem.

• *Parametrix to $D_t - \mathcal{D}_1 + F_3$*
Lemma 7. *The parametrix $E_4 = E_4(t,s)$ to the operator $D_t - \mathcal{D}_1 + F_3$ can be written as $E_4(t,s) = E_2(t,s)Q_4(t,s)$, where $E_2 = E_2(t,s)$ is the diagonal Fourier integral operator from Lemma 6 and $Q_4 = Q_4(t,s)$ is a diagonal pseudo-differential operator with symbol belonging to $W^1_\infty([0,T_0]^2, S^0_{1,0}(\mathbb{R}^n))$.*

To prove this result we follow the next steps:
• Application of Egorov's theorem, that is, conjugation of F_3 by Fourier integral operators, here we use the diagonal structure.
• For $t \in [0,T_0]$ with a sufficiently small T_0 we understand to which zone the Hamiltonian flow belongs to.

• *Parametrix to $D_t - \mathcal{D} + F_2$*
Lemma 8. *The parametrix $E_3 = E_3(t,s)$ to the operator $D_t - \mathcal{D} + F_2$ can be written as $E_3(t,s) = K(t)E_2(t,s)Q_4(t,s)K^\sharp(s)$, where K and its parametrix $K^\sharp$ having symbols from $L_\infty([0,T_0], S^0_{1,0}(\mathbb{R}^n)) \cap C^\infty((0,T_0]^2, S^0_{1,0}(\mathbb{R}^n))$ are the elliptic pseudo-differential operators from the above transformation.*

• *Parametrix to $D_t - \mathcal{D} + F_2 + P_\infty$*
Lemma 9. *The parametrix $E_1 = E_1(t,s)$ to the operator $D_t - \mathcal{D} + F_2 + P_\infty$ can be written as $E_1(t,s) = E_3(t,s)Q_1(t,s)$, where $Q_1 = Q_1(t,s)$ is a matrix pseudo-differential operator with symbol from $L_\infty([0,T_0]^2, S^{K_0}_{1-\varepsilon,\varepsilon}(\mathbb{R}^n)) \cap W^1_\infty([0,T_0]^2, S^{K_0+1+\varepsilon}_{1-\varepsilon,\varepsilon}(\mathbb{R}^n))$ for every small $\varepsilon > 0$. Here the constant K_0 describes the loss of derivatives coming from the pseudo-differential zone $Z_{pd}(2N)$ and the oscillations subzone $Z_{osc}(2N)$.*

To prove this result we follow the next steps:
• Egorov's theorem is not applicable because P_∞ has no diagonal structure.
• We have to use the properties of P_∞ after perfect diagonalization.
• The next result is a base to get a relation between the type of oscillations and the loss of derivatives.

Lemma 10. *The Fourier integral operator $P_\infty(t)E_3^\mp(t,s)$ is a pseudo-differential operator with the representation*

$$P_\infty(t)E_3^\mp(t,s)w(x) = \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}\tilde{r}^\mp(t,s,x,\xi)\hat{w}(\xi)d\xi \ ,$$

where the symbol satisfies the estimates

$$\left|\partial_x^\beta\partial_\xi^\alpha\tilde{r}^\mp(t,s,x,\xi)\right| \le \begin{cases} C_{\alpha\beta\varepsilon p}\left(\frac{1}{t}(\ln\frac{1}{t})^\gamma\right)^{p+1}\langle\xi\rangle^{-p+\varepsilon|\beta|-(1-\varepsilon)|\alpha|} \text{ in } Z_{reg}(2N), \\ C_{\alpha\beta\varepsilon}\Big(1+\left(\frac{1}{t}(\ln\frac{1}{t})^\gamma\right)^2\langle\xi\rangle^{-1}\Big)\langle\xi\rangle^{\varepsilon|\beta|-(1-\varepsilon)|\alpha|} \text{ in } Z_{osc}(2N), \\ C_{\alpha\beta\varepsilon}\langle\xi\rangle^{1+\varepsilon|\beta|-(1-\varepsilon)|\alpha|} \ \text{ in } \ Z_{pd}(2N), \end{cases}$$

for every $p \ge 0$, small $\varepsilon > 0$ and all $s \in [0,t]$.

