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RICCATI REPRESENTATION FOR ELEMENTS IN $H^{-1}(\mathbb{T})$ AND ITS APPLICATIONS

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Abstract. This paper is concerned with the spectral properties of the Schrödinger operator $L_q \overset{\text{def}}{=} -\frac{d^2}{dx^2} + q$ with periodic potential q from the Sobolev space $H^{-1}(\mathbb{T})$. We obtain asymptotic formulas and a priori estimates for the periodic and Dirichlet eigenvalues which generalize known results for the case of potentials $q \in L_0^2(\mathbb{T})$. The key idea is to reduce the problem to a known one – the spectrum of the impedance operator – via a nonlinear analytic isomorphism of the Sobolev spaces $H_0^{-1}(\mathbb{T})$ and $L_0^2(\mathbb{T})$.

1. Introduction

The present paper is devoted to the spectral properties of the Schrödinger operator $L_q \overset{\text{def}}{=} -\frac{d^2}{dx^2} + q$ with "singular" potential q from the Sobolev space $H^{-1}(\mathbb{T})$ viewed as an unbounded operator on $H^{-1}(\mathbb{T})$. Our approach is based on a nonlinear representation - clearly of independent interest - of the elements of the Sobolev space $H^{-1}(\mathbb{T})$ in terms of a unique function from $L_0^2(\mathbb{T})$, referred to as Riccati representation.

Let $\rho \in H^1(\mathbb{T})$ with $\min_{x\in\mathbb{T}} \rho(x) > 0$ and define $r \overset{\text{def}}{=} \rho'/\rho$. By the transformation $y \overset{\text{def}}{=} \rho\tilde{y}$ the equation $-\tilde{y}'' - 2r\tilde{y}' = \lambda\tilde{y}$ becomes the Schrödinger equation $-y'' + qy = \lambda y$ with $q = r' + r^2$. It was conjectured in [3] that the spectral properties of the Schrödinger operator L_q could be deduced from the spectral

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properties of the impedance operator $T_r(u) \stackrel{\text{def}}{=} -(\rho^2 u')'/\rho^2$, whose spectral theory is well-developed ([13, 1, 2, 9]). In the present paper we prove that for any $\alpha \geq 0$ the *Riccati map* $r \mapsto r' + r^2 - ||r||^2$ maps $H_0^\alpha(\mathbb{T})$ onto $H_0^{\alpha-1}(\mathbb{T})$ where $||r||^2 = \int_0^1 r^2(x)\, dx$ and $H_0^\alpha(\mathbb{T}) \stackrel{\text{def}}{=} \{f \in H^\alpha(\mathbb{T}) \mid \int_0^1 f\, dx = 0\}$. In fact it is a real-analytic isomorphism between $H_0^\alpha(\mathbb{T})$ and $H_0^{\alpha-1}(\mathbb{T})$ (see Section 3.). Our proof is elementary and based on the spectral properties of the Schrödinger operator. In particular, the properties of the first eigenvalue $\lambda_0(q)$ of the Schrödinger operator and the corresponding normalized eigenfunction $f_0(q)$ are essential for our approach.

The main result of this paper is summarized in the following theorem.

Theorem 1. *The Riccati map $R : L_0^2(\mathbb{T}) \to H_0^{-1}(\mathbb{T})$ is a real analytic isomorphism. For any $r \in L_0^2(\mathbb{T})$, the impedance operator T_r and the associated Schrödinger operator L_q with $q = R(r)$ have, up to a translation by $||r||^2$, the same periodic and Dirichlet spectrum. Moreover, $R^{-1}(q)$ is given by*

$$R^{-1}(q) = f_0'(\cdot, q)/f_0(\cdot, q)$$

where $f_0(\cdot, q)$ is an eigenfunction (in $H^1(\mathbb{T})$) corresponding to the first (lowest) periodic eigenvalue $\lambda_0(q)$ of L_q which can be proved to vanish nowhere.

The paper is organized as follows. In Section 2. we prove auxiliary results needed for the proof of Theorem 1 in Section 3.. Section 4. is devoted to applications of the Riccati representation needed in an essential way in subsequent work [4, 7]. In Section 4.1. we prove spectral results for the Hill operator L_q. The Dirichlet spectrum of L_q is considered in Section 4.2.. The results there generalize results of [17] to the case of singular potentials $q \in H^{-1}(\mathbb{T})$. Finally, we extend the notion of discriminant to potentials in $H^{-1}(\mathbb{T})$ and prove the isospectral invariance of the Riccati map (Section 4.4.). To make the paper self-contained we include Appendices A and B where we present results on the spectrum of T_r and L_q used in this paper.

The spectral theory of operators L_q with singular potentials $q \in H^\alpha(\mathbb{T})$, $\alpha > -1$, was developed in [3] and [5] and was partly motivated by the construction of action-angle coordinates on the phase space $H^\alpha(\mathbb{T})$ for the Korteweg-de Vries equation (KdV) – see the introduction in [3]. In [4] we extend the construction of action-angle coordinates for potentials in $H^{-1}(\mathbb{T})$ using the applications of Theorem 1 stated in Section 4.. In the subsequent paper [7] the Birkhoff coordinates are used to prove that KdV is well posed in $H^{-1}(\mathbb{T})$.

In the stage of finishing this paper we were informed by E. Korotyaev that at the same time but by different proofs he showed that the Riccati map is a real analytic isomorphism. Being interested rather in spectral problems than in applications to KdV, he used this result to solve an inverse problem ([11]). Note that there are no results in [11] concerning the Dirichlet spectrum of L_q.

This paper is an abridged version of [6].

The following notations are used throughout the paper. Denote by $h^{\beta,n}$ ($\beta \in \mathbb{R}, n \in \mathbb{Z}$) the Hilbert space of sequences $\{x_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ with finite norm $||x||_{\beta,n} \stackrel{\text{def}}{=} (\sum_{k\in\mathbb{Z}} \langle k+n\rangle^{2\beta} |x_k|^2)^{1/2}$, where $\langle s\rangle \stackrel{\text{def}}{=} |s|+1$. The scalar product in $h^{\beta,n}$ is given by $(x,y)_{\beta,n} \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} \langle k+n\rangle^{2\beta} x_k \bar{y}_k$. By definition, $h^\beta \stackrel{\text{def}}{=} h^{\beta,0}$.

By $\mathbb{T}_l$, $l>0$, we denote the one-dimensional torus $\mathbb{T}_l \stackrel{\text{def}}{=} \mathbb{R}/l\mathbb{Z}$. The Sobolev spaces $H^m(\mathbb{T}_l)$ and $H^m[0,1]$, $m \in \mathbb{N}$, are defined by $H^m(\Omega) \stackrel{\text{def}}{=} \{f : \Omega \to \mathbb{R} \mid f^{(k)} \in L^2(\Omega),\ k=0,1,...,m\}$ where $f^{(k)}$ is the k'th distributional derivative of the function f and Ω denotes the torus $\mathbb{T}_l$ or the interval $[0,1]$ respectively. The scalar product in $H^m(\Omega)$ is defined by $(f,g)_m \stackrel{\text{def}}{=} \sum_{k=0}^m (f^{(k)}, g^{(k)})$ where $(\cdot,\cdot)$ denotes the L^2-scalar product. For real $\alpha \geq 0$ the Sobolev spaces $H^\alpha(\mathbb{T}_l)$ and $H^\alpha[0,1]$ can be defined in a standard manner, for example, by interpolation (see [14], Chapter 1). By definition, $H^{-\alpha}(\Omega)$ is the dual space of $H^\alpha_c(\Omega)$, i.e. $H^{-\alpha}(\Omega) \stackrel{\text{def}}{=} (H^\alpha_c(\Omega))'$ where $H^\alpha_c(\Omega)$ is the closure in $H^\alpha(\Omega)$ of the space $C^\infty_c(\Omega)$ of smooth functions with compact support in the interior of Ω. The norm of f in $H^\alpha(\Omega)$ is denoted by $||f||_\alpha$ and for $\alpha=0$ we write $||f|| = ||f||_0$. The distributional derivative $\frac{d}{dx} : H^m(\Omega) \to H^{m-1}(\Omega)$, $m \in \mathbb{N}$, can be extended in a natural way to a bounded operator $\frac{d}{dx} : H^\alpha(\Omega) \to H^{\alpha-1}(\Omega)$ for arbitrary $\alpha \in \mathbb{R}$. By $H^\alpha_0(\mathbb{T}_l)$ we denote the linear subspace of elements $f \in H^\alpha(\mathbb{T}_l)$ with mean value zero, $[f] \stackrel{\text{def}}{=} \int_{\mathbb{T}_l} f dx = 0$. Note that the Sobolev spaces $H^\alpha(\mathbb{T}_l)$, $\alpha \in \mathbb{R}$, can be identified (up to an equivalence of the norms) with the space of Fourier series $\sum_{k\in\mathbb{Z}} \hat{f}_k e^{2i\pi x/l}$ whose Fourier coefficients $\{\hat{f}_k\}_{k\in\mathbb{Z}}$ have finite h^α-norm.

