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ON THE ZEROS OF THE SOLUTIONS TO NONLINEAR HYPERBOLIC EQUATIONS WITH DELAYS

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Abstract. We consider the nonlinear hyperbolic equation with delays

$$u_{xy} + \lambda u_{xy}(x-\sigma, y-\tau) + c(x, y, u, u_x, u_y) = f(x, y).$$

We obtain sufficient conditions for oscillation of the solutions of problems of Goursat in the case, where $\lambda \geq 0$.

1. Introduction

Oscillation theory for hyperbolic equations and especially with deviating arguments is still in initial stage. In [12] Yoshida pointed out that characteristic initial value problems for hyperbolic equations were considered in [1] — [4], [10, 11] and forced oscillations of solutions to hyperbolic equations were investigated by Kreith et al. [3] and Mishev [5].

Many results in the oscillation theory for equations with deviating arguments are similar to the respective results for equations without deviating arguments following the same ideas. In this paper we use the method due to Yoshida [12] to obtain the analogue of his result in the case with deviations. More exactly, we adapt the obtained there sufficient conditions in such a way that under them every solution to certain characteristic initial value problems for hyperbolic equations

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with constant deviations has a zero in bounded domains defined as in [12]. We also mention that this method leads to sufficient conditions for nonexistence of positive and monotonically increasing in every argument solutions as well as which are negative and monotonically decreasing in every argument solutions in certain sets if $-1 \le \lambda \le 0$ (see [7]).

We consider the following characteristic initial value problem

(1) $u_{xy}(x,y) + \lambda u_{xy}(x-\sigma, y-\tau) + c(x,y,u,u_x,u_y) = f(x,y),\ x>0, y>0,$

(2) $$\begin{aligned} u(x,y) &= p(x,y), \quad \forall\ (x,y) \in \mathbf{S_1} \equiv [-\sigma,\infty) \times [-\tau, 0],\\ u(x,y) &= q(x,y), \quad \forall\ (x,y) \in \mathbf{S_2} \equiv [-\sigma, 0] \times [-\tau,\infty),\end{aligned}$$

where λ, σ and τ are nonnegative numbers. We define

$$\begin{aligned} &Q_\rho = \{x>0, y>0 : Lx + k^2L^{-1}y > L\rho\ \}\\ &Q(t_1,t_2] = \{x>0, y>0 : Lt_1 < Lx + k^2L^{-1}y \le Lt_2\ \},\end{aligned}$$

where ρ and $t_1 \le t_2$ are nonnegative numbers and k and L are positive numbers.

We state the conditions (**H**):

H1. $c(x,y,\xi,\eta,\zeta) \in C(\overline{Q_\rho} \times \mathbf{R^3}, \mathbf{R})$;

H2. $f(x,y) \in C(\overline{Q_\rho}, \mathbf{R})$; $p(x,y)$, $q(x,y) \in C^1(\mathbf{R}^2, \mathbf{R})$;

H3. $p(x,y) = q(x,y) \quad \forall\ (x,y) \in [-\sigma,0] \times [-\tau,0]$;

H4. $\xi c(x,y,\xi,\eta,\zeta) \ge 0 \quad \forall\ (x,y,\xi,\eta,\zeta) \in \overline{Q_\rho} \times \mathbf{R}^3$.

For every solution $u \in D(Q_\rho) \equiv C^2(Q_\rho) \bigcap C^1(\overline{Q_\rho})$ to (1) we define

(3) $$w_n(t) = w(t) + \lambda w_d(t),$$

(4) where $$w(t) = t^{-1}\int_0^t u(\xi, L^2k^{-2}(t-\xi))d\xi,$$

(5) $$w_d(t) = t^{-1}\int_0^t u(\xi-\sigma, L^2k^{-2}(t-\xi)-\tau)d\xi$$

and $$W_n(\tilde t, t) = \tilde t w_n(\tilde t) + (w_n(\tilde t) + \tilde t w_n'(\tilde t))(t-\tilde t)$$