4.step: Conclusion

Using Lemma 9 and the backward transformation (from the steps of perfect diagonalization) we obtain the parametrix for $D_t - A$. The backward transformation does not bring an additional loss of derivatives. Therefore we can conclude the next result.

Theorem 3. *Let us consider*

$$u_{tt} - \sum_{k,l=1}^n a_{kl}(t,x)u_{x_kx_l} = 0, \quad u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x),$$

where the coefficients satisfy the conditions (4.1) and (4.2). The data φ, ψ belong to H^{s+1}, H^s, respectively. Then the following energy inequality holds:

$$E(u)|_{H^{s-s_0}}(t) \le C(T)E(u)|_{H^s}(0) \ \textit{for all}\ t \in (0,T], \tag{4.13}$$

where
- $s_0 = 0$ *if* $\gamma = 0$,
- s_0 *is an arbitrary small positive constant if* $\gamma \in (0,1)$,
- s_0 *is a positive constant if* $\gamma = 1$,
- *there doesn't exist a positive constant* s_0 *satisfying (3.1) if* $\gamma > 1$*, that is, we have an infinite loss of derivatives.*

It seems to be remarkable that we can prove the same relation between type of oscillations and loss of derivatives as in Theorem 1.

4.2. How to weaken C^2 regularity to keep the classification of oscillations

In the paper [12] the authors were interested in the question if there is something between (2.4) and (3.1) and the corresponding relation between the type of oscillations and the loss of derivatives. They supposed smoothness with respect to x. For simplicity of representation the authors studied the backward Cauchy problem

$$(\partial_t^2 - \Phi(t,x)\Delta)u = 0, \quad u(T,x) = \varphi(x),\ u_t(T,x) = \psi(x), \tag{4.14}$$

where $\Phi = \Phi(t,x) \in L_\infty((0,T);\mathcal{B}^\infty(\mathbb{R}^n))$ and $\Phi_0 \le \Phi(t,x)$ with a positive constant Φ_0. To understand the main results of [12] we have to introduce the next definitions.

Definition 7. (*Definition of admissible space of coefficients*) *Let T be a positive small constant, and let $\gamma \in [0,1]$ and $\beta \in [1,2]$ be real numbers. We define the weighted spaces of Hölder differentiable functions $\Lambda_\gamma^\beta = \Lambda_\gamma^\beta((0,T])$ as follows:*

$$\Lambda_\gamma^\beta((0,T]) = \Bigg\{ f = f(t,x) \in L_\infty((0,T);\mathcal{B}^k(\mathbb{R}^n)) \ : \\ \sup_{t\in(0,T]} \|f(t)\|_{\mathcal{B}^k(\mathbb{R}^n)} + \sup_{t\in(0,T]} \frac{\|\partial_t f(t)\|_{\mathcal{B}^k(\mathbb{R}^n)}}{t^{-1}(\ln t^{-1})^\gamma} \\ + \sup_{t\in(0,T]} \frac{\|\partial_t f\|_{M^{\beta-1}([t,T];\mathcal{B}^k(\mathbb{R}^n))}}{(t^{-1}(\ln t^{-1})^\gamma)^\beta} < \infty \quad \text{for all } k \ge 0 \Bigg\},$$

where $\|F\|_{M^{\beta-1}(I)}$ with a real number $\beta \in [1,2]$ and a closed interval I is defined by

$$\|F\|_{M^{\beta-1}(I)} = \sup_{\substack{s_1,s_2\in I\\ s_1\neq s_2}} \left\{ \frac{|F(s_1)-F(s_2)|}{|s_1-s_2|^{\beta-1}} \right\}.$$

The coefficient space Λ_0^1 was used in [5], Λ_1^2 was used in [6].