2. L_q associated to T_r

For any given $r \in L^2_0(\mathbb{T})$ denote by T_r the impedance operator

$$T_r(u) \stackrel{\text{def}}{=} -\frac{1}{\rho^2}(\rho^2 u')' = -u'' - 2ru' \tag{1}$$

on $L^2(\mathbb{T}_2)$ with domain $\texttt{Dom}(T_r) = H^2(\mathbb{T}_2)$. Here ρ is the absolutely continuous, 1-periodic, positive function given by $\rho(x) \stackrel{\text{def}}{=} \exp\left(\int\limits_0^x r(v)dv\right)$. In particular, $\rho \in H^1(\mathbb{T})$ and $\rho' = r\rho$. Note that T_r is an operator with compact resolvent, non-negative, and symmetric with respect to the inner product $(f,g)_\rho \stackrel{\text{def}}{=} \int_0^2 fg\rho^2 dx$

on $L^2(\mathbb{T}_2)$. Hence the spectrum $\mathtt{spec}(T_r)$ is discrete, real and non-negative. It turns out to be of the form $\mathtt{spec}(T_r) = \{0 = \tilde\lambda_0(r) < \tilde\lambda_1(r) \le \tilde\lambda_2(r) \le ...\}$. The corresponding eigenspaces are of finite dimension, and $\tilde\lambda_k(r) \to \infty$ as $k \to \infty$ (see Appendix A).

For any $q \in H^{-1}(\mathbb{T})$ we denote by L_q the Hill operator

$$L_q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q \tag{2}$$

viewed as an operator on the space $H^{-1}(\mathbb{T}_2)$ with domain $\mathtt{Dom}(L_q) = H^1(\mathbb{T}_2)$. The classical spectral theory of Hill's operator can be extended for such singular potentials (see Appendix B). It is proved in Appendix B, Lemma 6, that the spectrum of L_q is discrete, real, and of the form $\mathtt{spec}(L_q) = \{\lambda_0(q) \le \lambda_1(q) \le \lambda_2(q) \le ...\}$, the corresponding eigenspaces are of finite dimension, and $\lambda_k(q) \to \infty$ as $k \to \infty$.

Clearly, for any $r \in L_0^2(\mathbb{T})$, $r^2 - ||r||^2$ defines a bounded linear functional on $H^1(\mathbb{T})$ satisfying $\langle r^2 - ||r||^2, 1\rangle = 0$ where $||r||^2 \stackrel{\text{def}}{=} \int_0^1 r^2(x)dx$. Hence, $r^2 - ||r||^2$ is an element in $H_0^{-1}(\mathbb{T})$ and one can introduce the nonlinear map

$$R : L_0^2(\mathbb{T}) \to H_0^{-1}(\mathbb{T}),\ r \mapsto r' + r^2 - ||r||^2, \tag{3}$$

referred to as the Riccati map. The following result shows how the operators T_r and L_q are related if $q = R(r)$. It can be shown in a straightforward way (see [6]).

Lemma 1. *Let $r \in L_0^2(\mathbb{T})$ and assume that $q \in H_0^{-1}(\mathbb{T})$ satisfies Riccati's equation $q = R(r)$. Then*

(a) $\mathtt{spec}(T_r) = ||r||^2 + \mathtt{spec}(L_q)$;

(b) for any $k \ge 0$, the eigenspaces $V_{\lambda_k}(L_q)$ and $V_{\lambda_k+||r||^2}(T_r)$ have the same dimension;

(c) the $L^2(\mathbb{T})$-norm of r coincides with the absolute value of the first eigenvalue $\lambda_0(q)$ of Hill's operator L_q, i.e. $||r||^2 = |\lambda_0(q)| = -\lambda_0(q)$;

(d) the first eigenvalue $\lambda_0(q)$ of the operator L_q is simple and the corresponding eigenfunction normalized by $||f_0||^2 = 1$ and $f_0(0) > 0$ is $f_0 \stackrel{\text{def}}{=} \rho/||\rho||$. Hence, f_0 is in $H^1(\mathbb{T})$, doesn't vanish on $\mathbb{T}$ and satisfies $r = \frac{f_0'}{f_0}$.

Lemma 1 (d) implies the following result:

Corollary 1.. *The Riccati map $R : L^2_0(\mathbb{T}) \to H^{-1}_0(\mathbb{T})$ is injective. Moreover, if $q \in \texttt{range}(R)$ then $R^{-1}(q) = f_0'(\cdot, q)/f_0(\cdot, q)$.*

Remark 1.. *Note that the quotient f_0'/f_0 is independent of the normalization of f_0.*

3. Riccati's map

By definition, Riccati's map $R : L^2_0(\mathbb{T}) \to H^{-1}_0(\mathbb{T})$ is given by formula (3). For any $\alpha \geq 0$ denote by R_α the restriction of R to $H^\alpha_0(\mathbb{T}) \subset L^2_0(\mathbb{T})$, $R_\alpha \stackrel{\text{def}}{=} R|_{H^\alpha_0(\mathbb{T})}$.

Proposition 1.. *The Riccati map $R_\alpha : H^\alpha_0(\mathbb{T}) \to H^{\alpha-1}_0(\mathbb{T})$ is a diffeomorphism from $H^\alpha_0(\mathbb{T})$ to $H^{\alpha-1}_0(\mathbb{T})$.*

Proof. Let us consider first the case $\alpha = 0$. By definition, $R_0 = R$. First note that R is continuous. The claimed statement then follows from the following three assertions:

(i) R has dense image;

(ii) R is surjective (and hence bijective by Corollary 1);

(iii) R and R^{-1} are differentiable.

Let $q \in C^\infty_0(\mathbb{T}) \stackrel{\text{def}}{=} \{f \in C^\infty(\mathbb{T}) | \int_0^1 f(x)dx = 0\}$ and $\lambda_0(q)$ be the first eigenvalue of L_q. By the classical theory of Hill's equation, the corresponding eigenfunction f_0 doesn't have zeroes (see [15]). Hence, $L_q f_0 = \lambda_0(q) f_0$ can be rewritten as $q - \lambda_0(q) = \frac{f_0''}{f_0} = r' + r^2$ where $r \stackrel{\text{def}}{=} \frac{f_0'}{f_0}$. Integrating the last equality one gets $\lambda_0(q) = -||r||^2$, and therefore, $q = r' + r^2 - ||r||^2 = R(r)$. As $C^\infty_0(\mathbb{T})$-functions are dense in $H^{-1}_0(\mathbb{T})$, the image of the Riccati map is dense in $H^{-1}_0(\mathbb{T})$ proving item (i).

