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Pliska Stud. Math. Bulgar. **15** (2002), 131–148

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DIAGONALIZABLE COMPLEX SYSTEMS, REDUCED DIMENSION AND HERMITIAN SYSTEMS II

Jean Vaillant

Abstract. We consider a first order differential system. If its principal part $a(x,\xi)$ is hyperbolic – that means that the characteristic roots are real for every (x,ξ) – and if it is symmetric or hermitian, it is usual to construct an energy inequality; if the system is linear and C^{∞}, the Cauchy problem is C^{∞} is well-posed, for any zero order terms; in some non-linear cases, we have existence theorem. Moreover in the case of constant coefficients, the theorem by Kasahara and Yamaguti states the equivalence between strong hyperbolicity and uniformly (real) diagonalizability. So it is natural to study systems whose the principal part is diagonalizable or uniformly diagonalizable for each value of the variable x and to seek for conditions of symmetry or hermiticity. P. D. Lax in [12] gave an example of 3×3 system with constant coefficients, strongly hyperbolic and not equivalent to a symmetric system. G. Strang [7] stated that for 2×2 systems with constant coefficients, strong hyperbolicity and symmetry of the system in a convenient basis are equivalent. In [13] J. Vaillant defined the reduced dimension of a real $a(\xi)$; this definition is such that the reduced dimension of the system is equal to the reduced dimension of the determinant, if the system is diagonalizable; the reduced dimension of a polynomial was defined by Atiyah Bott and Gårding; in [13] it was stated that, if the reduced dimension of the principal part of the system is more than $m\left(\frac{m+1}{2}\right)$ and if the system is diagonalizable (some additional condition, in fact implied by the two first ones, as it will be proved by T. Nishitani [3], was satisfied), then the principal part is, in fact, symmetric in an convenient basis; we denote that the system is presymmetric:

2002 *Mathematics Subject Classification:* 35L40
Key words: strong hyperbolicity, symmetric, hermitian systems

there exists T such that $T^{-1}a(\xi)T$ is symmetric, for every ξ; the analogous result, in the case of complex coefficients, was obtained in the third cycle thesis of D. Schiltz.

Y. Oshime [6], in a series of papers studied completely the 3×3 diagonalizable real and complex system and characterized symmetric and hermitian system. In [3] T. Nishitani improved the result [13] and stated that, if the dimension $m \geq 3$, if the reduced dimension $d \geq m\binom{m+1}{2} - 1$ and if the system is diagonalizable, it is presymmetric; for $m = 3$ this result is optimum, by [6]. In [8] J. Vaillant stated for $m = 4$ and in [9] for general $m \geq 4$, that, if the system is strongly hyperbolic and if $d \geq m\binom{m+1}{2} - 2$, it is presymmetric.

T. Nishitani and J. Vaillant [4] stated in the case of variable coefficients, that, if for every x the previous conditions are satisfied, then the principal part is regularly presymmetric (that means there exists a regular-the same regularity as the coefficients-matrix $T(x)$ such that $T^{-1}(x)a(x,\xi)T(x)$ is symmetric for every (x,ξ); in fact they stated that, if $d \geq m\binom{m+1}{2} - \left[\frac{m}{2}\right]$ and if for every x, $a(x,\xi)$ is presymmetric then it is regularly presymmetric; that implies, thanks to the result with constant coefficients, the precedent result.

Then, J. Vaillant states in the case of complex coefficients that if the reduced dimension (in the real) $d_R(a) \geq m^2 - 2$ and if the system is diagonalizable, then it is prehermitian. The schedule of the proof will be published in the Proceedings of the Cortona colloquium (2001) and in the present paper.

We conjecture also that, if $d_R \geq m^2 - 3$, $m \geq 4$, and if the system is strongly hyperbolic, then the principal part is prehermitian; the result is, at the moment, is obtained for $m = 4$ (to appear).

1. Introduction

We study a first order system

$$a(D) = ID_0 + \sum_{k=1}^{n} \mathrm{a}_k D_k,$$

where I is the identity matrix and a_k is complex valued $m \times m$ matrix.

Let $a(\xi)$ the principal symbol $a(D)$:

$$a(\xi) = I\xi_0 + \sum_{k=1}^{n} \mathrm{a}_k \xi_k. \tag{1}$$

We define in an invariant manner the reduced dimension of a:

Definition 1. E is a real vector space of dimension $n+1$; F is a complex vector space of dimension m. a is a R-linear map from E to the vector space $\mathcal{L}(F,F)$ of the linear maps from F to F considered as a real vector space, $d(a) = \text{rank}(a) = \dim\,(\text{Im}\; a)$.

We have evident properties.

If a basis is chosen in F, $d(a)$ = dimension of the real vector subspace of $M(m, C)$ spanned by the matrix $(a^i_j(\xi))$.

We have also $d(a) = d({}^t a) = d(\overline{a})$.

We choose a basis in E, the first vector of which is a non characteristic vector N, $\det a(N) \neq 0$. Then:

$$a(\xi) = \xi_0 a(N) + a(\xi'),\ \xi' = (\xi_1, \ldots, \xi_n).$$

As usually, we can replace $a(N)$ by I and we obtain (1), then:

$d(a)$ = dimension of the real vector subspace of $M(m, C)$, spanned by I, Re a_1,…,Re a_n, Im a_1,…,Im a_n.

We introduce also the:

Definition 2. a is diagonalisable (R-diagonalizable) with respect to N if and only if:

i) $\forall \xi$, the zeroes of $\det(I\tau + a(\xi)) = 0$ are real,

ii) when τ has multiplicity μ, the dimension of the corresponding kernel of $I\tau + a(\xi)$ is μ.

That means, evidently, that:

i) the proper values of $a(\xi')$ are all real,

ii) the dimension of the proper space corresponding to a zero is equal to its multiplicity.

Definition 3. a is a prehermitian with respect to N, if and only if there exists a basis of E of first vector N and a basis of F such that in these bases the matrices $(a^i_j(\xi))$ are hermitian for every ξ.

We state the:

Theorem. *If a is diagonalizable with respect to N, if $d(a) \geq m^2 - 2$, then a is prehermitian with respect to N.*

That means, if we consider the matrix, there exists an invertible complex matrix such that:

$$T^{-1}(a^i_j(\xi))T \ \text{ is hermitian, } \forall \xi.$$

The cases of real matrix and variable coefficients were studied in a series of papers (consider [8], [9], [10], [4], [5] and their bibliography).

In the §2, we explain that the proof is divided in three parts. In this paper, half of the case II is studied in the §3 and the case III is studied in the §4, §5, §6. The cases I and half of the case II were considered in [11]

2. Schedule of the proof

For $m = 2$, Strang [7] obtained the result with the alone assumption of strong hyperbolicity.