(6) $$+ \int_{\tilde t}^t (t-s)\Big(p_n'(s) + L^2k^{-2}q_n'(L^2k^{-2}s)$$

$$+ L^2k^{-2}\int_0^s f(\xi, L^2k^{-2}(s-\xi))d\xi\Big)ds,$$

(7) where $$p_n(t) = p(t,0) + \lambda p(t-\sigma, -\tau)$$

(8) and $$q_n(L^2k^{-2}t) = q(0, L^2k^{-2}t) + \lambda q(-\sigma, L^2k^{-2}t-\tau).$$

We continue the introduction with the geometrical interpretation of all the above sets and the functions $w(t)$ and $w_d(t)$. This interpretation will be done in three steps:

I description of the sets Q_ρ and $Q(t_1, t_2]$;

II interpretation of the functions $w(t)$ and $w_d(t)$;

III explanation of the link between the set $Q(t_1, t_2]$ and the functions $w(t)$ and $w_d(t)$.

We deal with the family of lines:

$$l_t \ : y = L^2 k^{-2}(t - x), \quad t \geq 0. \tag{9}$$

Remark 1. *In fact, the equation of Descartes (9) shows that this is a family of lines of level, which depends on the parameter t. Moreover, a bigger parameter corresponds to a*

(10) *higher line. In other words, if* $\quad 0 \leq t_1 \leq t_2$

(11) *then the line* $\quad l_{t_1} \ : y = L^2 k^{-2}(t_1 - x)$

(12) *is above the line* $\quad l_{t_2} \ : y = L^2 k^{-2}(t_2 - x).$

Remark 2. *Now we are ready to describe the defined here sets. Let*

$$A_t = l_t \bigcap O_x \quad \text{and} \quad B_t = l_t \bigcap O_y.$$

Then the set Q_ρ *is the part of the first quadrant, which is above the line* $A_\rho B_\rho$*. Also, the set* $Q(t_1, t_2]$ *is the union of the open trapezoid* $A_{t_1} A_{t_2} B_{t_2} B_{t_1}$ *and the open segment* $A_{t_2} B_{t_2}$*.*

We continue with the second step. The family of lines (9) could be parametrized in the following way:

$$l_t \ : \left\{ \begin{array}{l} x = \xi, \quad \xi \ \in \ (-\infty, +\infty) \\ y = L^2 k^{-2}(t - \xi). \end{array} \right. \tag{13}$$

Obviously, (13) leads to the following parametrization of the segment $A_t B_t$:

$$A_t B_t : \left\{ \begin{array}{l} x = \xi, \quad \xi \ \in \ [0, t] \\ y = L^2 k^{-2}(t - \xi). \end{array} \right. \tag{14}$$

The comparison between (14) and (4) gives us that the function $w(t)$ is an average of the solution $u(x, y)$ on the segment A_tB_t. The function $w_d(t)$ is an average of the same solution on the segment

$$C_{t_*}D_{t_*}: \begin{cases} x = \xi - \sigma, & \xi \in [0, t] \\ y = L^2k^{-2}(t - \xi) - \tau, \end{cases} \tag{15}$$

where the points D_{t_*} and C_{t_*} correspond to the values 0 and t respectively, i. e.

$C_{t_*}(-\sigma, L^2k^{-2}t - \tau) = l_{t_*} \bigcap \{y = -\tau\}$

and $D_{t_*}(t - \sigma, -\tau) = l_{t_*} \bigcap \{x = -\sigma\}$.

(16) This segment is on the line $l_{t_*} : y = L^2k^{-2}(t_* - x)$,

(17) where $t_* = t - \sigma - L^{-2}k^2\tau = t - \sigma - \theta_*$, $\theta_* = L^{-2}k^2\tau$.

(18) Let us define the function $\theta(t) = t - \theta_*$, which is concerned with C_{t_*}.

