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OPTIMAL INTERPOLATION CONSTANT FOR THE GENERALIZED SCHRÖDINGER–NEWTON SYSTEM*

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In the present article we prove non-existence of radial solutions to the generalized Choquard equation of the form

$$\Delta u(x) + \omega u(x) = \left(\int_{\mathbb{R}^3} \frac{|u(y)|^p dy}{4\pi|y|} - \int_{\mathbb{R}^3} \frac{|u(y)|^p dy}{4\pi|x-y|} \right) |u(x)|^{p-2} u(x)$$

for $2 < p < 7/3$ and $\omega > 0$. The solutions can be associated with solutions to the Schrödinger–Newton system in $\mathbb{R}^3$

$$\begin{aligned} &\Delta u(x) + \omega u(x) = A(x)|u(x)|^{p-2}u(x) \\ &\Delta A(x) = |u(x)|^p, \end{aligned}$$

with a prescribed asymptotic behavior

$$\lim_{|x|\to\infty} A(x) = \int_{\mathbb{R}^3} \frac{|u(y)|^p dy}{4\pi|y|}$$

at infinity. Using the Kato result for the absence of embedded eigenvalues for short-range potential perturbations of the Laplace operator we show that any H^1 radial solution to the generalized Choquard equation is identically

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zero. Further, we propose a variational problem that will lead to generalized Choquard equation of the form

$$\Delta u(x) + \omega u(x) = \left(\delta \int_{\mathbb{R}^3} \frac{|u(y)|^p dy}{4\pi |y|} - \int_{\mathbb{R}^3} \frac{|u(y)|^p dy}{4\pi |x-y|} \right) |u(x)|^{p-2} u(x)$$

for $2 < p < 7/3$, $\delta \in [0, 1/2)$ and $\omega > 0$. The variational setting will give a radial decreasing H^1_{rad} solution to this equation.

1. Introduction and main results

The classical Schrödinger–Newton system (sometimes called also Schrödinger–Poisson system in 3D) can be written in the form

$$\begin{aligned} &\mathrm{i}\tfrac{d}{dt}\psi(t,x) + \Delta \psi(t,x) = A(t,x)\psi(t,x), \\ &\Delta A(t,x) = |\psi(t,x)|^2. \end{aligned} \tag{1}$$

This system can be connected with self-gravitating boson stars models, and moreover it is proposed as a model to explain the quantum wave function collapse (see for instance [3, 10]). Simplified model of type (1) can be derived by the aid of Born–Oppenheimer approximation of the N-body equations.

It is natural to consider the following generalized Schrödinger–Newton functional in $H^1(\mathbb{R}^3) \times \dot{H}^1(\mathbb{R}^3)$

$$\mathcal{E}(u, A) = \frac{1}{2}\|\nabla u\|^2_{L^2} + \frac{1}{2p}\|\nabla A\|^2_{L^2} + \frac{1}{p}\int_{\mathbb{R}^3} A(x)|u(x)|^p dx, \tag{2}$$

with $p \geq 2$, subject to the constraint condition

$$\frac{1}{2}\|u\|^2_{L^2} = \lambda. \tag{3}$$

The Euler–Lagrange equation for this variational problem is the generalized Schrödinger–Poisson system in $\mathbb{R}^3$

$$\begin{aligned} -\Delta u(x) + A(x)|u(x)|^{p-2}u(x) &= \omega u(x), \\ \Delta A(x) &= |u(x)|^p \end{aligned} \tag{4}$$

The above system becomes classical Schrödinger–Poisson system if $p = 2$. The assumption that the function A is in the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$ implies that A is defined modulo a constant as

$$A(x) = A(u)(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|u(y)|^p}{|x-y|} dy + C.$$

The uniqueness of the ground state for $p = 2$ is obtained in [1] and [6]. The approach in [1] is based on a specific choice of C that breaks the translation symmetry of the energy functional (2) and consequently the translation symmetry in (4). This specific choice is done in [1] so that

$$A(0) = 0. \tag{5}$$

The constraint (5) implies

$$A(u)(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\left(\frac{|u(y)|^p}{|y|} - \frac{|u(y)|^p}{|x-y|}\right)dy, \tag{6}$$

so we can reduce the system (4) to the following single equation

$$\Delta u(x) + \omega u(x) = \left(\int_{\mathbb{R}^3}\frac{|u(y)|^p dy}{4\pi|y|} - \int_{\mathbb{R}^3}\frac{|u(y)|^p dy}{4\pi|x-y|}\right)|u(x)|^{p-2}u(x). \tag{7}$$