Definition 8. *Let σ and γ be non-negative real numbers. Then we define the exponent-logarithmic scale $\mathcal{H}_{\gamma,\sigma}$ by the set of all functions $f \in L^2(\mathbb{R}^n)$ satisfying*

$$\|f\|_{\mathcal{H}_{\gamma,\sigma}} := \left(\int_{\mathbb{R}^n} \left| \exp\left(\sigma\left(\ln\langle\xi\rangle\right)^\gamma\right) \hat{f}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} < \infty,$$

where $\hat{f}$ denotes the Fourier image of f. In particular, we denote $\mathcal{H}_\gamma = \bigcup_{\sigma>0} \mathcal{H}_{\gamma,\sigma}$.

The main result of [12] is the following:

Theorem 4. *Let $\gamma \in [0,1]$ and $\beta \in (1,2]$. If $\Phi \in \Lambda_\gamma^\beta((0,T])$, then (4.14) is well-posed in $\mathcal{H}_\gamma$ on $[0,T]$, that is, there exist positive constants $C_{\gamma,\beta}$, σ and σ' with $\sigma \le \sigma'$ such that*

$$\|(\nabla u(t), u_t(t))\|_{\mathcal{H}_{\gamma,\sigma}} \le C_{\gamma,\beta} \|(\nabla\varphi, \psi)\|_{\mathcal{H}_{\gamma,\sigma'}}. \tag{4.15}$$

Remark 7. The statement of Theorem 4 yields the existence of energy solutions. Due to the finite loss of derivatives a sufficiently large σ' implies that these solutions are classical ones with respect to x if $\gamma = 1$. The solutions are $C^1([0,T])$ with respect to t. The second derivative with respect to t belongs to $L_\infty([0,T])$ because of the properties of Φ. The same relation between type of oscillations generated by $\gamma \ge 0$ and the loss of derivatives holds as in Theorems 1 and 3 independently of $\beta \in (1,2]$.

There are no problems to use the statements of Theorem 4 to prove the next result.

Theorem 5. *Let us consider the Cauchy problem (4.14), where the coefficient Φ belongs to $\Lambda_\gamma^\beta((0,T])$ with $\gamma \in [0,1]$ and $\beta \in (1,2]$. The data φ, ψ belong to H^{s+1}, H^s, respectively. Then the following energy inequality holds:*

$$E(u)|_{H^{s-s_0}}(t) \le C(T)E(u)|_{H^s}(0) \text{ for all } t \in [0,T], \tag{4.16}$$

where
- $s_0 = 0$ *if* $\gamma = 0$,
- s_0 *is an arbitrary small positive constant if* $\gamma \in (0,1)$,
- s_0 *is a positive constant if* $\gamma = 1$.

Sketch of the proof of Theorem 4

1.step: Tools

The zones $Z_{pd}(N)$ and $Z_{hyp}(N)$ are defined in a similar way as in Section 4.1. Suitable classes of symbols are introduced in the next definition.

Definition 9. *For real numbers m_1, m_2, $m_2 \ge 0$, we define $S_{m_2}^{m_1}$ and T^{m_1} as follows:*

$$S_{m_2}^{m_1} = \Big\{ a(t,x,\xi) \in L_{loc}^\infty((0,T); C^\infty(\mathbb{R}_{x,\xi}^{2n})) \\ : |\partial_x^\tau \partial_\xi^\eta a(t,x,\xi)| \le C_{\tau,\eta} \langle\xi\rangle^{m_1-|\eta|} \left(t^{-1}\left(\ln t^{-1}\right)^\alpha\right)^{m_2} \text{ in } Z_{hyp}(N) \Big\}$$

and

$$T^{m_1} = \Big\{ a(t,x,\xi) \in L^\infty((0,T); C^\infty(\mathbb{R}^{2n}_{x,\xi})) \;:\; |\partial_x^\tau \partial_\xi^\eta a(t,x,\xi)| \le C_{\tau,\eta} \langle \xi \rangle^{m_1 - |\eta|} \text{ in } Z_{pd}(N) \Big\}.$$

2.step: Regularization

Our goal is to carry out the first two steps of the perfect diagonalization procedure. But Φ doesn't belong to C^2 with respect to t. For this reason we define a regularization Φ_ρ of Φ.