To prove item (ii) take an arbitrary $q \in H^{-1}_0(\mathbb{T})$. It follows from item (i) that there exists a sequence $\{q_k\}_{k=1}^\infty \subset \texttt{range}(R)$ such that $q_k \to q$ $(k \to \infty)$ in $H^{-1}_0(\mathbb{T})$. Consider the sequence $\{r_k\}_{k=1}^\infty \subset L^2_0(\mathbb{T})$ such that $q_k = R(r_k)$. Item (c) of Lemma 1 shows that $||r_k||^2 = |\lambda_0(q_k)|$ where $\lambda_0(q_k)$ is the first eigenvalue of the Hill operator $L_{q_k} = -\frac{d^2}{dx^2} + q_k$. As a function of the potential $q \in H^{-1}_0(\mathbb{T})$ the first eigenvalue $\lambda_0(q)$ is continuous on $H^{-1}_0(\mathbb{T})$ – see Appendix B, Lemma 6. Therefore, $||r_k|| \to \sqrt{|\lambda_0(q)|}$ as $k \to \infty$. Hence, the sequence $\{r_k\}_{k=1}^\infty$ is bounded in $H^0(\mathbb{T})$.

Consider the map $S_0 : H^0(\mathbb{T}) \to H^{-1}(\mathbb{T})$, $r \mapsto r^2$. This map can be viewed as a composition of two maps $S_0 = \imath \circ S_1$, where $S_1 : H^0(\mathbb{T}) \to H^{-3/4}(\mathbb{T})$ is given by the formula $r \mapsto r^2$, and $\imath : H^{-3/4}(\mathbb{T}) \hookrightarrow H^{-1}(\mathbb{T})$ is the standard inclusion of Sobolev spaces. By Rellich's theorem $\imath$ is compact. As each Fourier coefficient $(\hat{r^2})_n$ of r^2 satisfies $|(\hat{r^2})_n| \le ||r||^2$ $(n \in \mathbb{Z})$ it follows that there exists $C > 0$ so that $||r^2||_{-3/4} \le C||r||^2$, hence the map S_1 is bounded. Therefore, there exists a subsequence $\{r_{k_j}\}_{j=1}^{\infty}$ of $\{r_k\}$ and an element $g \in H^{-1}(\mathbb{T})$ such that $r_{k_j}^2 \to g$ $(j \to \infty)$ in $H^{-1}(\mathbb{T})$. By the definition of Riccati's map, $r'_{k_j} = q_{k_j} - r_{k_j}^2 + ||r_{k_j}||^2 \in H_0^{-1}(\mathbb{T})$. Each of the terms on the right hand side of the latter equation converges in $H^{-1}(\mathbb{T})$. Hence r'_{k_j} converges to some element $s \in H_0^{-1}(\mathbb{T})$. Denote by r the unique element in $L_0^2(\mathbb{T})$ such that $s = r'$. As $||r - r_{k_j}|| \le const\, ||r' - r'_{k_j}||_{-1} \to 0$ for $j \to \infty$ and R is continuous it then follows that $q_{k_j} = R(r_{k_j}) \to R(r)$ $(j \to \infty)$ in $H^{-1}(\mathbb{T})$. Therefore, $q = R(r)$ and claim (ii) is proved.

Towards claim (iii) note that R is continuously differentiable. The corresponding property for R^{-1} follows from the identity $R^{-1}(q) = \frac{f_0'(\cdot,q)}{f_0(\cdot,q)}$ (see Lemma 1 (d)) as follows: the first eigenfunction $f_0(\cdot,q)$ considered as a map $H^{-1}(\mathbb{T}) \to H^1(\mathbb{T}_2)$ is continuously differentiable (in fact real analytic) in the variable q when normalized so that $\int_0^2 f_0^2(x,q)dx = 2$ and $f_0(0,q) > 0$ (see Appendix B, Lemma 6). Note that any one-periodic function $f \in H^1(\mathbb{T}_2)$ can be isometrically identified with a function in $H^1(\mathbb{T})$. Indeed, it follows from $f(x+1) = f(x)$ that the coefficients $\hat{f}_{2k+1}$ $(k \in \mathbb{Z})$ in the Fourier expansion $f = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{i\pi kx}$ of the function f on the interval $[0,2]$ all vanish and the condition $f \in H^1(\mathbb{T}_2)$ implies that $f \in H^1(\mathbb{T})$. In particular, we see that the normalized eigenfunction $f_0(\cdot,q)$ considered as a map $H^{-1}(\mathbb{T}) \to H^1(\mathbb{T})$ is continuously differentiable (in fact real analytic). This shows that R^{-1} is continuously differentiable.

The proof of the proposition for $\alpha > 0$ is straightforward. □

The following statement is a generalization of the corresponding classical result. Attempts to prove it using the classical approach (see [15]) fail at several stages.

Corollary 2. *For any $q \in H_0^{-1}(\mathbb{T})$, the first eigenvalue $\lambda_0(q)$ of L_q is simple. Any eigenfunction corresponding to $\lambda_0(q)$ is an element in $H^1(\mathbb{T})$ and doesn't vanish on $\mathbb{T}$. Moreover $||R^{-1}(q)||$ is a spectral invariant of L_q.*

Proof. Take $r = R^{-1}(q)$. By Lemma 1 (d) the first eigenvalue $\lambda_0(q)$ of L_q is simple and $f_0(x) = \rho(x)/||\rho||$ is an eigenfunction corresponding to $\lambda_0(q)$ where $\rho(x) = \exp(\int_0^x r(v)dv)$. Hence f_0 is an element in $H^1(\mathbb{T})$ and doesn't vanish on $\mathbb{T}$. It follows from Lemma 1 (c) that $||R^{-1}(q)||^2 = -\lambda_0(q)$ which is obviously a spectral invariant. □

Denote by $H_0^\alpha(\mathbb{T},\mathbb{C})$ the complexification of the (real) Sobolev space $H_0^\alpha(\mathbb{T})$. For complex-valued functions $r \in H_0^\alpha(\mathbb{T},\mathbb{C})$, $\alpha \ge 0$, the (complex) Riccati map is defined by the formula $R_\alpha(r) \stackrel{\text{def}}{=} r' + r^2 - \int_0^1 r^2(x)dx$. Using the same arguments as in the real case one concludes that R_α maps $H_0^\alpha(\mathbb{T},\mathbb{C})$ into $H_0^{\alpha-1}(\mathbb{T},\mathbb{C})$ and is an analytic map. As a consequence of Proposition 1 one easily obtains

Theorem 2. *For any $\alpha \geq 0$ there exist open neighborhoods $U \subset H_0^\alpha(\mathbb{T},\mathbb{C})$ and $W \subset H_0^{\alpha-1}(\mathbb{T},\mathbb{C})$ of $H_0^\alpha(\mathbb{T})$ and $H_0^{\alpha-1}(\mathbb{T})$ respectively such that the Riccati map $R_\alpha : U \to W$ is an analytic isomorphism.*

Proof of Theorem 1. The claimed results follow from Theorem 2 and Lemma 1.

4. Applications

This section contains several applications of the Riccati representation of the elements of the Sobolev space $H^{-1}(\mathbb{T})$: we give asymptotic formulas for the spectrum of the Hill operator with a singular potential and an a priori estimate of the potential q in terms of the gap lengths. Analogous asymptotic formulas are proved for the Dirichlet spectrum. Clearly, classical Floquet theory for Hill's operator $-\frac{d^2}{dx^2} + q$ for potentials q in $L^2(\mathbb{T})$ cannot be extended to potentials in $H^{-1}(\mathbb{T})$. Nevertheless we show that some features of the Floquet theory of Hill's operator can be extended without essential changes to the case of singular potentials from the Sobolev spaces $H^{-\alpha}(\mathbb{T})$, $0 < \alpha \leq 1$.