Remark 3. *Since $t \geq 0$, $\sigma \geq 0$, $\tau \geq 0$, $L > 0$ and $k > 0$, then (17) leads to*

$$t_* \leq t, \tag{19}$$

which is a particular case of (10). Hence, the line l_t is always above the line l_{t_*} according to Remark 1.

We pass to the third step of the geometrical interpretation, i. e. we shall investigate the link between the function $w_n(t)$ and the set $Q(t_1, t_2]$. More precisely, we are interested if the open segments A_tB_t and $C_{t_*}D_{t_*}$ lie in $Q(t_1, t_2]$.

We apply all of the above remarks to conclude that the condition:

$$0 \leq t_1 \leq t \leq t_2 \tag{20}$$

guarantees both that the open segment A_tB_t is in $Q(t_1, t_2]$ and that the open segment $A_{t_2}B_{t_2}$ is above the open segment $A_{t_*}B_{t_*}$. In fact, the combination of our conjecture (20) and the inequality (19), which is always fulfilled according to Remark 3 leads to

$$0 \leq t_* \leq t \leq t_2. \tag{21}$$

(22) Hence, we do not know previously if $0 \leq t_1 \leq t_*$

and we should suppose (22) additionally to be sure that the open segment $A_{t_*}B_{t_*}$ is above the open segment $A_{t_1}B_{t_1}$. Let us rewrite (22) taking in attention (17):

(23) $0 \leq t_1 \leq t - \sigma - L^{-2}k^2\tau$,

(24) which is equivalent to the following inequality : $0 \leq \tilde{t}_1 \leq t$,

(25) where $\tilde{t}_1 = t_1 + \sigma + L^{-2}k^2\tau = t_1 + \sigma + \theta_*$.

We extract the above considerations in a special remark, which will be applied in the next section essentially. More exactly, the nonnegative numbers $t_1 \leq t_2$ will be fixed but arbitrary in the second section.

Remark 4. *Let*

$$0 \leq \tilde{t}_1 \leq t \leq t_2. \tag{26}$$

Then the open segments A_tB_t and $A_{t_*}B_{t_*}$ lie in $Q(t_1, t_2]$.

We denote by U_t and V_t the points on the line l_t, which correspond to the values σ and $\theta(t)$. We do not know in general if

$$\sigma \in [0, t] \tag{27}$$

$$\text{and} \quad \theta(t) \in [0, t] \tag{28}$$

but we shall prove the following lemma taking in attention Remark 4.

Lemma 1. *If (26) is fulfilled then (27) and (28) are also fulfilled.*

Proof. We replace (25) in (26):

$$0 \leq t_1 + \sigma + \theta_* \leq t \leq t_2. \tag{29}$$

Every addend in the left hand side of (29) is nonnegative:

$$t_1 \geq 0, \quad \sigma \geq 0 \quad \text{and} \quad \theta_* \geq 0. \tag{30}$$

Hence, the first conclusion of the simultaneous consideration of (29) and (30) is

$$0 \leq \sigma \leq t,$$

which is just (27). The second one is concerned with (18) also:

$$0 \leq t_1 \leq t - \theta_* \leq t \tag{31}$$

and then (28) follows from (31) immediately. $\square$

Definition 1. *We say that the function $\varphi(t)$ is* **oscillating** *when $t \to \infty$, if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty} t_n = \infty \qquad \textit{and} \qquad \varphi(t_n) = 0. \tag{32}$$

Definition 2. *We say that the function* $\varphi(t)$ *is* **eventually positive** *(***eventually negative***), if there exists* $c = const$ *such that*

$$\varphi(t) > 0 \quad (\varphi(t) < 0) \qquad \forall\, t \in [c, \infty). \tag{33}$$

Remark 5. *In fact, we use the obvious fact that if there is a constant* c *such that* $f(t)$ *changes its sign infinitely many times when* $t \in [c, \infty)$ *and* $f(t) \in C([c, \infty), \mathbf{R})$*, then the function* $f(t)$ *is oscillating.*

Remark 6. *Since in all the bellow our results are for the classical solutions of concrete problems, then the solutions are only of the mentioned three types. It means that it is enough to show that an equation has nor eventually positive neither eventually negative solutions to establish that it has only oscillating solutions.*

Here this is possible only in the cases, where $\lambda \geq 0$. We especially underline that the paper of Yoshida [12] is devoted to the situation $\lambda = 0$. Our present publication is a generalization of Yoshida [12]. More exactly, the similar results in [12] follow directly from all in the bellow.