Definition 10. *Let $\chi = \chi(s) \in \mathcal{B}^\infty(\mathbb{R})$ be an even non-negative function having its support on $(-1,1)$. Let this function satisfy $\int \chi(s)ds = 1$. Moreover, let the function $\mu = \mu(r) \in C^\infty([0,\infty))$ satisfy $0 \le \mu(r) \le 1$, $\mu(r) = 1$ for $r \ge 2$ and $\mu(r) = 0$ for $r \le 1$. We define the pseudo-differential operator $\Phi_\rho = \Phi_\rho(t,x,D_x)$ with the symbol*

$$\Phi_\rho(t,x,\xi) = \mu\left(t\langle\xi\rangle(N(\ln\langle\xi\rangle)^\gamma)^{-1}\right)\phi_\rho(t,x,\xi) + \left(1 - \mu\left(t\langle\xi\rangle(N(\ln\langle\xi\rangle)^\gamma)^{-1}\right)\right)\Phi_0,$$

where

$$\phi_\rho(t,x,\xi) = \langle\xi\rangle \int \Phi(s,x)\chi\left((t-s)\langle\xi\rangle\right)ds.$$

The properties of this regularization due to the symbol classes $S^{m_1}_{m_2}$ and T^{m_1} are described in the next lemma.

Lemma 11. *The regularization $\Phi_\rho = \Phi_\rho(t,x,\xi)$ has the following properties:*
(i) $\Phi_\rho(t,x,\xi) \ge \Phi_0$;
(ii) $\Phi_\rho(t,x,\xi) \in S^0$;
(iii) $\partial_t \Phi_\rho(t,x,\xi) \in S^0_1 \cap T^{-\infty}$;
(iv) $\partial_t^2 \Phi_\rho(t,x,\xi) \in S^{-\beta+2}_\beta \cap T^{-\infty}$;
(v) $\Phi(t,x) - \Phi_\rho(t,x,\xi) \in S^{-\beta}_\beta \cap T^0$.

3.step: Diagonalization procedure

Carrying out the first two steps of perfect diagonalization and a suitable transformation by an elliptic pseudo-differential operator which takes into consideration the special structure of our starting problem (see (4.12)) we arrive at

$$(D_t - A - B - R)U = 0, \tag{4.17}$$

where $B \in S_{\beta}^{-\beta+1}$, $R \in S^0$ and

$$A := \sqrt{\Phi_\rho}\langle D_x\rangle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

4.step: Application of sharp Gårding's inequality for matrix-valued operators

Let us define $\theta_0 = \theta_0(t,\xi)$, $\theta_1 = \theta_1(t,\xi)$ and $\theta = \theta(t,\xi)$ as follows:

$$\begin{aligned}\theta_0(t,\xi) := {} & \mu(t\langle\xi\rangle(N(\ln\langle\xi\rangle)^\gamma)^{-1})\langle\xi\rangle^{-\beta+1}\left(t^{-1}\left(\ln t^{-1}\right)^\gamma\right)^\beta \\ & + (1-\mu(t\langle\xi\rangle(N(\ln\langle\xi\rangle)^\gamma)^{-1}))\langle\xi\rangle,\end{aligned}$$

$$\theta_1(t,\xi) := \mu(t\langle\xi\rangle(N(\ln\langle\xi\rangle)^\gamma)^{-1})\left(\ln t^{-1}\right)^\gamma + (1-\mu(t\langle\xi\rangle(N(\ln\langle\xi\rangle)^\gamma)^{-1}))(\ln t_\xi^{-1})^\gamma$$

and

$$\theta(t,\xi) := \theta(t,\xi;K) = K(1+\theta_0(t,\xi)+\theta_1(t,\xi))$$

with a positive real constant K. We define $V = V(t,x)$ by

$$V(t,x) := e^{-\int\limits_t^T \theta(s,D_x)ds} U(t,x).$$

Such a type of transformation was used in [2]. The operator $e^{-\int_t^T \theta(s,D_x)ds}$ describes the loss of derivatives which appears for our starting problem (4.14) in a natural way. The main point is to determine the symbol $\theta = \theta(t,\xi)$ describing in consequence our relation between classification of oscillations and related loss of derivatives.