4.1. Periodic spectrum

In this paragraph we prove four results on the spectrum of Hill's operator.

Theorem 3. *The spectrum of Hill's operator $L_q = -\frac{d^2}{dx^2}+q$ on $H^{-1}(\mathbb{T}_2)$ with singular potential $q \in H^{-1}(\mathbb{T})$ is discrete, $\mathtt{spec}(L_q) = \{\lambda_0(q) < \lambda_1(q) \leq \lambda_2(q) < ...\}$, $\lambda_k(q) \to \infty$ as $k \to \infty$. The eigenvalues are totally ordered, $\lambda_{2k-1}(q) \leq \lambda_{2k}(q)$ and $\lambda_{2k}(q) < \lambda_{2k+1}(q)$, where the equality $\lambda_{2k-1}(q) = \lambda_{2k}(q)$ means that the corresponding eigenspace has two dimensions. Otherwise, the corresponding eigenspaces are one-dimensional. The eigenvalues $\lambda_{2k-1}(q) \leq \lambda_{2k}(q)$ with k odd correspond to anti-periodic eigenfunctions, i.e. $f(x+1) = -f(x)$ and those with k even to periodic ones, i.e. $f(x+1) = f(x)$.*

Proof. By Theorem 1 there exists $r \in L_0^2(\mathbb{T})$ such that $q = R(r)$. Theorem 3 then follows from Lemma 1 and the spectral properties of the impedance operator T_r (see Appendix A, §5.1.). □

For any $k \geq 0$, denote by $\gamma_k(q) \stackrel{\text{def}}{=} \lambda_{2k}(q) - \lambda_{2k-1}(q)$ the *k'th gap-length* and by $\gamma(q)$ the sequence $\{\gamma_k(q)\}_{k\geq 1}$. The following two theorems are applications of results in [9] concerning the spectrum of the impedance operator T_r for $r \in H_0^0(\mathbb{T})$.

Theorem 4. *For any $q \in H_0^{-1}(\mathbb{T})$, $\{\gamma_k(q)\}_{k\geq 1}$ belongs to the sequence space h^{-1}.*

Theorem 5. *There exists a constant $c > 0$ such that for every potential $q \in H_0^{-1}(\mathbb{T})$*

$$||q||_{-1} \leq c||\gamma(q)||_{-1}(1 + c||\gamma(q)||_{-1})^3 \tag{4}$$

Proof of Theorems 4 and 5. Take $r \stackrel{\text{def}}{=} R^{-1}(q)$. By Lemma 1, the operators L_q and T_r have, up to a translation, the same spectrum. Hence these operators have the same gap-lengths. Theorem 4 thus follows from Theorem 1.1 in [9].

Using $q = R(r) \stackrel{\text{def}}{=} r' + r^2 - ||r||^2$, the Cauchy-Schwartz inequality, and the easily verified inequalities $||r'||_{-1} \leq ||r||$ and $||r^2||_{-1} \leq c_1||r||^2$ for some constant $c_1 > 0$, one concludes that there is a constant $c_2 > 0$ so that for any $r \in L_0^2(\mathbb{T})$, and $q \in H_0^{-1}(\mathbb{T})$ with $q = R(r)$,

$$||q||_{-1} = ||R(r)||_{-1} \leq ||r'||_{-1} + ||r^2||_{-1} + ||r||^2 \leq c_2||r||(1 + c_2||r||). \tag{5}$$

By Theorem 1.2 in [9] for the impedance operator T_r, there exists $c_3 > 0$ so that for any $r \in L_0^2(\mathbb{T})$ and $q \in H_0^{-1}(\mathbb{T})$ with $q = R(r)$

$$||r|| \leq c_3||\gamma(q)||_{-1}(1 + c_3||\gamma(q)||_{-1}). \tag{6}$$

Combining these last two estimates, Theorem 5 follows.

Remark 2. *By Lemma 1 (d) and (6) we obtain the following estimate of the first eigenvalue $\lambda_0(q)$ in the terms of the sequence of gap lengths*

$$\sqrt{|\lambda_0(q)|} \leq c_3||\gamma(q)||_{-1}(1 + c_3||\gamma(q)||_{-1}).$$

4.2. Dirichlet spectrum

Consider the operator $L_q^{Dir} = -\frac{d^2}{dx^2} + q$ on $H^{-1}[0,1] = (H_c^1[0,1])'$ with $q \in H^{-1}(\mathbb{T})$ and domain $\mathtt{Dom}(L_q^{Dir}) = H_{Dir}^1[0,1]$ (see Appendix B, §6.2.). First we need some auxiliary results which again can be proved in a straightforward way.

Lemma 2. *For given $q \in H_0^{-1}(\mathbb{T})$, let $r \stackrel{\text{def}}{=} R^{-1}(q) \in L_0^2(\mathbb{T})$ where R^{-1} is the inverse of Riccati's map. Then*

(a) $\mathtt{spec}(L_q^{Dir}) = \mathtt{spec}(T_r^{Dir}) - ||r||^2$;

(b) for any $k \geq 1$, the eigenspaces $V_{\mu_k}(L_q^{Dir})$ and $V_{\mu_k + ||r||^2}(T_r^{Dir})$ have the same dimension.

The next two theorems generalize results of [17].

Theorem 6. *The spectrum of L_q^{Dir} is discrete* $\mathtt{spec}(L_q^{Dir}) = \{\infty < \mu_1(q) < \mu_2(q) < ...\}$*, the corresponding eigenspaces are one-dimensional, and $\mu_k \to \infty$ as $k \to \infty$.*

P r o o f. The claimed results follow from the spectral properties of the operator T_r^{Dir} (see Appendix A, §5.2.) and Lemma 2. □

Theorem 7. *A sequence $\{\infty < \sigma_1 < \sigma_2 < ...\}$, $\sigma_k \to \infty$ as $k \to \infty$, is the Dirichlet spectrum of L_q^{Dir} for some $q \in H^{-1}(\mathbb{T})$, if and only if $\sigma_k = const + (k\pi + s_k)^2$ where $\{s_k\}_{k\geq 1} \in h^0$.*

Before proving Theorem 7 let us state Theorem 8. Given $q \in H_0^{-1}(\mathbb{T})$ let $r \stackrel{\text{def}}{=} R^{-1}(q)$. The eigenvalues $\tilde{\mu}_k$ of T_r^{Dir} satisfy $0 < \tilde{\mu}_1 < \tilde{\mu}_2 < ...$ and, according to Lemma 2, $\tilde{\mu}_k = \mu_k + ||r||^2$ where $\mu_{k\geq 1}$ is the spectrum of L_q^{Dir}. Introduce the sequence $\omega(q) \stackrel{\text{def}}{=} \{(\ln(\tilde{\mu}_k^{1/2})\}_{k\geq 1}$. Unlike the eigenfunctions of L_q^{Dir}, an eigenfunction $\tilde{g}_k$ of the operator T_r^{Dir} corresponding to the eigenvalue $\tilde{\mu}_k = \mu_k + ||r||^2$ is in $H^2[0,1]$. Hence we can define $\kappa(q) \stackrel{\text{def}}{=} \{\kappa_k(q)\}$, $\kappa_k(q) \stackrel{\text{def}}{=} \ln\left|\frac{\tilde{g}_k'(1)}{\tilde{g}_k'(0)}\right|$. Note that $\kappa_k(q)$ is independent of the normalization of $\tilde{g}_k$ and for q in $L_0^2(\mathbb{T})$ coincides with $\ln|g_k'(1)/g_k'(0)|$ where $g_k \in H^2[0,1]$ is an eigenfunction of L_q^{Dir} corresponding to the eigenvalue μ_k. Let us consider the map

$$H_0^{-1}(\mathbb{T}) \ni q \mapsto (\omega(q), \kappa(q)). \tag{7}$$

To discuss its properties introduce

$$S \stackrel{\text{def}}{=} \{\{\ln(k\pi + s_k)\}_{k\geq 1} | \; \{s_k\}_{k\geq 1} \in h^0, 0 < k\pi + s_k < (k+1)\pi + s_{k+1}\}.$$

Theorem 8. *The mapping $q \mapsto (\mu(q), \kappa(q))$ is a real analytic isomorphism onto $S \times h^0$.*

Proof of Theorem 7 and 8. The stated results follow directly from Corollary 5.5 and Corollary 5.6 in [2] together with Lemma 2 and Theorem 2.

