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NEW TYPES OF TWO COMPONENT NLS-TYPE EQUATIONS

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We study MNLS related to the D.III-type symmetric spaces. Applying to them Mikhailov reduction groups of the type $\mathbb{Z}_r \times \mathbb{Z}_2$ we derive new types of 2-component NLS equations. These are **not** counterexamples to the Zakharov-Schulman theorem because the corresponding interaction Hamiltonians depend not only on $|u_k|^2$, but also on $u_1 u_2^* + u_1^* u_2$.

1. Introduction

The non-linear Schrodinger equation

$$i\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + |u|^2 u = 0, \quad u = u(x,t) \tag{1}$$

was first solved by Zakharov and Shabat in 1971 [27]. Since then, it has found numerous applications [6, 26]. The first multi-component NLS with applications to physics is the Manakov model [21, 22]:

$$\begin{aligned} &i\frac{\partial u_1}{\partial x} + \frac{1}{2}\frac{\partial^2 u_1}{\partial x^2} + (|u_1|^2 + |u_2|^2)u_1 = 0, \\ &i\frac{\partial u_2}{\partial x} + \frac{1}{2}\frac{\partial^2 u_2}{\partial x^2} + (|u_1|^2 + |u_2|^2)u_2 = 0. \end{aligned} \tag{2}$$

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It is natural to look for other types of multi-component generalizations. Such generalizations were analyzed in [14] and this work can be viewed as its continuation. There is a close relationship between MNLS equations and homogeneous and symmetric spaces [7].

Soon after the pioneer paper by Zakharov and Shabat [27], Manakov [21, 21] proposed a two-component NLS model. Both NLS models have important applications in nonlinear optics, plasma physics, hydrodynamics etc. Manakov model was easily generalized to N-components known as vector NLS; it has also non-Euclidean version (see [20]):

$$
\begin{aligned}
& i\frac{\partial u_1}{\partial x} + \frac{1}{2}\frac{\partial^2 u_1}{\partial x^2} + (|u_1|^2 - |u_2|^2)u_1 = 0, \\
& i\frac{\partial u_2}{\partial x} + \frac{1}{2}\frac{\partial^2 u_2}{\partial x^2} + (|u_1|^2 - |u_2|^2)u_2 = 0.
\end{aligned} \tag{3}
$$

Here we should mention another famous paper by Zakharov and Schulman [28] where they prove a theorem classifying the integrable two-component NLS systems. They request that the interaction Hamiltonian depends only on $|u_1|^2$ and $|u_2|^2$ and prove that eqs. (2) and (3) are the only integrable MNLS.

The next step in studying multicomponent NLS equations is based on the important idea of Fordy and Kulish relating the MNLS equations to the symmetric spaces [7].

The present paper is continuation of [7] and a sequel of papers [10, 12, 13, 14, 11, 15, 16, 17, 18] in which Mikhailov's reduction group [23] was applied on the MNLS thus deriving new versions with small number of components. Below we limit ourselves with the MNLS related to the D.III-type symmetric spaces and using Mikhailov reduction groups of the type $\mathbb{Z}_r \times \mathbb{Z}_2$ derive new types of 2-component NLS equations. These are **not** counterexamples to the Zakharov-Schulman theorem because the corresponding interaction Hamiltonians depend not only on $|u_k|^2$, but also on $u_1 u_2^* + u_1^* u_2$.

In Section 2 we collect preliminary facts about the D.III symmetric spaces and the types of reductions that will be applied to the Lax pairs. In Section 3 we analyze two types of $\mathbb{Z}_4$-reductions of MNLS related to the algebra $so(8)$. In Section 4 we analyze $\mathbb{Z}_5$-reductions of MNLS related to the algebra $so(10)$. To each of these cases we relate a new two-component NLS equation. In Section 5 we formulate the consequences of these reductions for the Jost solutions and the scattering matrix, We finish with discussions and conclusions.

2. Preliminaries

2.1. Symmetric spaces and $\mathbb{Z}_2$-gradings

We assume that reader is familiar with the basic properties of the simple Lie groups and Lie algebras, see [19]. The Cartan-Weyl basis of the algebras of the $\mathrm{D_r}$-series, $r \geq 4$ are given in the Appendix.