At first, for $m = 3$, we prove the:

Lemma 2.1. *If a is diagonalizable with respect to N, if $d(a) \geq 7$, then there exists a $\xi' \neq 0$, such that $deta(\xi_0, \xi')$ has a multiple zero in ξ_0 (or $a(\xi')$ has a multiple proper value).*

Proof. It is quite similar to the proofs of [1], [2]; consider also the remark 2.3. □

Then we have a result by Oshime (if $m = 3$):

Lemma 2.2. [6] *If a is diagonalizable with respect to N, if $d(a) \geq 7$ and if there exists a ξ' such that $\det a(\xi_0, \xi')$ has a multiple zero in ξ_0, then a is prehermitian with respect to N.*

The theorem is obtained for $m = 3$.

We consider now the general case. We denote also by ϕ^i_j the entries of the matrix a.

We prove, by an adaptation of [3], that, thanks to the diagonalizability of a, we can assume:

i) for $p < q$, real and imaginary parts of $\phi^p_q \in$ span { real and imaginary parts of ϕ^j_i, $i > j\} = V$,

ii) for $1 \leq i \leq m$, $\phi^j_i(\xi) = \xi_0 + \chi_i(\xi') + i\lambda_i(\xi')$, where χ_i and λ_i are real linear forms; moreover: $\lambda_i \in V$.

So: $d(a) =$ dim real vector space spanned by $\{V, \ldots, \xi_0 + \chi_i, \ldots\}$. The number of elements of this set is at most m^2.

Remark 2.3. If ξ' cancels all the $\phi^1(\xi')$, $2 \leq i \leq m$, then $\lambda_1(\xi') = 0$.

We distinguish three cases:

I. $\dim_R V = m^2 - m$ (all the linear forms $\phi^i_j \in V$ have their real and imaginary parts linearly independent in $\mathbf{R}$),

II. $\dim_R V = m^2 - m - 1$ (one form of V depends linearly on the others),

III. $\dim_R V = m^2 - m - 2$ (two forms of V depend linearly on the others).

We will use frequently the

Lemma 2.4. *If b is prehermitian, there exists a hermitian and definite positive matrix H such that:*

$$bH = H^t\bar{b}. \tag{2}$$

P r o o f. There exists T such that

$$T^{-1}bT = \overline{{}^t(T^{-1}bT)} = {}^t\,\overline{T}{}^t\overline{b}{}^t\overline{T}^{-1};$$

we denote $H = T^t\overline{T}$; H is hermitian define positive and we obtain the result. □

3. Case II$_2$

One form ϕ^i_j, $i > j$, has its real or its imaginary part depending on the other forms; $m-2$ forms χ_i at less are independent; we consider the case where $m-2$ forms χ_i are independent; the other case is analogous but simpler. By a convenient choice of basis in E and F, we can write:

$$\phi^i_j(\xi') = \xi^i_j + i\eta^i_j,\ i > j,\ i \geq 4.$$

We distinguish

Case II$_1$: One dependent form is in the third line, second column [11],

Case II$_2$: $\phi^3_j(\xi') = \xi^3_j + i\eta^3_j$, $1 \leq j \leq 2$.

Case II$'_2$: $\phi^2_1(\xi') = \sum_{k,\ell} c^{2k}_{1\ell}\xi^\ell_k + \sum_{k,\ell} d^{2k}_{1\ell}\eta^\ell_k + d^{21}_{12}\eta^2_1 + i\eta^2_1$,

Case II$''_2$: $\phi^2_1(\xi') = \xi^2_1 + i\left(\sum_{k,\ell} e^{2k}_{1\ell}\xi^\ell_k + \sum_{k,\ell} f^{2k}_{1\ell}\eta^\ell_k + e^{21}_{12}\xi^2_1\right)$.

At first, we study the case II$'_2$. We have:

$$\begin{aligned}
\phi^i_i(\xi) &= \xi_0 + \chi_i + i\left(\sum_{k,\ell} e^\ell_{ik}\xi^k_\ell + \sum_{k,\ell} f^\ell_{ik}\xi^k_\ell + f^1_{i2}\eta^2_1\right),\ 2 \leq i \leq m,\ \chi_m = 0,\\
\phi^1_1(\xi) &= \xi_0 + \sum_2^{m-1} c_{1k}\xi_k + \sum_{k,\ell} c^\ell_{1k}\xi^k_\ell + \sum_{k,\ell} d^\ell_{1k}\eta^k_\ell + d^1_{12}\eta^2_1\\
&\quad + i\left(\sum_{k,\ell} e^\ell_{1k}\xi^k_\ell + \sum_{k,\ell} f^\ell_{1k}\eta^k_\ell + f^1_{12}\eta^2_1\right),
\end{aligned}$$

the ξ^k_ℓ, η^k_ℓ, $1 \leq \ell < k \leq m$, $(k,\ell) \neq (2,1)$, η^2_1, χ_k are new coordinates (or independent variables).

We have also, for $i < j$:

$$\phi^i_j(\xi') = \sum_{k,\ell} c^{i\ell}_{jk}\xi^k_\ell + \sum_{k,\ell} d^{i\ell}_{jk}\eta^k_\ell + d^{i1}_{j2}\eta^2_1 + i\left(\sum_{k,\ell} e^{i\ell}_{jk}\xi^k_\ell + \sum_{k,\ell} f^{i\ell}_{jk}\eta^k_\ell + f^{i1}_{j2}\eta^2_1\right).$$

We denote: $c^i_j = c^{ii}_{jj}$; $H = (h_{uv})$, $h_{uu} = h_u$.

Lemma 3.1. *i)* $e^k_{i\ell} = f^k_{i\ell} = f^1_{i2} = 0$, $\forall k, \ell$, $(k,\ell) \neq (k', m)$, $1 \leq k' \leq m-1$,

ii) If $1 \leq i < j \leq m-1$, $(i,j) \neq (1,2)$, *then*

$$\begin{aligned}
\phi^i_j(\xi') &= c^i_j(\xi^j_i - i\eta^j_i) + \sum_k c^{ik}_{jm}\xi^m_k + \sum_k d^{ik}_{jm}\eta^m_k + i\left(\sum_k e^{ik}_{jm}\xi^m_k + \sum_k f^{ik}_{jm}\eta^m_k\right),\\
&\quad 1 \leq k \leq m-1,\\
\phi^1_2(\xi') &= k^1_2\left(\sum_{k,\ell} c^{2k}_{1\ell}\xi^\ell_k + \sum_{\ell,k} d^{2k}_{1\ell}\eta^\ell_k + d^{21}_{12}\eta^2_1 - i\eta^2_1\right)\\
&\quad + \sum_{k'} c^{1k'}_{2m}\xi^m_{k'} + \sum_{k'} d^{1k'}_{2m}\eta^m_{k'} + i\left(\sum_{k'} e^{1k'}_{2m}\xi^m_{k'} + \sum_{k'} f^{1k'}_{2m}\eta^m_{k'}\right),\\
&\quad (\ell,k) \neq (k',m),\ 1 \leq k \leq m-1.
\end{aligned}$$

iii) $c^1_j = k^1_2 c^2_j$, $j \leq m-1$, $c^i_j = c^i_k c^k_j$, $1 \leq i \leq j \leq m-1$, $k \neq 2$; *all the considered* c *and* k^1_2 *are positive.*

Proof. Let $\xi^m_1 = \eta^m_1 = \cdots = \xi^m_{m-1} = \eta^m_{m-1} = 0$; we get immediately i) for $i = m$.