4.3. Discriminant of Hill's operator

The potentials $q \in H_0^{-1}(\mathbb{T})$ are too singular for L_q to admit fundamental solutions. Hence the Floquet matrix cannot be defined in this situation. However, it turns out that the trace $\Delta(\lambda, q)$ of the Floquet matrix, referred to as discriminant, can still be defined as we will explain now. Recall that the discriminant $\tilde{\Delta}$ is defined for $\tilde\lambda \in \mathbb{C}$ and $r \in L^2_0(\mathbb{T})$ arbitrary, by $\tilde\Delta(\tilde\lambda, r) \stackrel{\text{def}}{=} u_1(1,\tilde\lambda,r) + u_2'(1,\tilde\lambda,r)$ where $u_1(x,\tilde\lambda,r)$ and $u_2(x,\tilde\lambda,r)$ are the fundamental solutions of the equation

$$-u'' - 2ru' = \tilde\lambda u. \tag{8}$$

For $q \in L^2_0(\mathbb{T})$ the discriminant $\Delta(\lambda,q)$ is well defined and related to $\tilde\Delta(\tilde\lambda,r)$ as follows. Define $r \stackrel{\text{def}}{=} R^{-1}(q)$ and $\rho(x) \stackrel{\text{def}}{=} \exp(\int_0^x r(v)dv)$. As $q \in L^2(\mathbb{T})$, the equation

$$-y'' + qy = \lambda y \tag{9}$$

admits fundamental solutions $y_1(x,\lambda,q)$ and $y_2(x,\lambda,q)$, i.e. solutions satisfying the initial conditions $y_1(0,\lambda,q) = y_2'(0,\lambda,q) = 1$ and $y_1'(0,\lambda,q) = y_2(0,\lambda,q) = 0$. It is easy to see that for $j = 1,2$, the functions $\tilde y_j(x,\tilde\lambda,r) \stackrel{\text{def}}{=} y_j(x,\lambda,q)/\rho(x)$ are solutions of (8) with $\tilde\lambda = \lambda + ||r||^2$ and $||r||^2 = \int_0^1 r^2(x)dx$. As $\rho(0) = 1$, one obtains $\tilde y_1(0,\tilde\lambda,r) = \tilde y_2'(0,\tilde\lambda,r) = 1$, $\tilde y_2(0,\tilde\lambda,r) = 0$, and $\tilde y_1'(0,\tilde\lambda,q) = -\rho'(0)$. Hence, the fundamental solutions u_1 and u_2 of equation (8) are related to $\tilde y_1$ and $\tilde y_2$ by

$$u_1(x,\tilde\lambda,r) = \tilde y_1(x,\tilde\lambda,r) + \rho'(0)\tilde y_2(x,\tilde\lambda,r)$$

and

$$u_2(x,\tilde\lambda,r) = \tilde y_2(x,\tilde\lambda,r).$$

Using that $\rho(1) = \rho(0) = 1$ and $\rho'(0) = \rho'(1)$ we obtain

$$\begin{aligned}\tilde\Delta(\tilde\lambda,r) &\stackrel{\text{def}}{=} u_1(1,\tilde\lambda,r) + u_2'(1,\tilde\lambda,r)\\ &= (y_1/\rho + \rho'(0)y_2/\rho)|_{(x=1,\lambda,q)} + (y_2'/\rho - y_2\rho'/\rho^2)|_{(x=1,\lambda,q)}\\ &= y_1(1,\lambda,q) + y_2'(1,\lambda,q) \stackrel{\text{def}}{=} \Delta(\lambda,q). \qquad (10)\end{aligned}$$

Hence, we can *define* $\Delta(\lambda,q)$ for $q \in H_0^{-1}(\mathbb{T})$ by the latter identity.

Definition 1. *For any $q \in H_0^{-1}(\mathbb{T})$ and $\lambda \in \mathbb{C}$,*

$$\Delta(\lambda,q) \stackrel{\text{def}}{=} \tilde\Delta(\lambda + ||r||^2, r) \tag{11}$$

where $r = R^{-1}(q)$, $||r||^2 = \int_0^1 r^2(x)dx$.

It follows directly from Theorem 2 and the properties of $\tilde\Delta(\tilde\lambda,r)$ that $\Delta(\lambda,q)$ is an analytic function on $\mathbb{C}\times W$ where $W \subset H_0^{-1}(\mathbb{T},\mathbb{C})$ is the neighborhood of $H_0^{-1}(\mathbb{T})$ given by Theorem 2, and that the zeroes of $\Delta(\lambda,q)^2 - 4$ are precisely the eigenvalues of L_q, i.e. for any $\lambda \in \mathbb{C}$,

$$\Delta(\lambda,q)^2 - 4 = 0 \textit{ if and only if } \lambda \in \mathtt{spec}(L_q).$$

4.4. Isospectral invariance of Riccati's map

For any $q \in H_0^{-1}(\mathbb{T})$ denote by $\mathtt{Iso}(L_q)$ the set of potentials $p \in H_0^{-1}(\mathbb{T})$ such that $\mathtt{spec}(L_p) = \mathtt{spec}(L_q)$, i.e.

$$\mathtt{Iso}(L_q) \stackrel{\text{def}}{=} \{p \in H_0^{-1}(\mathbb{T}) | \, \mathtt{spec}(L_p) = \mathtt{spec}(L_q)\}.$$

Similarly for any $r \in L_0^2(\mathbb{T})$, denote

$$\mathtt{Iso}(T_r) \stackrel{\text{def}}{=} \{u \in L_0^2(\mathbb{T}) | \, \mathtt{spec}(T_u) = \mathtt{spec}(T_r)\}.$$

Theorem 9. *For every $r \in L_0^2(\mathbb{T})$*

$$R(\mathtt{Iso}(T_r)) = \mathtt{Iso}(L_{R(r)}).$$

Proof. Let $r \in L_0^2(\mathbb{T})$ and $q \stackrel{\text{def}}{=} R(r) \in H_0^{-1}(\mathbb{T})$. To see that $\mathtt{Iso}(L_{R(r)}) \subset R(\mathtt{Iso}(T_r))$ take any $p \in \mathtt{Iso}(L_q)$ and set $u \stackrel{\text{def}}{=} R^{-1}(p)$. By Lemma 1 (a) and (d),

$$\mathtt{spec}(T_u) = -\lambda_0(p) + \mathtt{spec}(L_p) = -\lambda_0(q) + \mathtt{spec}(L_q) = \mathtt{spec}(T_r).$$

Conversely, take $u \in \mathtt{Iso}(T_r)$ and let $p \stackrel{\text{def}}{=} R(u)$. By the definition of the isospectral set $\mathtt{Iso}(T_r)$ we obtain that $\mathtt{spec}(T_u) = \mathtt{spec}(T_r)$. It follows from Lemma 1 (a) that $\mathtt{spec}(L_p) = \mathtt{spec}(T_u) - ||u||^2$ and $\mathtt{spec}(L_q) = \mathtt{spec}(T_r) - ||r||^2$. By Corollary 4 (Appendix A) the L^2-norm $||r||$ of the potential $r \in L_0^2(\mathbb{T})$ is a spectral invariant of the impedance operator T_r. Hence, it follows from $\mathtt{spec}(T_u) = \mathtt{spec}(T_r)$, that $||u||^2 = ||r||^2$ and therefore $\mathtt{spec}(L_p) = \mathtt{spec}(L_q)$. This completes the proof of Theorem 9. □