Obviously,

$$tw_n(t) = I_1(t) + \lambda E_1(t) + I_2(t) + I_3(t) + \lambda E_3(t), \tag{34}$$

$$\text{where} \quad I_1(t) = \int_0^{\sigma} u(\xi, L^2k^{-2}(t-\xi))d\xi, \tag{35}$$

$$E_1(t) = \int_0^{\sigma} p(\xi - \sigma, L^2k^{-2}(t-\xi) - \tau)d\xi, \tag{36}$$

$$I_2(t) = \int_{\sigma}^{\theta(t)} \big(u(\xi, L^2k^{-2}(t-\xi)) + \lambda u(\xi - \sigma, L^2k^{-2}(t-\xi) - \tau)\big)\, d\xi, \tag{37}$$

$$I_3(t) = \int_{\theta(t)}^{t} u(\xi, L^2k^{-2}(t-\xi))d\xi \tag{38}$$

$$\text{and} \quad E_3(t) = \int_{\theta(t)}^{t} q(\xi - \sigma, L^2k^{-2}(t-\xi) - \tau)d\xi. \tag{39}$$

Lemma 1 guarantees that the integrals $I_1(t)$, $I_2(t)$ and $I_3(t)$ are linear integrals on the segments A_tU_t, U_tV_t and V_tB_t respectively, which are inside the first quadrant. Similarly, the integrals $E_1(t)$ and $E_3(t)$ are linear integrals on the segments $B_{t_*}D_{t_*}$ and $C_{t_*}A_{t_*}$, which are outside of the first quadrant.

2. Main Results

The present results illustrate the important role of the function $W_n(t_1, t_2)$.

Theorem 1. *Suppose that $\lambda \geq 0$ as well as the conditions (H) hold. Let t_1 be a number with $t_1 > \rho$ and $u \in D(Q_\rho)$ be a solution to the problem (1), (2). If there is a number $t_2 > \tilde{t}_1$ such that*

$$W_n(t_1, t_2) \leq 0, \tag{40}$$

$$E_1(t_2) \geq 0 \quad \text{and} \quad E_3(t_2) \geq 0 \tag{41}$$

then the solution u is not positive in $Q(t_1, t_2]$.

Proof. Assume to the contrary, i. e. that:

$$\exists\ t_2 > \tilde{t}_1 : \quad W_n(t_1, t_2) \leq 0 \tag{42}$$

$$\text{and that} \quad u(x, y) > 0 \quad \forall\ (x, y) \in Q(t_1, t_2] \tag{43}$$

holds for the same t_1 and t_2. The proof consists of two parts. We prove that

$$t_2 w_n(t_2) > 0 \tag{44}$$

in the first part. Then we prove that

$$t_2 w_n(t_2) \leq W_n(t_1, t_2) \tag{45}$$

in the second one.

We obtain the needed contradiction since the simultaneous consideration of (42), (44) and (45) leads to the impossible inequalities:

$$0 < t_2 w_n(t_2) \leq W_n(t_1, t_2) \leq 0. \tag{46}$$

□

Let us begin with the first part, i. e. let us assume that (43) is fulfilled.