Our first main result is the following

Theorem 1. *Assume that $p \in (2, 7/3)$. If $u \in H^1_{rad}(\mathbb{R}^3)$ is a positive decreasing solution to (7), then $u = 0$.*

Note that (6) implies

$$\mathcal{E}(u, A(u)) = \frac{1}{2}\|\nabla u\|^2_{L^2} + \frac{1}{p}M(|u|^p)\|u\|^p_{L^p} - \frac{1}{2p}D(|u|^p, |u|^p), \tag{8}$$

where we use the notations

$$D(f,g) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f(x)g(y)}{|x-y|}dxdy, \quad M(f) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{f(x)\,dx}{|x|}. \tag{9}$$

Our next step is to study the following variational problem associated with the functional (8). Consider the generalized functional

$$E_\mu(u) = \frac{1}{2}\|\nabla u\|^2_{L^2} + \frac{\mu}{p}\|u\|^p_{L^p} - \frac{1}{2p}D(|u|^p, |u|^p), \quad \mu \geq 0 \tag{10}$$

and consider the following minimization problem

$$I(\lambda, \mu, \delta) = \inf_{u \in S(\lambda,\mu)}\left\{E_{\delta\mu}(u)\right\}, \tag{11}$$

where $\delta \in [0, 1)$ and

$$S(\lambda, \mu) = \left\{u \in H^1(\mathbb{R}^3); \|u\|^2_{L^2} = 2\lambda, M(|u|^p) \geq \mu\right\}. \tag{12}$$

The next goal is to show that for any $\lambda > 0$ and $\delta \in [0,1)$ there exists a unique $\mu = \mu(\lambda,\delta) > 0$, such that a minimizer $u_{\lambda,\mu,\delta}$ of $I(\lambda,\mu,\delta)$ for $\mu \in [0,\mu(\lambda,\delta)]$ exists and

$$M(|u_{\lambda,\mu(\lambda,\delta)}|^p) = \mu(\lambda,\delta). \tag{13}$$

It is easy to see the following rescaling property

Lemma 1. *If $4/3 < p$, then for any $\lambda > 0, \mu \geq 0$, we have the properties:*

1. *the set $S(\lambda,\mu)$ is nonempty;*
2. *for any $\kappa > 0$ and any $a \in \mathbb{R}$, we have*

$$u \in S(\lambda,\mu) \Longleftrightarrow u_\kappa(x) = \kappa^a u(\kappa x) \in S(\lambda\kappa^{2a-3}, \mu\kappa^{pa-2}). \tag{14}$$

For this reason we can consider only the case $\lambda = 1$.

Lemma 2. *If $p \in [2, 7/3)$, then for any $\mu > 0, \delta \in [0,1/2)$ we have*

$$S(1,\mu) \cap \left\{u \in H^1(\mathbb{R}^3); E_{\mu\delta}(u) < 0\right\} \neq \emptyset. \tag{15}$$

P r o o f. For $u \in H^1_{rad}(\mathbb{R}^3)$ we have the equalities

$$2\delta M(|u|^p)\|u\|^p_{L^p} - D(|u|^p,|u|^p) =$$

$$= \frac{2\delta}{4\pi}\int_{\mathbb{R}^3}\frac{|u(x)|^p dx}{|x|}\int_{\mathbb{R}^3}|u(y)|^p dy - \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p dxdy}{\max(|x|,|y|)} =$$

$$= \frac{2\delta-1}{4\pi}\int_{\mathbb{R}^3}\int_{|y|<|x|}\frac{|u(x)|^p|u(y)|^p dxdy}{|x|} +$$

$$+ \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{|y|>|x|}\left(\frac{2\delta}{|x|} - \frac{1}{|y|}\right)|u(x)|^p|u(y)|^p dxdy.$$

These relations show that one can find for any $\delta \in [0,1/2)$ a radial u_δ so that

$$2\delta M(|u_\delta|^p)\|u_\delta\|^p_{L^p} - D(|u_\delta|^p,|u_\delta|^p) < 0. \tag{16}$$

A rescaling argument completes the proof of the Lemma. □

Our next main result is the following.