Corollary 3. *For every potential $q \in H_0^{-1}(\mathbb{T})$, the isospectral set $\mathtt{Iso}(L_q)$ is compact in $H_0^{-1}(\mathbb{T})$.*

Proof. First we prove that for any $r \in L_0^2(\mathbb{T})$ the isospectral set $\mathtt{Iso}(T_r)$ is compact. Let $\{r_k\}_{k \geq 0}$ be a sequence in $\mathtt{Iso}(T_r)$. It follows from the spectral invariance of the L^2-norm of r that $||r_k|| = ||r||$ for any $k \geq 1$ (Corollary 4). Hence, there exists a subsequence $\{r_{k_j}\}_{j \geq 1}$ that converges weakly to some element $u \in L_0^2(\mathbb{T})$. By Lemma 2.4 in [9], the sequence of discriminants $\tilde{\Delta}(\tilde{\lambda}, r_k)$ of T_{r_k} converges to $\tilde{\Delta}(\tilde{\lambda}, u)$ as $k \to \infty$ uniformly on bounded subsets of $\mathbb{C}$. On the other side, the spectral invariance of r_k shows that $\tilde{\Delta}(\tilde{\lambda}, r_k) = \tilde{\Delta}(\tilde{\lambda}, r)$ and therefore $\tilde{\Delta}(\tilde{\lambda}, r) = \tilde{\Delta}(\tilde{\lambda}, u)$. In particular, $\mathtt{spec}(T_r) = \mathtt{spec}(T_u)$ and $||u|| = ||r||$. As r_{k_j} converges weakly to u, $||u - r_{k_j}||^2 = 2||u||^2 - 2(u, r_{k_j}) \to 0$ as $j \to \infty$. Therefore, the isospectral set $\mathtt{Iso}(T_r)$ is compact.

Theorem 9 and the continuity of the Riccati map $R : L_0^2(\mathbb{T}) \to H^{-1}(\mathbb{T})$ then imply the compactness of the isospectral sets $\mathtt{Iso}(L_q)$, $q \in H_0^{-1}(\mathbb{T})$. □

4.5. Complex potentials

In a straightforward way many of the previous results can be extended for complex potentials in some open neighborhood W of $H_0^{-1}(\mathbb{T})$ in $H_0^{-1}(\mathbb{T},\mathbb{C})$. As an example we mention the following theorem which can be proved using the same arguments as in the proof of Lemma 1.

Denote by U and W the neighborhoods given by Theorem 2.

Theorem 10. *For given $q \in W \subset H_0^{-1}(\mathbb{T},\mathbb{C})$, let $r \stackrel{\text{def}}{=} R^{-1}(q) \in U \subset L_0^2(\mathbb{T},\mathbb{C})$ where R^{-1} is the inverse of Riccati's map $R : U \to W$. Then*

(a) $\texttt{spec}(L_q) = \texttt{spec}(T_r) - \int_0^1 r(x)^2 dx$;

(b) for any $k \geq 0$, the eigenspaces $V_{\lambda_k}(L_q)$ and $V_{\tilde{\lambda}_k}(T_r)$ have the same dimension where $\tilde{\lambda}_k \stackrel{\text{def}}{=} \lambda_k + \int_0^1 r(x)^2 dx$.

In view of Theorem 10 one can reformulate results on the spectrum of the impedance operator T_r with $r \in U$ in terms of the corresponding result for the operator L_q with $q = R(r) \in W$, and vice versa.

A complex analogue of Lemma 2 can be easily proved as well.

5. Appendix A: Impedance operator

An impedance operator is a Sturm-Liouville operator of a special type and is treated in numerous articles and books – see [13, 1, 2, 9]. For the convenience of the reader we recall those properties needed in the main part of this paper.

5.1. Periodic problem

For any $r \in L_0^2(\mathbb{T})$ denote by ρ the element in $H^1(\mathbb{T})$ satisfying $\rho' = r\rho$ and $\rho(0) = 1$. Then ρ is a one-periodic, absolutely continuous, positive function given by $\rho(x) = \exp(\int_0^x r(v)dv)$. The periodic impedance operator T_r is defined on the Hilbert space $L^2(\mathbb{T}_2)$ with domain $\texttt{Dom}(T_r) = H^2(\mathbb{T}_2)$ by the formula

$$T_r(u) \stackrel{\text{def}}{=} -(\rho^2 u')'/\rho^2 = -u'' - 2ru'. \tag{12}$$

Note that the operator T_r is positive and symmetric with respect to the $L^2(\mathbb{T}_2)$-inner product $(f,g)_\rho \stackrel{\text{def}}{=} \int_0^2 fg\rho^2 dx$.

As T_r has compact resolvent the spectrum of T_r is discrete. It turns out (see [13, 12]) that $\texttt{spec}(T_r)$ is of the form $\texttt{spec}(T_r) = \{0 = \tilde{\lambda}_0(r) < \tilde{\lambda}_1(r) \leq \tilde{\lambda}_2(r) \leq ...\}$, the corresponding eigenspaces are of dimension 1 or 2, and $\tilde{\lambda}_k(r) \to \infty$ as $k \to \infty$. For $k \geq 0$ even, the eigenfunctions $\tilde{f}_{2k-1}$ and $\tilde{f}_{2k}$ are periodic while for k

odd, $\tilde{f}_{2k-1}$ and $\tilde{f}_{2k}$ are anti-periodic. If $\lambda_{2k-1}(r) = \lambda_{2k}(r)$ then the corresponding eigenvalue has multiplicity 2 (cf. [12]). Indeed, the impedance operator T_r can be transformed by a change of the variable $y = y(x) \stackrel{\text{def}}{=} \int_0^x (1/\rho^2(s))\, ds$ to the linear operator $-\rho^{-4}\frac{d^2}{dy^4}$ on the torus $\mathbb{T}_l$ with period $l \stackrel{\text{def}}{=} y(1)$ whose spectral properties are established in [12].

Lemma 3. *The first eigenvalue $\tilde{\lambda}_0(r) = 0$ of T_r is simple and the corresponding eigenspace is spanned by the constant function $\tilde{f}_0 \stackrel{\text{def}}{=} 1/||\rho||$.*

Proof. Assume that $u \in \ker(T_r)$. Integrating by parts we obtain

$$0 = (T_r(u), u)_\rho \stackrel{\text{def}}{=} -\int_0^2 (\rho^2 u')' u dx = \int_0^2 \rho^2 (u')^2 dx. \tag{13}$$

As ρ is positive, $u' \equiv 0$ and hence u is constant. This shows that the dimension of $\ker(T_r)$ is equal to one and Lemma 3 is proved. □

The discriminant $\tilde{\Delta}$ of the impedance operator T_r is defined for $\tilde{\lambda} \in \mathbb{C}$ and $r \in L_0^2(\mathbb{T})$ arbitrary, by $\tilde{\Delta}(\tilde{\lambda}, r) \stackrel{\text{def}}{=} u_1(1, \tilde{\lambda}, r) + u_2'(1, \tilde{\lambda}, r)$ where $u_1(x, \tilde{\lambda}, r)$ and $u_2(x, \tilde{\lambda}, r)$ are the fundamental solutions of the equation $-u'' - 2ru' = \tilde{\lambda} u$. It follows from the results in [1] that the discriminant $\tilde{\Delta}(\tilde{\lambda}, r)$ is an analytic function on $\mathbb{C} \times L_0^2(\mathbb{T})$. The following lemma can be proved in straightforward way using the results in [1].