Lemma 2. *Let all the conditions of Theorem 1 be satisfied as well as the inequality (43). Then (44) holds.*

Proof. Since (43) holds, then together with Remark 4 and Lemma 1 we

$$\text{conclude that} \quad u(\xi, L^2 k^{-2}(t_2 - \xi)) > 0 \quad \forall\ \xi \in [0, t_2] \tag{47}$$

$$\text{and} \quad u(\xi - \sigma, L^2 k^{-2}(t_2 - \xi) - \tau)) > 0 \quad \forall\ \xi \in [\sigma, \theta(t_2)]. \tag{48}$$

First, we combine (47) and (48) with (35), (37) and (38): $I_1(t_2)+I_2(t_2)+I_3(t_2) =$

$$(49)\quad = \int_0^{t_2} u(\xi, L^2k^{-2}(t_2-\xi))d\xi + \lambda \int_\sigma^{\theta(t_2)} u(\xi-\sigma, L^2k^{-2}(t_2-\xi)-\tau)d\xi > 0.$$

Further, we replace (49) and (41) in (34) in the particular case, where $t = t_2$ to establish (44). □

Then we continue with the second part.

Lemma 3. *Let the conditions (H) be satisfied and let the function u be a solution of the problem (1), (2). Then*

$$(50)\quad \begin{aligned}(tw_n(t))'' &= p_n'(t) + L^2k^{-2}q_n'(L^2k^{-2}t) + \\ &+ L^2k^{-2}\int_0^t \big(u_{xy}(\xi, L^2k^{-2}(t-\xi)) + \lambda u_{xy}(\xi-\sigma, L^2k^{-2}(t-\xi)-\tau)\big)\, d\xi.\end{aligned}$$

Lemma 4. *Let all the conditions of Theorem 1 hold and let the function u be a positive solution of the problem (1), (2) in $Q(t_1,t_2]$. Then (45) is fulfilled.*

P r o o f. First, we apply the condition H4 to (43) and establish that

$$(51)\quad c(x,y,\xi,\eta,\zeta) \geq 0 \quad \forall\ (x,y) \in Q(t_1,t_2].$$

Secondary, we apply (51) to (1):

$$u_{xy}(\xi, L^2k^{-2}(t-\xi)) + \lambda u_{xy}(\xi-\sigma, L^2k^{-2}(t-\xi)-\tau) \leq f(\xi, L^2k^{-2}(t-\xi)),$$

i.e.

$$(52)\quad (tw_n(t))'' \leq p_n'(t) + L^2k^{-2}q_n'(L^2k^{-2}t) + L^2k^{-2}\int_0^t f(\xi, L^2k^{-2}(t-\xi))d\xi$$

because of Lemma 3. Further, we integrate two times the above inequality. □

Theorem 2. *Suppose that $\lambda \geq 0$ as well as the conditions (H) hold. Let t_1 be a number with $t_1 > \rho$ and $u \in D(Q_\rho)$ be a solution to the problem (1), (2). If there is a number $t_2 > \tilde{t}_1$ such that*

$$(53)\quad W_n(t_1,t_2) \geq 0,$$

$$(54)\quad E_1(t_2) \leq 0 \quad \text{and} \quad E_3(t_2) \leq 0$$

then the solution u is not negative in $Q(t_1,t_2]$.

P r o o f. Assume to the contrary, i. e. that:

$$\exists\ t_2 > \tilde{t}_1 \colon \quad W_n(t_1, t_2) \geq 0 \quad \text{and} \quad u(x, y) < 0 \quad \forall\ (x, y) \in Q(t_1, t_2]$$

holds for the same t_1 and t_2. This time we establish that:

$$\mathbf{1.} \quad t_2 w_n(t_2) < 0 \quad \text{and} \quad \mathbf{2.} \quad t_2 w_n(t_2) \geq W_n(t_1, t_2)$$

to obtain a contradiction. □

Theorem 3. *Suppose that $\lambda \geq 0$ as well as the conditions (H) hold. Let t_1 be a number with $t_1 > \rho$ and $u \in D(Q_\rho)$ be a solution to the problem (1), (2). If there is a number $t_2 > \tilde{t}_1$ such that*

$$W_n(t_1, t_2) = 0, \tag{55}$$

$$E_1(t_2) = E_3(t_2) = 0, \tag{56}$$

then the solution u has a zero in $Q(t_1, t_2]$.