Theorem 2. *Suppose* $2 \leq p < 7/3$. *For any* $\lambda > 0$ *and any* $\delta \in [0, 1/2)$ *one can find unique* $\mu(\lambda) \in (0, \infty)$, *such that for any* $\mu \in [0, \mu(\lambda)]$ *one can find radial positive minimizer*

$$u(x) = u_{\lambda,\mu,\delta}(x) \in H^1_{rad}(\mathbb{R}^3),$$

such that

$$E_{\mu\delta}(u_{\lambda,\mu,\delta}) = I(\lambda, \mu, \delta) = \min_{u \in S(\lambda,\mu)} \{E_{\mu\delta}(u)\}$$

and

$$M(|u_{\lambda,\mu,\delta}|^p) \begin{cases} > \mu, & \text{if } 0 \leq \mu < \mu(\lambda); \\ = \mu(\lambda), & \text{if } \mu = \mu(\lambda). \end{cases} \tag{17}$$

The special case of equation (7) with $p = 2$, is commonly referred to as the stationary Choquard equation and it arises in an approximation to Hartree–Fock theory for a one component plasma (for more information, see [7]). The existence and uniqueness of the ground state solution is proved by Lieb in [6], while Choquard, Stubbe and Vuffray obtained the same result in [1] for the equivalent Schrödinger–Newton system (4). Recently, the ground state solutions to the generalized nonlinear Choquard problem (7) with $p > 2$ and dimention $n \geq 3$ have been studied by many authors (for example, see [2, 4, 8, 9]).

2. Asymptotic behaviour of the solution

The relation (6) in the radial case becomes

$$A(r) = \int_0^r \left(\frac{1}{s} - \frac{1}{r}\right) u^p(s) s^2 ds > 0, \tag{18}$$

while the system (4) can be rewritten as

$$\begin{aligned} u'' + \frac{2}{r}\, u' + \omega u &= A u^{p-1}, \\ A'' + \frac{2}{r}\, A' &= u^p. \end{aligned} \tag{19}$$

First we can see that $\omega > 0$. Indeed, from (18) we know that $A > 0$ so from

$$-\Delta u + A u^{p-1} = \omega u$$

we see that $\omega > 0$. After rescaling we can assume $\omega = 1$.

We can rewrite (19) as integral equation

$$u(r) = \alpha + \int_0^r \left(\frac{1}{s} - \frac{1}{r}\right) u(s)(A(s)u^{p-2}(s) - 1)s^2 ds, \tag{20}$$

where $\alpha = u(0) > 0$.

It is not difficult to see that any radial H^1 solution to (7) satisfies the following rough estimates.

Lemma 3. *If $u(r) = u(r, \alpha)$ is a $H^1(\mathbb{R}^3)$ radial positive solution to (7), then it satisfies the estimates*

$$|u(r)| + (1+r)^{-3}|u'(r)| + (1+r)^{-2}|A(r)| + (1+r)^{-1}|A'(r)| \le C. \tag{21}$$

The proof follows easily from the properties

$$\lim_{r\to\infty} u(r) = 0, \quad u'(r) \le 0$$

and the integral equation (20) combined with

$$\begin{aligned} u'(r) &= \frac{1}{r^2}\int_0^r u(s)(A(s)u^{p-2}(s) - 1)s^2 ds, \\ A'(r) &= \frac{1}{r^2}\int_0^r u^p(s)s^2 ds, \end{aligned} \tag{22}$$

so we omit it.

The first improvement of the rough a priori estimates (21) can be done by using the radial lemma of Strauss [11] and use the implication

$$u(|x|) \in H^1(\mathbb{R}^3) \Longrightarrow |u(r)| \le \frac{C}{r}, \ \forall r > 0. \tag{23}$$

Further, the integral equations in (22) give the following upper bounds

$$|A'(r)| \le \begin{cases} Cr^{1-p}, & \text{if } 2 < p \le 3 \text{ and } r > 1; \\ Cr^{-2}, & \text{if } 3 \le p \le 5 \text{ and } r > 1, \end{cases} \tag{24}$$

so an integration in r gives

$$|A(r)| \le C \tag{25}$$

and we can conclude that $V(r) = A(r)u(r)^{p-2}$ obeys the estimates

$$|V(r)| = |A(r)u(r)^{p-2}| \le Cr^{2-p}. \tag{26}$$

Now we are ready to derive the Gaussian bound of the solution.