The matrix b obtained by removing the last line and the last column of $a(\xi)$ is prehermitian by an easy induction; there exists H (Lemma 2.4) such that:

$$bH = H^t\bar{b}.$$

If for some k' we have not: all the $c_{1k} = 0$, $k \neq k'$ and $c_{1k'} = 1$, then H is diagonal and the result is easy.

If $\forall k \neq k'$, $c_{1k} = 0$ and $c_{1k'} = 1$, k' fixed, then: $h_{uv} = 0$, except $h_{1k'}$.

We consider, at first, the cases $k' \neq 2$, by change of lines and columns, it is sufficient to consider, for instance, $k' = m-1$. We explicit (2). By considering the entries in the i-th line, i-th column, $2 \leq i \leq m-2$, we obtain i) for $2 \leq i \leq m-2$. The entry in the $(m-1)$-th line, $(m-1)$-th column gives

$$\mathrm{Im}\left[\left(\xi^{m-1}_1 + i\eta^{m-1}_1\right) h_{1m-1} + ih_{m-1}\left(\sum_{k,\ell} e^k_{m-1\ell}\xi^\ell_k + \sum_{k,\ell} f^k_{m-1\ell}\eta^\ell_k + f^1_{m-12}\eta^2_1\right)\right] = 0,$$

$(k,\ell) \neq (k', m)$, $1 \leq k' \leq m-1$.

We obtain, for $3 \leq i < m-1$

$$\begin{aligned}\phi^i_{m-1}(\xi') &= c^i_{m-1}\left(\xi^{m-1}_i - i\eta^{m-1}_i\right) - \frac{h_{1m-1}}{h_{m-1}}\left(\xi^i_1 + i\eta^i_1\right) + \sum_k c^{ik}_{m-1m}\xi^m_k \\ &+ \sum_k d^{ik}_{m-1m}\eta^m_k + i\left(\sum_k e^{ik}_{m-1m}\xi^m_k + \sum_k f^{ik}_{m-1m}\eta^m_k\right),\end{aligned}$$

$1 \leq k \leq m-1$; $c^i_{m-1} = h_i/h_{m-1}$,

$$\begin{aligned}\phi^2_{m-1}(\xi') &= c^2_{m-1}\big(\xi^{m-1}_2 - i\eta^{m-1}_2\big) - \frac{h_{1m-1}}{h_{m-1}}\left(\sum_{k,\ell} c^{2\ell}_{1k}\xi^k_\ell + \sum_{k,\ell} d^{2\ell}_{1k}\eta^k_\ell + d^{21}_{12}\eta^2_1 + i\eta^2_1\right) \\ &+ \sum_{k'} c^{2k'}_{m-1m}\xi^m_{k'} + \sum_{k'} d^{2k'}_{m-1m}\eta^m_{k'} + i\left(\sum_{k'} e^{2k'}_{m-1m}\xi^m_{k'} + \sum_{k'} f^{2k'}_{m-1m}\eta^m_{k'}\right),\end{aligned}$$

$(k,\ell) \neq (k',m)$, $1 \leq k' \leq m-1$; $c^2_{m-1} = h_2/h_{m-1}$,

$$\begin{aligned}\phi^1_{m-1}(\xi') &= c^1_{m-1}\left(\xi^{m-1}_1 - i\eta^{m-1}_1\right) - \frac{h_{1m-1}}{h_{m-1}}\left(\sum_{k,\ell} c^{\ell}_{1k}\xi^k_\ell + \sum_{k,\ell} d^{\ell}_{1k}\eta^k_\ell + d^1_{12}\eta^2_1\right) \\ &+ \sum_{k'} c^{1k'}_{m-1m}\xi^m_{k'} + \sum_{k'} d^{1k'}_{m-1m}\eta^m_{k'} + i\left(\sum_{k'} e^{1k'}_{m-1m}\xi^m_{k'} + \sum_{k'} f^{1k'}_{m-1m}\eta^m_{k'}\right),\end{aligned}$$

$(k,\ell) \neq (k',m)$, $1 \leq k' \leq m-1$; $c^1_{m-1} = h_1/h_{m-1}$,

We obtain, for $2 \leq i \leq j \leq m-2$:

$$\phi^i_j(\xi') = c^i_j\left(\xi^j_i - i\eta^j_i\right) + \sum_k c^{ik}_{jm}\xi^m_k + \sum_k d^{ik}_{jm}\eta^m_k + i\left(\sum_k e^{ik}_{jm}\xi^m_k + \sum_k f^{ik}_{jm}\eta^m_k\right),$$

$1 \leq k \leq m-1$; $c^i_j = h_i/h_j$.

For $3 \leq j \leq m-2$:

$$\begin{aligned}\phi^1_j(\xi') &= c^1_j\left(\xi^j_1 - i\eta^j_1\right) + \frac{h_{1m-1}}{h_{m-1}}\left(\xi^{m-1}_j + i\eta^{m-1}_j\right) \\ &+ \sum_k c'^{1k}_{jm}\xi^m_k + \sum_k d'^{1k}_{jm}\eta^m_k + i\left(\sum_k e'^{1k}_{jm}\xi^m_k + \sum_k f'^{1k}_{jm}\eta^m_k\right),\end{aligned}$$

$1 \leq k \leq m-1$, $c'^1_j = \dfrac{h_1 h_{m-1} - |h_{1m-1}|^2}{h_j h_{m-1}}$,

$$\phi^1_2(\xi') = k'^1_2 \left(\sum_{k,\ell} c^{2\ell}_{1k} \xi^k_\ell + \sum_{k,\ell} d^{2\ell}_{1k} \eta^k_\ell + d^{21}_{12} \eta^2_1 - i\eta^2_1 \right) + \frac{h_{1m-1}}{h_{m-1}} \left(\xi^{m-1}_2 + i\eta^{m-1}_2 \right)$$
$$+ \sum_k c'^{1k}_{2m} \xi^m_k + \sum_k d'^{1k}_{2m} \eta^m_k + i \left(\sum_k e'^{1k}_{2m} \xi^m_k + \sum_k f'^{1k}_{2m} \eta^m_k \right),$$

$1 \leq k \leq m-1$, $k'^1_2 = \dfrac{h_1 h_{m-1} - |h_{1m-1}|^2}{h_2 h_{m-1}}$, $c'^1_j = k'^1_2 c^2_j$, $j \leq m-2$, $c^i_j = c^i_k c^k_j$, $2 \leq i < j \leq m-1$.