P r o o f. Everything here follows from Remark 6 and the above two theorems. Really, the condition (55) satisfies both (40) and (53). Also, the condition (56) satisfies both (41) and (54). Hence, we could apply Theorems 1 and 2. These theorems make us sure that the problem (1), (2) has nor positive solution in $Q(t_1, t_2]$ neither negative solution in $Q(t_1, t_2]$. Finally, we apply Remark 6 to finish the proof. □

Theorem 4. *Suppose that $\lambda \geq 0$ as well as the conditions (H) and*

$$p(x, y) = q(x, y) \equiv 0, \quad \forall\ (x, y) \in \mathbf{S_1} \bigcup \mathbf{S_2} \tag{57}$$

are satisfied. Let t_1 be a number with $t_1 > \rho$ and $u \in D(Q_\rho)$ be a solution to the problem (1), (2). If there is a number $t_2 > \tilde{t}_1$ such that (55) holds then the solution u has a zero in $Q(t_1, t_2]$.

P r o o f. Since (57) guarantees that the functions $p(x, y)$ and $q(x, y)$ satisfy (56), then the present theorem is true because it is a particular case of the previous one. □

Theorem 5. *Suppose that $\lambda \geq 0$ and (H) are fulfilled and let there exist two sequences $\{\tau_m\}_{m=1}^{\infty}$ and $\{\theta_m\}_{m=1}^{\infty}$ such that*

$$\lim_{m \to \infty} \tau_m = +\infty \quad \text{and} \quad \lim_{m \to \infty} \theta_m = +\infty, \tag{58}$$

which satisfy the following conditions:

$$E_1(\theta_m) = E_3(\theta_m) = 0, \tag{59}$$

$$\theta_m > \tilde{\tau_m}, \quad \textit{where} \quad \tilde{\tau_m} = \tau_m + \sigma + \theta_* \tag{60}$$

$$\textit{and} \quad W_n(\tau_m, \theta_m) = 0 \tag{61}$$

$\forall\, m \in \mathbf{N}$. *Then every solution* $u \in D(Q_\rho)$ *to the problem (1), (2) is oscillating in* Q_ρ.

P r o o f. Since all the conditions of Theorem 3 are fulfilled in the particular case, where

$$t_1 = \tau_m \quad \text{and} \quad t_2 = \theta_m \quad \forall\, m\ \in \mathbf{N}$$

then the solution u has a zero in $Q(\tau_m, \theta_m]$ $\forall\, m \in \mathbf{N}$. In fact, the last one means just that the function u is oscillating. □

Theorem 6. *Let* $\lambda \geq 0$. *Assume the conditions (H) hold as well as the condition*

$$E_1(t) = E_3(t) = 0 \quad \forall\, t \geq 0. \tag{62}$$

Then every solution $u \in D(Q_\rho)$ *to the problem (1), (2) is oscillating in* Q_ρ *if*

$$\liminf_{t\to\infty} \int_T^t (1 - (s/t))\, (p_n'(s) + L^2k^{-2}q_n'(L^2k^{-2}s) + \\ + L^2k^{-2} \int_0^s f(\xi, L^2k^{-2}(s-\xi))d\xi)ds = \, -\infty \quad \textit{and} \tag{63}$$

$$\limsup_{t\to\infty} \int_T^t (1 - (s/t))\, (p_n'(s) + L^2k^{-2}q_n'(L^2k^{-2}s) + \\ + L^2k^{-2} \int_0^s f(\xi, L^2k^{-2}(s-\xi))d\xi)ds = \, +\infty \tag{64}$$

for all large T.

P r o o f. The hypotheses give us that for every arbitrary fixed first argument $\tilde{t}_*$ of $W_n(\tilde{t}, t)$ there exist two second arguments t_{*1} and t_{*2} in which this continuous function of two arguments has opposite signs. Consequently, from the well known theorem there exists a number t_* between t_{*1} and t_{*2} such that $W_n(\tilde{t}_*, t_*) = 0$ and we apply Theorem 5. □

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