Lemma 4. *If $u(r) = u(r, \alpha)$ is a $H^1(\mathbb{R}^3)$ radial positive solution to* (20), *then it satisfies the estimates*

$$|u(r)| \le Ce^{-\delta r^2}, \ |u'(r)| \le Ce^{-\delta r^2} \tag{27}$$

for $r \ge 1$.

Proof. We know that

$$V(r) = A(r)u(r)^{p-2} = o(1), \ r \to \infty, \tag{28}$$

due to (26). Set

$$Z(r) = \frac{-u'(r)}{u(r)}.$$

Then we use the ordinary differential equation (19) and see that

$$Z' = Z^2 + 1 - \frac{2}{r}Z - V. \tag{29}$$

Therefore, for $r > r_0 \gg 1$ we get the inequality

$$Z' + \frac{2}{r}Z \ge \frac{1}{2},$$

which can be rewritten as

$$(r^2 Z)' \ge \frac{r^2}{2}.$$

Integrating the last inequality over (r_0, r), we find

$$r^2 Z(r) \ge \frac{r^3}{6} - C_0, \ C_0 = \frac{r_0^3}{6} - r_0^2 Z(r_0),$$

so taking $r > r_1 \gg r_0$, we can write

$$Z(r) = \frac{-u'(r)}{u(r)} \ge \frac{r}{8}.$$

Integrating again, the last inequality gives

$$|u(r)| \le Ce^{-r^2/16}. \tag{30}$$

To evaluate $|u'(r)|$ from above we have to estimate $Z(r)$ from above for $r > r_0 \gg 1$. It is not difficult to see that the domain

$$\mathcal{U}_{r_0} = \{(r, z) \in \mathbb{R} \times \mathbb{R}; r > r_0, z > 4\}$$

is forbidden for the orbit $(r, Z(r))$ with $r > r_0$. Indeed, if the orbit enters $\mathcal{U}_{r_0}$ we can use the inequality

$$Z' \geq \frac{Z^2}{2}$$

and the qualitative study of this inequality will lead to a blow-up of $Z(r)$, which is impossible, due to the assumption that u is a radial decreasing solution in $H^1(\mathbb{R}^3)$. This observation shows that $Z(r) = -u'(r)/u(r) \leq 4$, so we can use (30) and arrive at

$$|u'(r)| \leq Ce^{-r^2/16}. \tag{31}$$

This completes the proof of Lemma 4. □

3. Proof of Theorem 1.

If $(u(r), A(r))$ is a radial positive solution to (19), then we can use the Gaussian bounds (27), so setting

$$V(r) = Au^{p-2} \sim O(e^{-\delta_1 r^2})$$

for $r \to \infty$ we see that $u \in H^1_{rad}(\mathbb{R}^3)$ is a solution to the equation

$$-\Delta u + Vu = u. \tag{32}$$

Now we are in position to apply the result due to Kato [5] and deduce from the fact that $V(r)$ decays exponentially and u is exponentially decreasing function satisfying (32), that $u = 0$.

4. Idea of the proof of Theorem 2

We apply concentrated compactness argument combined with the following.

Lemma 5. *Assume $p \in [2, 7/3)$. One can find positive constants $C_1 < C_2$ so that if $u \in S(1, \mu)$, $\mu > 0$ and $E_{\mu\delta}(u) \leq C_1$, then*

$$\|\nabla u\|^2_{L^2} + M(|u|^p)\|u\|^p_{L^p} + D(|u|^p, |u|^p) \leq C_2.$$

P r o o f. Take any $u \in S(1, \mu)$. We aim to show that one can find $C_1 > 0$, independent of μ, such that

$$\frac{1}{2}\|u\|^2_{L^2} = 1, \Longrightarrow E_{\mu\delta}(u) \geq -C_1. \tag{33}$$