We transform $a(\xi)$ by the invertible matrix:

$$I + E^1_{m-1}$$

that means, we consider

$$(I + E^1_{m-1})^{-1} a(\xi)(I + E^1_{m-1});$$

in the matrix E^1_{m-1}, all the entries are zero, except in the first line, $(m-1)$th column where there is $e^1_{m-1} = h_{m-1}/h_m$.

We define:

$$c'^1_{m-1} = \frac{h_1 h_{m-1} - |h_{1m-1}|^2}{(h_{m-1})^2}$$

and we obtain the result (we denote now $c' = c$; $k' = k$).

If $c_{12} = 1$, $c_{1k} = 0$, $k \neq 2$, we have similar calculus. □

Lemma 3.2. *We assume $c_{12} \neq 0$. Then:*

i) $e^\ell_{ik} = f^\ell_{ik} = 0$, $\forall i \geq 3$, $e^\ell_{1m} + e^\ell_{2m} = 0$, $f^\ell_{1m} + f^\ell_{2m} = 0$,

ii) $\phi^i_j(\xi') = c^i_j(\xi^j_i - i\eta^j_i)$, $\forall i, j$, $3 \leq i < j$.

Proof. We consider the coefficient of $c_{12}\chi^2_2 \prod_{k \neq 2} \chi_k$ in det a(ξ'):

$$i \left(\sum_\ell e^\ell_{mm} \xi^m_\ell + \sum_\ell f^\ell_{mm} \eta^m_\ell \right);$$

it is real; so we have i) for $i = m$.

We consider the coefficient of $c_{12}\chi^2_2 \xi_0 \prod \chi_k$ $k \neq 2$, $k \neq k'$ in det $a(\xi)$:

$$i \left(\sum_\ell e^\ell_{k'm} \xi^m_\ell + \sum_\ell f^\ell_{k'm} \eta^m_\ell \right);$$

it is real ; and we obtain the end of i).

Now we consider the coefficient of $c_{12}\chi_2^2\xi_0\prod\chi_k$ $k\neq 2$, $k\neq k'$ in det $a(\xi')$: and we obtain:

$$\phi_m^{k'}(\xi') = c_m^{k'}\,(\xi_{k'}^m - i\eta_{k'}^m)\,.$$

We consider the coefficient of $c_{12}\chi_2^2\xi_0\prod\chi_k$ $k\neq i$, $k\neq j$ in det $a(\xi)$ and we obtain ii) for $3\leq i<m$. □

Lemma 3.3. *We assume $c_{12}\neq 0$, then:*

i) $i\in\{1,2\}$,

$$\begin{aligned}\phi_m^i(\xi') \;=\;& c_m^i(\xi_i^m - i\eta_i^m) + \sum_k c_{mm-1}^{ik}\xi_k^{m-1} + c_{mm}^{im-1}\xi_{m-1}^m\\ &+\sum_k d_{mm-1}^{ik}\eta_k^{m-1} + d_{mm}^{im-1}\eta_{m-1}^m\\ &+ i\left(\sum_k e_{mm-1}^{ik}\xi_k^{m-1} + e_{mm}^{im-1}\xi_{m-1}^m + \sum_k f_{mm-1}^{ik}\eta_k^{m-1} + f_{mm}^{im-1}\eta_{m-1}^m\right),\end{aligned}$$

$1\leq k\leq m-2$.

ii) $i\in\{1,2\}$, $i<j\leq m-2$, $(i,j)\neq(1,2)$,

$$\phi_j^i(\xi') = c_j^i(\xi_i^j - i\eta_i^j) + c_{jm}^{im-1}\xi_{m-1}^m + d_{jm}^{im-1}\eta_{m-1}^m + i\left(e_{jm}^{im-1}\xi_{m-1}^m + f_{jm}^{im-1}\eta_{m-1}^m\right)$$

$$\begin{aligned}\phi_2^1(\xi') \;=\;& k_2^1\left(\sum_{k,\ell} c_{1\ell}^{2k}\xi_k^\ell + \sum_{k,\ell} d_{1\ell}^{2k}\eta_k^\ell + d_{12}^{21}\eta_1^2 - i\eta_1^2\right)\\ &+ c_{2m}^{1m-1}\xi_{m-1}^m + d_{2m}^{1m-1}\eta_{m-1}^m + i\left(e_{2m}^{1m-1}\xi_{m-1}^m + f_{2m}^{1m-1}\eta_{m-1}^m\right),\end{aligned}$$

$(k,\ell)\neq(m-1,m)$.

iii) $c_m^1 = k_2^1c_m^2 = c_j^1c_m^j$, $2<j<m$, $j\neq m-1$, $c_m^2 = c_j^2c_m^j$, $j\neq m-1$.

P r o o f. Let $\xi_1^{m-1} = \eta_1^{m-1} = \cdots = \xi_{m-2}^{m-1} = \eta_{m-2}^{m-1} = \xi_{m-1}^m = \eta_{m-1}^m = 0$.

The submatrix $(m-1)\times(m-1)$: b obtained by cancelling the $(m-1)$th line and the $(m-1)$th column in $a(\xi)$ is diagonalizable; its reduced dimension is more than: $(m-1)^2-2$; by induction, it is prehermitian; by the Lemma 2.4, there exists a hermitian matrix H such that:

$$bH = H^t\bar{b}. \tag{3}$$

We consider the variables χ_k and we obtain:
if $c_{12}\neq 1$ or $c_{1k}\neq 0$ for some $k\neq 2$, then H is diagonal;
if $c_{12}=1$ and $c_{1k}=0$, $\forall k\neq 2$, then $h_{uv}=0$, $u\neq v$, except h_{12}.

In the first case, we get easily the lemma.