Lemma 4. *The discriminant function $\tilde{\Delta}(\tilde{\lambda}, r)$ is a spectral invariant of the impedance operator T_r. The set of zeroes $\tilde{\lambda}_k$ of the equation $\tilde{\Delta}(\tilde{\lambda}, r)^2 = 4$, counted with their multiplicities, coincides with the spectrum of T_r.*

Lemma 4 together with Corollary 1.2 in [10] then lead to the following

Corollary 4. *The L^2-norm $||r||$ of $r \in L_0^2(\mathbb{T})$ is a spectral invariant of the impedance operator T_r.*

5.2. Dirichlet problem

The Dirichlet problem for the impedance operator has been considered by many authors – see e.g. [1, 2]. The operator T_r^{Dir} is defined on $L^2[0,1]$ with domain $\mathtt{Dom}(T_r^{Dir}) = H^2_{Dir}[0,1] \stackrel{\text{def}}{=} \{f \in H^2[0,1] \mid f(0) = f(1) = 0\}$. By definition, the operator T_r^{Dir} acts on elements $u \in H^2_{Dir}[0,1]$ by the formula $T_r^{Dir}(u) \stackrel{\text{def}}{=} -(\rho^2 u')'/\rho^2 = -u'' - 2ru'$ where $\rho(x) \stackrel{\text{def}}{=} \exp(\int_0^x r(v) dv)$. The spectrum

$\mathtt{spec}(T_r^{Dir})$ is called *Dirichlet spectrum* of T_r. It is known that $\mathtt{spec}(T_r^{Dir})$ is discrete, all eigenvalues are simple, and $\mathtt{spec}(T_r^{Dir}) = \{0 < \mu_1(r) < \mu_2(r) < ...\}$ where $\mu_k(r) \to \infty$ as $k \to \infty$. Other spectral properties of T_r^{Dir}, including the solution of an inverse problem, were established in [1, 2]. We refer the reader to these papers.

6. Appendix B: Schrödinger operator

6.1. Periodic problem

Take $q \in H^{-1}(\mathbb{T}) \stackrel{\text{def}}{=} (H^1(\mathbb{T}))'$ and consider Hill's operator

$$L_q = -\frac{d^2}{dx^2} + q \tag{14}$$

on $H^{-1}(\mathbb{T}_2)$ with domain $\mathtt{Dom}(L_q) = H^1(\mathbb{T}_2)$. The elements of $H^{-1}(\mathbb{T})$ can be considered as elements of $H^{-1}(\mathbb{T}_2)$ as follows: to any element $u \in H^{-1}(\mathbb{T})$ with Fourier expansion $u = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{i2k\pi x}$, we assign the unique element in $H^{-1}(\mathbb{T}_2)$ given by the Fourier series $\sum_{k\in\mathbb{Z}} \hat{s}_k e^{ik\pi x}$ with $\hat{s}_{2k} \stackrel{\text{def}}{=} \hat{u}_k$ and $\hat{s}_{2k+1} \stackrel{\text{def}}{=} 0$. The operator L_q acts on elements $u \in H^1(\mathbb{T}_2)$ by $L_q u \stackrel{\text{def}}{=} -u'' + qu$ where the multiplication qu is viewed as an element of $H^{-1}(\mathbb{T}_2)$ according to the formula $\langle qu, v\rangle \stackrel{\text{def}}{=} \langle q, uv\rangle$. The brackets $\langle\cdot,\cdot\rangle$ denote the dual pairing between the elements of $H^{-1}(\mathbb{T}_2) \stackrel{\text{def}}{=} (H^1(\mathbb{T}_2))'$ and $H^1(\mathbb{T}_2)$. As the multiplication map $H^1(\mathbb{T}_2) \times H^1(\mathbb{T}_2) \to H^1(\mathbb{T}_2)$ given by $u \cdot v \stackrel{\text{def}}{=} uv$ is continuous the linear functional qu is continuous as well. It can be easily seen that L_q induces a bounded operator $L_q : H^1(\mathbb{T}_2) \to H^{-1}(\mathbb{T}_2)$. Considered as an operator on $H^{-1}(\mathbb{T}_2)$, with domain $\mathtt{Dom}(L_q) = H^1(\mathbb{T}_2)$, L_q is an unbounded operator.

The operators L_q with singular potentials $q \in H^{-\alpha}(\mathbb{T}_2)$, $0 < \alpha \le 1$, have been considered in [16, 3]. In order to make this paper self-contained we review the case $\alpha = 1$ treated in [16] and prove the auxiliary facts used in the main part of the paper. In our presentation, we mainly follow [16], §1.5.1.

For $M > 0$, $r > 0$, and $n \in \mathbb{N}$ introduce the sets

$$Ext_M \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} \mid \mathtt{Re}(\lambda) \le |\mathtt{Im}(\lambda)| - M\} \tag{15}$$

$$Vert_n(r) \stackrel{\text{def}}{=} \{\lambda = n^2\pi^2 + z \in \mathbb{C} \mid |\mathtt{Re}(z)| \le n\pi^2, |z| \ge r\}. \tag{16}$$

Via the Fourier transform, we identify L_q with the operator $\hat{L}_v$ on h^{-1} with domain $\mathtt{Dom}(\hat{L}_v) = h^1$. The operator $\hat{L}_v$ acts on the sequences $x = \{x_k\}_{k\in\mathbb{Z}} \in h^1$ by $D + V$ where D and V are the infinite matrices $D \stackrel{\text{def}}{=} (k^2\pi^2\delta_{kl})_{k,l\in\mathbb{Z}}$ and

$V \stackrel{\text{def}}{=} (v(k-l))_{k,l\in\mathbb{Z}}$, δ_{kl} is the Kronecker delta and $v(k)$ are the Fourier coefficients of the potential $q \in H^{-1}(\mathbb{T})$ viewed as an element in $H^{-1}(\mathbb{T}_2)$. The proof of the following result can be found in [16].

Lemma 5. *For every $v \in h^{-1}$ there exist a neighborhood $U(v) \subset h^{-1}$ of v and constants $M > 0$ and $n_0 \in \mathbb{N}$ such that for every $u \in U(v)$, the sets Ext_M and $Vert_n(n)$ $(n > n_0)$ are contained in the resolvent set $\texttt{resol}(\hat{L}_u)$ of $\hat{L}_u$. The resolvent $(\lambda\mathbf{1} - \hat{L}_u)^{-1} \in \mathcal{L}(h^{-1}, h^1)$, considered as a function of (λ, u) on $Ext_M \times U(v)$ or $Vert_n(n) \times U(v)$ with $n > n_0$, is continuous in (λ, u) and for every $u \in U(v)$ holomorphic in λ. Moreover, for any smooth contour $\Gamma \subset Ext_M \cup \bigcup_{n>n_0} Vert_n(n)$ and integer $l \geq 0$, the operator $Q^l_\Gamma(q) \stackrel{\text{def}}{=} \frac{1}{2\pi i}\oint_\Gamma \lambda^l(\lambda\mathbf{1} - \hat{L}_u)^{-1}d\lambda \in \mathcal{L}(h^{-1}, h^1)$ is analytic when viewed as a map $U(v) \to \mathcal{L}(h^{-1}, h^1)$.*

Lemma 6. *(i) For any $q \in H^{-1}(\mathbb{T})$, L_q has a compact resolvent.*

(ii) The spectrum of the Hill operator L_q with potential $q \in H^{-1}(\mathbb{T})$ is discrete, $\texttt{spec}(L_q) = \{\lambda_0(q) \leq \lambda_1(q) \leq \lambda_2(q) \leq ...\}$, the corresponding eigenspaces are of finite dimension, and $\lambda_k(q) \to \infty$ as $k \to \infty$.