To verify this property, we can use the relation

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |u(x)|^p |u(y)|^p \frac{dxdy}{|x-y|} = \left\| (-\Delta)^{-1/2} |u|^p \right\|^2_{L^2}$$

and via Sobolev embedding

$$\left\| (-\Delta)^{-1/2} g \right\|_{L^2} \leq C \|g\|_{L^{6/5}},$$

and the Gagliardo–Nirenberg interpolation inequality to get the estimate

$$D(|u|^p, |u|^p) \leq C \|u\|_{L^2}^{5-p} \|\nabla u\|_{L^2}^{3p-5}. \tag{34}$$

In this way we have the inequality

$$E_\nu(u) \geq \frac{1}{2} \|\nabla u\|^2_{L^2} - C \|\nabla u\|_{L^2}^{3p-5},$$

due to the normalization assumption in (33). Now, we can set $Y = \|\nabla u\|^2_{L^2}$ and rewrite the above estimate as follows

$$E_\nu(u) \geq \varphi(Y), \ \varphi = \frac{Y}{2} - C Y^{(3p-5)/2}.$$

Since the function $\varphi(Y) : Y \in [0, \infty) \to \mathbb{R}$ is bounded from below for $(3p-5)/2 < 1$ or $p < 7/3$, we find

$$E_\nu(u) \geq -C_1,$$

so (33) is established with a constant $C_1 > 0$ independent of $\nu > 0$.

Careful analysis of the previous argument shows that $\varphi(Y) < C_1$ implies $Y \leq C_2$ and hence, there exist positive constant $C_1 < C_2$ so that

$$\frac{\|u\|^2_{L^2}}{2} = 1, \ E_\nu(u) \leq C_1 \Longrightarrow \ \|\nabla u\|^2_{L^2} \leq C_2. \tag{35}$$

Finally, using a combination of Hardy inequality and Gagliardo–Nirenberg interpolation inequality for $2 \leq p \leq 4$, we obtain

$$M(u^p) = \int_{\mathbb{R}^3} \frac{u^p(x)dx}{|x|} \leq C \left\| \frac{u}{|x|} \right\|_{L^2} \|u\|^{p-1}_{L^{2(p-1)}} \leq C \|\nabla u\|_{L^2}^{\frac{3p-4}{2}} \|u\|_{L^2}^{\frac{4-p}{2}}. \tag{36}$$

This inequality and (34) imply

$$u \in S(1,\nu), \ E(u) \leq C_1 \Longrightarrow \nu + \|u\|^p_{L^p} + D(|u|^p, |u|^p) \leq C_2. \tag{37}$$

This completes the proof of the Lemma. □

References

[1] P. Choquard, J. Stubbe, M. Vuffray. Stationary solutions of the Schrödinger–Newton model an ODE approach. *Differential Integral Equations*, **21**, (2008), 665-679.

[2] S. Cingolani, M. Clapp, S. Secchi. Multiple solutions to a magnetic nonlinear Choquard equation. *Z. Angew. Math. Phys.*, **63**, (2012), 233-248.

[3] L. Diosi. Gravitation and quantum-mechanical localization of macro-objects. *Physics Letters A*, **105** (1984), 199–202.

[4] H. Genev, G. Venkov. Soliton and blow-up solutions to the time-dependent Schrödinger–Hartree equation. *Discrete and continuous dynamical systems, Ser. S* , **5** (2012), 903–923.

[5] T. Kato. Growth properties of solutions of the reduced wave equation with a variable coefficient. *Comm. Pure and Applied Math.*, **12** (1959), 403–425.

[6] E. Lieb. Existence and uniqueness of the minimizing solution of Choquards nonlinear equation. *Studies in Applied Math.*, **57** (1977), 93–105.

[7] E. Lieb, B. Simon. The Hartree-Fock theory for Coulomb systems. *Comm. Math. Phys.*, **53**, (1977), 185–194.

[8] L. Ma, L. Zhao. Classification of positive solitary solutions of the nonlinear Choquard equation. *Arch. Ration. Mech. Anal.*, **195** (2010), 455-467.

[9] V. Moroz, J. Van Schaftingen. Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics. *J. Funct. Anal.*, **265** (2013), 153-184.

[10] R. Penrose. On Gravity's role in quantum state reduction. *General Relativity and Gravitation*, **28**, (1996), 581–600.

[11] A. Strauss. Existence of solitary waves in higher dimension. *Comm. Math. Phys.*, **55**, (1977), 149–162.

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