In the second case, we explicit the entries in the third line, second column in (2) and we obtain again:

$$h_{12} = 0$$

and the result. □

Lemma 3.4. *We assume $c_{12} \neq 0$, then:*

$$\phi^i_j(\xi') = c^i_j \left(\xi^j_i - i\eta^j_i\right), \ \forall i < j, \ (i,j) \neq (1,2).$$

Proof. We consider the coefficient of $\prod \chi_k$ in det $a(\xi')$; it is real and we obtain:

$$i \in \{1,2\}, \ \phi^i_m(\xi') = c^i_m \left(\xi^m_i - i\eta^m_i\right).$$

We consider the coefficient of $\xi_0 \prod_{k \neq m-1} \chi_k$ in det $a(\xi)$ and we obtain

$$i \in \{1,2\}, \ \phi^i_{m-1}(\xi') = c^i_{m-1} \left(\xi^{m-1}_i - i\eta^{m-1}_i\right).$$

We consider the coefficient of $\xi_0 \chi_{m-1} \prod \chi_k$, $k \neq m-1$, and for $k' \neq m-1$, $k \neq k'$, we obtain

$$i \in \{1,2\}, \ \phi^i_{k'}(\xi') = c^i_{k'} \left(\xi^{k'}_i - i\eta^{k'}_i\right).$$

□

Lemma 3.5. *We assume $c^1_2 \neq 0$, then:*
i) $e^{m-1}_{1m} = f^{m-1}_{1m} = e^{m-1}_{2m} = f^{m-1}_{2m} = 0$.
ii) $\phi^1_2(\xi') = k^1_2 \left(\sum_{k,\ell} c^{1k}_{2\ell} \xi^\ell_k + \sum_{k,\ell} d^{1k}_{2\ell} \eta^\ell_k - i\eta^2_1\right)$.
iii) $c^1_m = c^1_{m-1} c^{m-1}_m$, $c^2_m = c^2_{m-1} c^{m-1}_m$.

Proof. Let $\xi^3_1 = \eta^3_1 = \xi^3_2 = \eta^3_2 = \cdots = \xi^4_3 = \eta^4_3 = \cdots = \xi^m_3 = \eta^m_3 = 0$.

By considering the $(m-1) \times (m-1)$ submatrix b obtained by removing the third lines and columns, we obtain the result. □

Lemma 3.6. *We assume $c_{12} \neq 0$, then $a(\xi)$ is prehermitian.*

We have obtained:

$$c^i_j = c^i_k c^k_j, \ \forall i,j,k, \ (i,k) \neq (1,2),$$

$$c^1_j = k^1_2 c^2_j; \text{ all the } c^i_j \text{ and } k^1_2 > 0.$$

Finally, we transform $a(\xi)$ by the diagonal matrix $(1, \frac{1}{\sqrt{k^1_2}}, \frac{1}{\sqrt{c^1_3}}, \ldots, \frac{1}{\sqrt{c^1_m}})$.

We assume now that there exist some k, $k \neq 2$, such that $c_{1k} \neq 0$; by change of lines and columns, we can assume $c_{13} \neq 0$.

Lemma 3.7. *If $c_{13} \neq 0$, then $a(\xi)$ is prehermitian.*
Proof. It is quite similar to the case $c_{12} \neq 0$. $\square$
Lemma 3.8. *If $\forall k$, $c_{1k} = 0$, then:*
i) $e^j_{im} = f^j_{im} = 0$, $\forall j \geq 3$.
ii) $\phi^1_2(\xi') = k^1_2 \left(\sum_{k,\ell} c^{2k}_{1\ell}\xi^k_\ell + \sum_{k,\ell} d^{2k}_{1\ell}\eta^k_\ell + d^{21}_{12}\eta^2_1 - i\eta^2_1\right)$
$+ \sum_{k'} c^{1k'}_{2m}\xi^m_{k'} + \sum_{k'} d^{1k'}_{2m}\eta^m_{k'} + i\left(\sum_{k'} e^{1k'}_{2m}\xi^m_{k'} + \sum_{k'} f^{1k'}_{2m}\eta^m_{k'}\right)$,
$(k,\ell) \neq (k',m)$, $k' \in \{1,2\}$.
iii) $3 \leq i < j \leq m-1$

$$\phi^i_j(\xi') = c^i_j(\xi^j_i - i\eta^j_i) + \sum_k c^{ik}_{jm}\xi^m_k + \sum_k d^{ik}_{jm}\eta^m_k + i\left(\sum_k e^{ik}_{jm}\xi^m_k + \sum_k f^{ik}_{jm}\eta^m_k\right),$$

$k \in \{1,2\}$.
iv) $3 \leq i \leq m-1$

$$\phi^i_m(\xi') = c^i_m\left(\xi^m_i - i\eta^m_i\right) + \sum_{k,\ell} c^{ik}_{m\ell}\xi^\ell_k + \sum_{k,\ell} d^{ik}_{m\ell}\eta^\ell_k + i\left(\sum_{k,\ell} e^{ik}_{m\ell}\xi^\ell_k + \sum_{k,\ell} f^{ik}_{m\ell}\eta^\ell_k\right),$$

k,ℓ *such that* $3 \leq \ell \leq m$, $k \in \{1,2\}$.
v) $c^i_j = c^i_k c^k_j$, $3 \leq i < j \leq m$; $c^i_j > 0$.
Proof. Let $\xi^\ell_k = \eta^\ell_k = 0$, $3 \leq \ell \leq m$, $k \in \{1,2\}$.
The matrix 2×2: (ϕ^i_j), $i,j \in \{1,2\}$ and the matrix $(m-2)\times(m-2)$: (ϕ^i_j), $i,j \in \{3,\dots,m\}$ are diagonalizable; their reduced dimension is such that they are prehermitian; thanks to the Lemma 2.4, with H diagonal, we obtain easily the result. $\square$
Lemma 3.9. *If $\forall k$, $c_{1k} = 0$, then:*
$e^k_{i\ell} = f^k_{i\ell} = 0$, $\forall i \neq 1$, $i \neq m$, $e^k_{1m} + e^k_{mm} = 0$, $f^k_{1m} + f^k_{mm} = 0$, $k \in \{1,2\}$.
Proof. Let $\xi^{m-1}_1 = \eta^{m-1}_1 = \cdots = \xi^{m-1}_{m-2} = \eta^{m-1}_{m-2} = 0$

$$c^{m-1}_m(\xi^m_{m-1} - i\eta^m_{m-1}) + \sum_{k,\ell} c^{m-1k}_{m\ell}\xi^\ell_k + \sum_{k,\ell} d^{m-1k}_{m\ell}\eta^\ell_k + i\left(e^{m-1k}_{m\ell}\xi^\ell_k + f^{m-1k}_{m\ell}\eta^\ell_k\right) = 0$$

$\ell \geq 3$, $\ell \neq m-1$, $k \in \{1,2\}$; (we use $c^{m-1}_m \neq 0$).
We obtain immediately: $e^k_{m-1\ell} = f^k_{m-1\ell} = 0$.
The submatrix $b = (m-1)\times(m-1)$ obtained by cancelling the $(m-1)$th line and the $(m-1)$th column is prehermitian. There exists a matrix H, [Lemma 2.4], such that (2) is satisfied.