The operator equation (4.17) is transformed to

$$(\partial_t - P_0 - P_1)\,V = 0, \tag{4.18}$$

where

$$P_0 = iA + K\,(1+\theta_0(t,D_x))\,I + iB + iR$$

and

$$P_1 = K\theta_1(t,D_x)I + i\left[e^{-\int\limits_t^T \theta(s,D_x)ds}, A+B+R\right] e^{\int\limits_t^T \theta(s,D_x)ds}.$$

The choice of θ_0 and θ_1 is organized with a sufficiently large K in such a way that sharp Gårding's inequality is applicable to the matrix-valued operators P_0

and P_1. Multiplying V on both sides of (4.18) and integrating over R^n, we have $(\partial_t V, V) = (P_0 V, V) + (P_1 V, V)$. It follows that

$$\frac{d}{dt}\|V\|^2 = 2\Re\,(P_0 V, V) + 2\Re\,(P_1 V, V)\,.$$

The application of sharp Gårding's inequality yields

$$2\Re\,(P_0 V, V) \geq -C_0\|V\|^2, \quad 2\Re\,(P_1 V, V) \geq -C_1\|V\|^2.$$

Finally we deduce

$$\|V(t)\|^2 \leq e^{C(T-t)}\|V(T)\|^2 \leq e^{CT}\|V(T)\|^2.$$

This inequality yields together with

$$\int_0^T \theta(s,\xi)ds \leq C(\ln\langle\xi\rangle)^\gamma$$

the statements of Theorem 4. $\square$

5. Concluding remarks

Let us mention further results which are obtained for model problems with non-Lipschitz behavior and more problems could be of interest.

Remark 8. (*Lower regularity with respect to* x) The results and the approach from [12] motivate to study the question how to weaken the regularity with respect to x (compare with [8]). From this paper we understand to which class the remainder should belong after diagonalization. Thus pseudo-differential operators with symbols of finite smoothness or maybe paradifferential operators should be used.

Remark 9. (*Mixing of different non-regular effects*) The survey article [4] gives results if we mix different non- regular effects as Hölder regularity of $a = a(t)$ on $[0,T]$, order of degeneracy at $t = 0$, and L_p integrability of a weighted derivative on $[0,T]$. Among all these results we mention only that one which guarantees C^∞ well- posedness of (1.3) if $a = a(t)$ satisfies $t^q\partial_t a \in L_p(0,T)$ for $q + 1/p = 1$.

Remark 10. (*Quasi-linear models*) Quasi-linear models with behavior of suitable derivatives as $O(\frac{1}{t})$ was studied in [16]. Here the log-effect from (3.1) could not be observed.

Remark 11. (*Applications to Kirchhoff type equations*)
A nice application of non-Lipschitz theory with behavior $a'(t) = O((T-t)^{-1})$ for $t \to T-0$ to Kirchhoff equations was described in [14]. The assumed regularity of data could be weakened in [10] by proving that these very slow oscillations (in the language of Definition 1) produce no loss of derivatives (see Theorem 1).

Remark 12. (*p-Evolution equations*) The paper [1] is devoted to the Cauchy problem for p- evolution equations with LogLip coefficients. Due to an information of Prof. Cicognani there exist joint discussions with Prof. Colombini about p-evolution equations with coefficients behaving like $ta'(t) \le C$ on $(0,T]$. An interesting question is to find p-evolution models with log-effect from (3.1).

Acknowledgments: The author would like to express many thanks to Prof's Gramchev (Cagliari), Popivanov and Slavova (both Sofia) for the organization of *Conference on Partial Differential Equations on Spaces with Geometrical Singularities* held at the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences in Sofia in the period $2-7$ September 2002.

The author thanks DFG for financial support.

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