(iii) As functions of the potential $q \in H^{-1}(\mathbb{T})$, the k'th eigenvalue $\lambda_k(q)$: $H^{-1}(\mathbb{T}) \to \mathbb{R}$, $q \mapsto \lambda_k(q)$ is continuous.

(iv) Suppose that the eigenvalue $\lambda_k(q)$ is simple for some $q \in H^{-1}(\mathbb{T})$. Then there exists a neighborhood $U(q) \subset H^{-1}(\mathbb{T})$ of q such that for any $u \in U(q)$, the k-th eigenvalue $\lambda_k(u)$ is simple and the corresponding eigenfunction $f_k(\cdot, u)$ (normalized so that $\int_0^2 f_k^2(x, u)dx = 2$ and $f(x_0, u) > 0$, where $x_0 \in [0, 2]$ is chosen so that $f_k(x_0, q) > 0$ for the given potential q) is analytic as a map $U(q) \to H^1(\mathbb{T}_2)$.

Remark 3. *We improve on Lemma 6 in Section 4. (see Theorem 3).*

Remark 4. *An analogue of Lemma 6 is true for potentials $q \in H^{\alpha-1}(\mathbb{T})$ with arbitrary $\alpha \geq 0$. In particular, the eigenvalues $\lambda_k(q)$ are continuous with respect to the norm in $H^{\alpha-1}(\mathbb{T})$. If $\lambda_k(q)$ is simple, the corresponding normalized eigenfunction $f_k(\cdot, q)$ is analytic as a map from a neighborhood of q in $H^{\alpha-1}(\mathbb{T})$ to $H^{\alpha+1}(\mathbb{T}_2)$. The proofs are the same as in the case $\alpha = 0$.*

Essentially the same arguments prove the following extension of Lemma 6 *(ii)*.

Theorem 11. *The spectrum of Hill's operator* $L_q = -\frac{d^2}{dx^2} + q$ *on* $H^{-1}(\mathbb{T}_2, \mathbb{C})$ *with singular potential* $q \in H^{-1}(\mathbb{T}, \mathbb{C})$ *is discrete,* $\mathtt{spec}(L_q) = \{\mathtt{Re}(\lambda_0(q)) \le \mathtt{Re}(\lambda_1(q)) \le \mathtt{Re}(\lambda_2(q)) \le ...\}$, *the corresponding eigenspaces are of finite dimension, and* $\mathtt{Re}(\lambda_k(q)) \to \infty$ *as* $k \to \infty$.

6.2. Dirichlet problem

In this subsection we set up the Dirichlet problem for the operator $L_q \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + q$ on $[0,1]$ with potential $q \in H^{-1}(\mathbb{T})$. In order to make our presentation self-contained, we give some definitions and auxiliary facts on this problem. Further details (including the case $q \in H^{-\alpha}(\mathbb{T})$, $\alpha < 1$) can be found in [3], §3.

Define the operator L_q^{Dir} on the Sobolev space $H^{-1}[0,1]$ with domain $\mathtt{Dom}(L_q^{Dir}) = H^1_{Dir}[0,1]$. By definition, $H^1_{Dir}[0,1] \stackrel{\text{def}}{=} \{f \in H^1[0,1] \,|\, f(0) = f(1) = 0\}$ and $H^{-1}[0,1] \stackrel{\text{def}}{=} (H^1_{Dir}[0,1])'$ – see Section 1. where the definition of the Sobolev space $H^1[0,1]$ is recalled. In a natural way, the elements of $H^1_{Dir}[0,1]$ can be identified with elements in $H^1(\mathbb{T})$, the corresponding inclusion map $\imath : H^1_{Dir}[0,1] \to H^1(\mathbb{T})$ being continuous. For any $u \in H^1_{Dir}[0,1]$, qu is defined to be the functional in $H^{-1}[0,1] \stackrel{\text{def}}{=} (H^1_{Dir}[0,1])'$ given by the formula

$$\langle qu, v\rangle \stackrel{\text{def}}{=} \langle q, \imath(u)\imath(v)\rangle \,.$$

The mapping $H^{-1}[0,1] \times H^1_{Dir}[0,1] \to H^{-1}[0,1]$, $(q,u) \mapsto qu$, is continuous as is the operator $\frac{d^2}{dx^2} : H^1[0,1] \to H^{-1}[0,1]$. For any $u \in H^1_{Dir}[0,1]$ we set $L_q^{Dir} u \stackrel{\text{def}}{=} -\frac{d^2 u}{dx^2} + qu$. In this way, L_q^{Dir} is a bounded operator $L_q^{Dir} : H^1_{Dir}[0,1] \to H^{-1}[0,1]$. Considered as an operator on $H^{-1}[0,1]$, L_q^{Dir} is an unbounded operator.

Lemma 7. *The operator* L_q^{Dir} *has compact resolvent. As a consequence, the spectrum of* L_q^{Dir} *is discrete, the eigenspaces are of finite dimension, and in every compact set* $K \subset \mathbb{C}$ *there are finitely many eigenvalues.*

P r o o f. As in §3.2 in [3] we identify the operator L_q^{Dir} with an operator on an appropriate sequence space. Then Lemma 7 can be proved using the same arguments as in the proof of Lemma 6. $\square$

REFERENCES

[1] C. Coleman, J. McLaughlin. Solution of the inverse spectral problem for an impedance with integrable derivative I. *Comm. Pure Appl. Math*, **46** (1993), 145–184.

[2] C. Coleman, J. McLaughlin. Solution of the inverse spectral problem for an impedance with integrable derivative II. *Comm. Pure Appl. Math*, **46** (1993), 185–212.

[3] T. Kappeler, C. Möhr. Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator with singular potentials. *J. Funct. Anal.*, **186** (2001), no. 1, 62–91.

[4] T. Kappeler, C. Möhr, P. Topalov. Birkhoff coordinates for KdV on phase spaces of distributions. Preprint Series, Institute of Mathematics, University of Zurich, 2003.

[5] T. Kappeler, J. Pöschel. KdV & KAM. preliminary version, 2001.

[6] T. Kappeler, P. Topalov. Riccati representation for elements in $H^{-1}(\mathbb{T})$ and its applications. Preprint Series, Institute of Mathematics, University of Zurich, 2002.

[7] T. Kappeler, P. Topalov. Global Well-Posedness of KdV in $H^{-1}(\mathbb{T}, \mathbb{R})$. Preprint.

[8] T. Kato. *Perturbation Theory of Linear Operators*, Springer, 1980.

[9] E. Korotyaev. Periodic weighted operators. SFB-288 preprint 388.

[10] E. Korotyaev. Inverse problems for periodic "weighted" operators. *J. Funct. Anal.*, **170** (2000), 188–218.

[11] E. Korotyaev. Characterization of spectrum for Schrödinger operator with periodic distributions. Preprint, 2002.

[12] M. Krein. The basic propositions of the theory of λ-zones of stability of a canonical system of linear differential equation with periodic coefficients. In memory of A.A. Andronov, p. 413–498, Izdat. Acad. Nauk SSSR, Moscow, 1955.

[13] M. Lyapunov. Sur une équation transcendante et les équations differentielles linéaires du second ordre à coefficients périodiques. *C.R.Acad. Sci.*, Paris CXXXVIII, **18** (1899), 1085-8.

[14] J.-L. Lions, E. Magenes. *Non-homogeneous Boundary Value Problems and Applications I.* Grundlehren, Vol. 181, Springer-Verlag, 1972

[15] W. Magnus, S. Winkler. *Hill's equation*, Interscience, 1966.

[16] C. Möhr. Thesis at the University of Zürich, 2001.

[17] J. Pöschel, E. Trubowitz. *Inverse Spectral Theory*, Academic Press, 1987.

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