We denote $H = (h_{uv})$, $h_{uu} = h_u$; we verify: $h_{uv} = 0$, $\forall u \neq v$, except h_{1m-1}; by considering the entries in the ith line, ith column in (2), we obtain:

$$e^k_{i\ell} = f^k_{i\ell},\ 2 \leq i \leq m-2.$$

□

Lemma 3.10. *If* $\forall k$, $c_{1k} = 0$, *then*

i) $\phi^i_m(\xi') = c^i_m\left(\xi^m_i - i\eta^m_i\right) + c^{i1}_{mi}\xi^i_1 + d^{i1}_{mi}\eta^i_1 + i\left(e^{i1}_{mi}\xi^i_1 + f^{i1}_{mi}\eta^i_1\right)$, $3 \leq i \leq m$.

ii) $\phi^1_j(\xi') = c^1_j\left(\xi^j_1 - i\eta^j_1\right) + c^{1j}_{jm}\xi^m_j + d^{1j}_{jm}\eta^m_j + i\left(e^{1j}_{jm}\xi^m_j + f^{1j}_{jm}\eta^m_j\right)$, $3 \leq j \leq m$.

iii) $e^{i1}_{mi} + e^{1i}_{im} = 0$; $d^{i1}_{mi} + d^{1i}_{im} = 0$; $c^{i1}_{mi} + f^{1i}_{im} = 0$; $c^{1i}_{im} + f^{i1}_{mi} = 0$, $3 \leq i \leq m-1$.

P r o o f. The coefficient of $\prod \chi_k$ in det $a(\xi')$ is real; the coefficient of $\chi_2\xi_0\prod\chi_k$, $k \neq 2$, $k \neq k$ in det $a(\xi)$ is real; by considering the difference between these coefficients, we obtain the result. □

Let $\xi^{m-1}_1 = \eta^{m-1}_1 = \cdots = \xi^{m-1}_{m-2} = \eta^{m-1}_{m-2} = \xi^{m-1}_m = \eta^{m-1}_m = 0$ in $a(\xi)$ and consider the submatrix $(m-1)\times(m-1)$ obtained by removing the $(m-1)$th line and column; it is prehermitian and there exists H [Lemma 2.4] such that (2) is satisfied; we verify: $h_{uv} = 0$, $\forall u, v$, except h_{1m-1}; we explicit (2); we transform $a(\xi)$ by the invertible matrix $I + E^1_m$; all the elements of E^1_m are zero except in the first line, mth column where we have: $\frac{h_{1m-1}}{h_{m-1}}$; we denote:

$$\begin{aligned} c'^1_j &= \frac{h_1 h_{m-1} - |h_{1m-1}|^2}{h_j h_m},\ j \neq 2,\ j \neq m-1, \\ k'^1_2 &= \frac{h_1 h_{m-1} - |h_{1m-1}|^2}{h_2 h_m}. \end{aligned}$$

We obtain the:

Lemma 3.11. *If* $\forall k$, $c_{1k} = 0$, *then:*

i) $e^k_{i\ell} = f^k_{i\ell} = 0$,

ii) $\phi^1_2(\xi') = k'^1_2\left(\sum_{k,\ell} c^{2k}_{1\ell}\xi^\ell_k + \sum_{k,\ell} d^{2k}_{1\ell}\eta^\ell_k - i\eta^2_1\right)$
$+ \sum_{k'} c^{2k'}_{1m-1}\xi^{m-1}_{k'} + \sum_{k'} d^{2k'}_{1m-1}\eta^{m-1}_{k'} + c^{1m-1}_{2m}\xi^m_{m-1} + d^{1m-1}_{2m}\eta^m_{m-1}$
$+ i\left(e^{1m-1}_{2m}\xi^m_{m-1} + f^{1m-1}_{2m}\eta^m_{m-1}\right)$
$(k,\ell) \neq (m-1,k')$ *and* $(k,\ell) \neq (m-1,m)$; $1 \leq k' \leq m-2$.

$$
\begin{aligned}
3\le j\le m-2,\quad & \phi_j^1(\xi')=c_j^1\left(\xi_1^j-i\eta_1^j\right),\\
& \phi_{m-1}^1(\xi')=c_{m-1}^1\left(\xi_1^{m-1}-i\eta_1^{m-1}\right)+c_{m-1m}^{1m-1}\xi_{m-1}^m\\
& +d_{m-1m}^{1m-1}\eta_{m-1}^m+i\left(e_{m-1m}^{1m-1}\xi_{m-1}^m+f_{m-1m}^{1m-1}\eta_{m-1}^m\right),\\
1\le i\le 2,\quad & \phi_m^i(\xi')=c_m^i(\xi_i^m-i\eta_i^m)+\sum_k c_{mm-1}^{ik}\xi_k^{m-1}+c_{mm}^{im-1}\xi_{m-1}^m\\
& +\sum_k d_{mm-1}^{ik}\eta_k^{m-1}+d_{mm}^{im-1}\eta_{m-1}^m+i\Big(\sum_k e_{mm-1}^{ik}\xi_k^{m-1}\\
& +e_{mm}^{im-1}\xi_{m-1}^m+\sum_k f_{mm-1}^{ik}\eta_k^{m-1}+f_{mm}^{im-1}\eta_{m-1}^m\Big),\\
& 1\le k\le m-2,\\
3\le i\le m-2,\quad & \phi_m^i(\xi')=c_m^i\left(\xi_i^m-i\eta_i^m\right),\\
& \phi_m^{m-1}(\xi')=c_m^{m-1}\left(\xi_{m-1}^m-i\eta_{m-1}^m\right)+c_m^{m-1}\xi_{m-1}^m\\
& +d_m^{m-1}\eta_{m-1}^m+i\left(e_m^{m-1}\xi_{m-1}^m+f_m^{m-1}\eta_{m-1}^m\right),\\
3\le j\le m-2,\quad & \phi_j^2(\xi')=c_j^2\left(\xi_i^j-i\eta_i^j\right)+c_{jm}^{2m-1}\xi_{m-1}^m\\
& +d_{jm}^{2m-1}\eta_{m-1}^m+i\left(e_{jm}^{2m-1}\xi_{m-1}^m+f_{jm}^{2m-1}\eta_{m-1}^m\right),\\
& \phi_{m-1}^2(\xi')=c_{m-1}^2(\xi_2^{m-1}-i\eta_2^{m-1})+\sum_k c_{m-1m}^{2k}\xi_k^m\\
& +\sum_k d_{m-1m}^{2k}\eta_k^m+i\left(\sum_k e_{m-1m}^{2k}\xi_k^m+\sum_k f_{m-1m}^{2k}\right)\eta_k^m,\\
& 1\le k\le m-1,\\
3\le i<j\le m-1,\quad & \phi_j^i(\xi')=c_j^i(\xi_i^j-i\eta_i^j)+\sum_k c_{jm}^{ik}\xi_k^m+\sum_k d_{jm}^{ik}\eta_k^m\\
& +i\left(\sum_k e_{jm}^{ik}\xi_k^m+\sum_k f_{jm}^{ik}\eta_k^m\right),\\
& k\in\{1,2\},
\end{aligned}
$$

iii) $c_m'^1=k_2'^1c_m^2,\ c_m'^1=c_j'^1c_m^j,\ c_m^2=c_j^2c_m^j,\ c>0.$

Lemma 3.12. *We assume* $\forall k,\ c_{1k}=0$

$$\phi_m^1(\xi')=c_m^1(\xi_1^m-i\eta_1^m).$$

Proof. In det $a(\xi')$ the coefficient of $\prod_u \chi_u$ is real. $\square$

Lemma 3.13. *We assume* $\forall k,\ c_{1k} = 0$,

$$\phi_2^1(\xi') = k_2^1 \phi_1^2(\xi'),\ \phi_m^2(\xi') = c_m^2(\xi_2^m - i\eta_2^m).$$

Proof. Let: $\xi_i^j = \eta_i^j = 0$, $j \geq 3$, $i \in \{1,2\}$, as in the Lemma 3.8; we obtain:

$$c_{2m}^{1m-1} = k_2'^1 c_{1m}^{2m-1},\ d_{2m}^{1m-1} = k_2'^1 d_{1m}^{2m-1},\ e_{2m}^{1m-1} = f_{1m}^{2m-1} = 0.$$

The coefficient of $\xi_0 \prod_{k\neq 2} \chi_k$ in det $a(\xi)$ is real; we deduce $k_2^1 = k_2'^1$ and the value of ϕ_2^1; then we obtain ϕ_m^2. $\square$

Lemma 3.14. *If* $\forall k,\ c_{1k} = 0$,
$3 \leq j \leq m-1,\ \phi_j^2(\xi') = c_j^2(\xi_2^j - i\eta_2^j),\quad 3 \leq i < j \leq m-1,\ \phi_j^i(\xi') = c_j^2(\xi_i^j - i\eta_i^j).$

Proof. We consider in det $a(\xi)$ the reality of the coefficient of $\xi_0^2 \prod_k \chi_k$, $k \neq k'$. $\square$

Lemma 3.15. *If* $\forall k,\ c_{1k} = 0$, *a is prehermitian.*

Proof. Let $\xi_i^j = \eta_i^j = 0$, except: ξ_1^{m-1}, η_1^{m-1}, we construct a multiple zero in ξ_0 in det $a(\xi)$ and we get: $c_{mm-1}^{m-11} = d_{mm-1}^{m-11} = e_{mm-1}^{m-11} = f_{mm-1}^{m-11} = 0$. Considering the relations satisfied by the c_j^i and k_2^1, we obtain easily the result. $\square$

The case II_2'' is reducible to the case II_2': we transform $a(\xi)$ by the unitary diagonal matrix where the entry in the first line, first column is equal to the complex number i; the others are 1.

4. Case III_1

Two forms Re ϕ_j^i, Im $\phi_{i'}^{j'}$, $i > j$, $i' > j'$ depend linearly on the other forms; the forms χ_i are independent.

We distinguish as the first case, the case where Re ϕ_2^3 and Im ϕ_2^3 depend linearly on the other forms Re ϕ_j^i, Im ϕ_i^j, $i > j$; we denote this case III_1. The cases where Re ϕ_j^i and Im $\phi_{i'}^{j'}$, $i > j$, are dependent can be reduced to this case.

In the case III_1, by a convenient choice of coordinates, we can denote: If: $i > j$, $(i,j) \neq (3,2)$, $\phi_j^i(\xi') = \xi_i^j + i\eta_i^j$,

$$\begin{aligned}
\phi_2^3(\xi') &= \sum_{k,\ell} c_{2\ell}^{3k}\xi_k^\ell + \sum_{k,\ell} d_{2\ell}^{3k}\eta_k^\ell + i\left(\sum_{k,\ell} e_{2\ell}^{3k}\xi_k^\ell + \sum_{k,\ell} f_{2\ell}^{3k}\eta_k^\ell\right),\\
\phi_i^i(\xi') &= \xi_0 + \chi_i + i\left(\sum_{k,\ell} e_{i\ell}^{k}\xi_k^\ell + \sum_{k,\ell} f_{i\ell}^{k}\eta_k^\ell\right),\\
&\quad \ell > k,\ (\ell,k) \neq (3,2).
\end{aligned}$$

Lemma 4.1. $e^k_{i\ell} = f^k_{i\ell} = 0$.

P r o o f. We can assume $\chi_m = 0$; the coefficient of $\prod \chi_u$ in det $a(\xi')$ is real; so: $e^k_{m\ell} = 0$. The coefficient of $\xi_0 \prod_{u\neq v} \chi_u$ in det $a(\xi)$ is real; so $e^\ell_{vk} = 0$, $1 \le v \le m-1$.

We have also $i < j$,

$$\phi^i_j(\xi') = \sum_{k,\ell} c^{ik}_{j\ell}\xi^\ell_k + \sum_{k,\ell} d^{ik}_{j\ell}\eta^\ell_k + i\left(\sum_{k,\ell} e^{ik}_{j\ell}\xi^\ell_k + \sum_{k,\ell} f^{ik}_{j\ell}\eta^\ell_k\right).$$

We denote: $c^i_j = c^{ii}_{jj}$, $i < j$. □

Lemma 4.2. *$\forall(i,j)$, $i<j$, $(i,j)\neq(2,3)$:*

i) $\phi^i_j(\xi') = c^i_j(\xi^j_i - i\eta^j_i)$.

ii) $\big(\sum c^{3k}_{2\ell}\xi^\ell_k + \sum d^{3k}_{2\ell}\eta^\ell_k\big)\big(\sum e^{2k}_{3\ell}\xi^\ell_k + \sum f^{2k}_{3\ell}\eta^\ell_k\big)$
$+\ \big(\sum c^{2k}_{3\ell}\xi^\ell_k + \sum d^{2k}_{3\ell}\eta^\ell_k\big)\big(\sum e^{3\ell}_{2k}\xi^\ell_k + \sum f^{3\ell}_{2k}\eta^\ell_k\big) = 0.$

P r o o f. We consider the coefficient of $\prod_{u\neq u'}\chi_u$ in det $a(\xi')$ and we have the lemma for ϕ^i_m.

We consider the coefficient of $\xi_0\prod\chi_k$ in det $a(\xi)$, where $k\in K$; K subset of $\{1,2,\dots,m-1\}$ such that:

$$\text{card } K = m-3.$$

If $K\neq C\{2,3\}$, we obtain i), if $K = C\{2,3\}$, we obtain ii). □

Lemma 4.3. *$a(\xi)$ is prehermitian.*

P r o o f. We distinguish four cases:

i) $\sum c^{3\ell}_{2k}\xi^k_\ell + \sum d^{3\ell}_{2k}\eta^k_\ell \not\equiv 0$,

i$_1$) and divides $\sum c^{2\ell}_{3k}\xi^k_\ell + \sum d^{2\ell}_{3k}\eta^k_\ell$.

Then $\phi^2_3 = k^2_3\overline{\phi}^3_2$, $k^2_3\in R$.

Let: $\xi^m_1 = \eta^m_1 = \dots = \xi^m_{m-1} = \eta^m_{m-1} = 0$; the $(m-1)\times(m-1)$ matrix b obtained by cancelling the last line and the last column of $a(\xi)$ is diagonalizable and its reduced dimension is more than: $(m-1)^2-2$; by induction b is prehermitian and thanks to the Lemma 2.4, there exists b such that:

$$bH = H^t\overline{b};$$

we verify H is diagonal and we obtain if: $1\le i<j\le m-1$ then

$$c^i_j = c^i_k c^k_j,\ (i,k)\neq(2,3),\ c^2_j = k^2_3 c^3_j,\ c^i_j > 0,\ k^2_3 > 0. \tag{4}$$

By an analogous manner, we obtain (3) if: $2\le i<j\le m$.

Then, as usually, we obtain: $a(\xi)$ is prehermitian.

i$_2$) $\sum c_2^3 \cdots + \sum d_2^3 \cdots$ divides $\sum e_2^3 \cdots + \sum f_2^3 \cdots$. We obtain

$$\begin{aligned}\phi_2^3(\xi') &= \big(\sum c_{2k}^{3\ell}\xi_\ell^k + \sum d_{2k}^{3\ell}\eta_\ell^k\big)(1+i\lambda),\\ \phi_3^2(\xi') &= \big(\sum c_{3k}^{2\ell}\xi_\ell^k + \sum d_{3k}^{2\ell}\eta_\ell^k\big)(1-i\lambda),\end{aligned}$$

$\lambda \in R$.

As in the case i$_1$), we obtain:

$$\phi_3^2 = k_3^2\overline{\phi}_2^3,$$

the relations satisfied by the c_j^i and k_3^2 and $a(\xi)$ is prehermitian.

ii) $\sum c_2^3 \cdots + \sum d_2^3 \cdots \equiv 0$,

ii$_1$) $\sum e_2^3 \cdots + \sum f_2^3 \cdots \not\equiv 0$.

Then $\sum c_3^2 \cdots + \sum d_3^2 \cdots \equiv 0$ as in the cases i) we obtain:

$$\phi_3^2 = k_3^2\overline{\phi}_2^3$$

and the same results.

ii$_2$) $\phi_2^3(\xi') \equiv 0$.

We obtain $\phi_3^2(\xi') \equiv 0$. We construct relations between the c_j^i and by an easy calculus, we obtain: $a(\xi)$ is prehermitian. □

5. Case III$_2$

Two forms Re ϕ_j^i, Im ϕ_j^i, $i > j$ in the same line or in the same column depend on the others forms, the χ_i are independent, $\chi_m = 0$.

In this case by a convenient choice of coordinates, we can denote:
If: $i > j$, $(i,j) \neq (3,2)$, $(i,j) \neq (4,2)$,

$$\begin{aligned}\phi_j^i(\xi') &= \xi_i^j + i\eta_i^j,\\ \phi_2^3(\xi') &= \sum_{k,\ell} c_{2\ell}^{3k}\xi_k^\ell + \sum_{k,\ell} d_{2\ell}^{3k}\eta_k^\ell + d_{23}^{32}\eta_2^3 + i\eta_2^3,\\ \phi_4^2(\xi') &= \sum_{k,\ell} c_{2\ell}^{4k}\xi_k^\ell + \sum_{k,\ell} d_{2\ell}^{4k}\eta_k^\ell + d_{24}^{42}\eta_2^4 + i\eta_2^4.\end{aligned}$$

We have, as in Lemmas 4.1, 4.2:

$$\phi_i^i(\xi) = \xi_0 + \chi_i$$

and: $\forall(i,j)$, $i<j$, $(i,j)\neq(2,3)$, $(i,j)\neq(2,4)$, $\phi^i_j(\xi')=c^i_j(\xi^j_i-i\eta^j_i)$. We have also:

$$\mathrm{Im}\Big[\left(\sum_{k,\ell}c^{ik}_{2\ell}\xi^\ell_k+\sum d^{ik}_{2\ell}\eta^\ell_k d^{i2}_{2i}\eta^i_i\right)\left(\sum_{k,\ell}e^{2k}_{i\ell}\xi^\ell_k+\sum f^{2k}_{i\ell}\eta^\ell_k f^{2i}_{ii}\eta^i_2\right)$$
$$+\eta^i_2\left(\sum_{k,\ell}c^{2k}_{i\ell}\xi^\ell_k+\sum d^{2k}_{i\ell}\eta^\ell_k d^{22}_{ii}\eta^i_2\right)\Big]=0$$

for $i\in\{3,4\}$.

We deduce: $i\in\{3,4\}$ $\quad\phi^2_i(\xi')=k^2_i\overline{\phi}^i_2(\xi')$ as in the Lemma 4.3, we obtain that $a(\xi)$ is prehermitian (we have only to pay attention to the special case $m=4$).

6. Case III$_3$

In this case, by a convenient choice of coordinates, we can denote:
If $i>j$, $(i,j)\neq(3,2)$, $(i,j)\neq(4,3)$,

$$\begin{aligned}\phi^i_j(\xi') &= \xi^j_i+i\eta^j_i\\ \phi^3_2(\xi') &= \sum_{k,\ell}c^{3k}_{2\ell}\xi^\ell_k+\sum_{k,\ell}d^{3k}_{2\ell}\eta^\ell_k+d^{32}_{23}\eta^3_2+i\eta^3_2,\\ \phi^3_4(\xi') &= \sum_{k,\ell}c^{4k}_{3\ell}\xi^\ell_k+\sum_{k,\ell}d^{4k}_{3\ell}\eta^\ell_k+d^{43}_{34}\eta^4_3+i\eta^4_3.\end{aligned}$$

We have, as before:

$$\begin{aligned}i<j,\ \phi^i_j(\xi') &= c^i_j(\xi^j_i-i\eta^j_i),\ (i,j)\neq(2,3),\ (i,j)\neq(3,4),\\ \phi^2_3(\xi') &= k^2_3\overline{\phi}^3_2(\xi');\quad \phi^2_4(\xi')=k^2_4\overline{\phi}^4_2(\xi').\end{aligned}$$

We obtain finally: $a(\xi)$ is prehermitian; (we pay attention to the case $m=4$).

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