Here will briefly remind the well known facts about the D.III-type symmetric spaces which is $SO^*(2r)/U(r)$ and the structure of its local coordinates [19]. Each of these symmetric spaces is generated by a Cartan involution which induces a $\mathbb{Z}_2$-grading on the Lie algebra which in our case is $\mathfrak{g} \simeq so(2r)$. The root system of $\mathfrak{g}$ is:

$$\Delta = \Delta^+ \cup (-\Delta^+), \qquad \Delta^+ \equiv \{e_i - e_j, \quad e_i + e_j, \quad 1 \leq i < j \leq r\}, \tag{4}$$

The $\mathbb{Z}_2$-grading is induced by the Cartan element $J = \sum_{s=1}^r H_s$. It induces a $\mathbb{Z}_2$-grading of $so(2r) \equiv \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ as follows:

$$\mathfrak{g}^{(0)} \equiv \{X \in \mathfrak{g} \colon [J, X] = 0\}, \qquad \mathfrak{g}^{(1)} \equiv \{Y \in \mathfrak{g} \colon JY + YJ = 0\}. \tag{5}$$

This grading splits the set of the positive roots $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$ into subsets:

$$\Delta_0^+ \equiv \{e_i - e_j, \quad 1 \leq i < j \leq r\}, \qquad \Delta_1^+ \equiv \{e_i + e_j, \quad 1 \leq i < j \leq r\} \tag{6}$$

i.e.

$$\alpha \in \Delta_0^+ \quad \text{iff} \quad \alpha(J) = 0, \qquad \beta \in \Delta_1^+ \quad \text{iff} \quad \beta(J) = 2. \tag{7}$$

We will need also the co-adjoint orbit in $\mathfrak{g}$ passing through J which coincides with the linear functionals (depending on x and t) over the linear subspace $\mathfrak{g}^{(1)}$. We will denote it by $\mathcal{M}_J$; a generic $Q(x,t)$ element in it is provided by:

$$Q(x,t) = \sum_{\beta \in \Delta_1^+} (q_\beta(x,t) E_\beta + p_\beta(x,t) E_{-\beta}) = \begin{pmatrix} 0 & \boldsymbol{q} \\ \boldsymbol{p} & 0 \end{pmatrix}. \tag{8}$$

where $\boldsymbol{q}(x,t)$ and $\boldsymbol{p}(x,t)$ are $r \times r$ block matrices. For simplicity and definitiveness we will assume that the $q_\beta(x,t)$ and $p_\beta(x,t)$ are Schwartz-type functions of their variables.

2.2. Reductions

An important and systematic tool to construct new integrable NLEE is the so-called reduction group [23]. It will be instructive to start with the local $\mathbb{Z}_2$-reductions:

$$
\begin{aligned}
&(9) && A_1U^\dagger(x,t,\kappa_1\lambda^*)A_1^{-1} = U(x,t,\lambda), && A_1V^\dagger(x,t,\kappa_1\lambda^*)A_1^{-1} = V(x,t,\lambda),\\
&(10) && A_2U^T(x,t,\kappa_2\lambda)A_2^{-1} = -U(x,t,\lambda), && A_2V^T(x,t,\kappa_2\lambda)A_2^{-1} = -V(x,t,\lambda),\\
&(11) && A_3U^*(x,t,\kappa_1\lambda^*)A_3^{-1} = -U(x,t,\lambda), && A_3V^*(x,t,\kappa_1\lambda^*)A_3^{-1} = -V(x,t,\lambda),\\
&(12) && A_4U(x,t,\kappa_2\lambda)A_4^{-1} = U(x,t,\lambda), && A_4V(x,t,\kappa_2\lambda)A_4^{-1} = V(x,t,\lambda).
\end{aligned}
$$

The consequences of these reductions and the constraints they impose on the FAS and the Gauss factors of the scattering matrix are well known, see [23, 26]. Since we are dealing with D_r-algebras we take into account that $X \to -X^T$ is an inner automorphism. This means that reduction (9) is equivalent to reduction (11) and reduction (10) is equivalent to reduction of (12). Therefore it will be enough to consider only reductions (11) and (12).

Along with $\mathbb{Z}_2$ we will need also $\mathbb{Z}_p$-reductions:with $p > 2$. If p is odd, we can use only reductions of type (12).

$$
\begin{aligned}
&(13) && B_1U^\dagger(x,t,\kappa_1(\lambda^*))B_1^{-1} = U(x,t,\lambda), && B_1V^\dagger(x,t,\kappa_1(\lambda^*))B_1^{-1} = V(x,t,\lambda),\\
&(14) && B_4U(x,t,\kappa_4(\lambda))B_4^{-1} = U(x,t,\lambda), && B_4V(x,t,\kappa_4(\lambda))B_4^{-1} = V(x,t,\lambda).
\end{aligned}
$$

where the functions κ_3 and κ_4 if applied p times satisfy $\kappa_j(\kappa_j(...\kappa_j(\lambda)...)) = \lambda$, $j = 3, 4$. In addition we will use also $\mathbb{Z}_r$-reductions of the form

$$
A_1U(x,t,\epsilon\lambda)A_1^{-1} = U(x,t,\lambda), \quad A_1V(x,t,\epsilon\lambda)A_1^{-1} = V(x,t,\lambda) \tag{15}
$$

where $\epsilon = \pm 1$ and $A_1^r = \mathbb{1}$ with $r > 2$; if r is odd then $\epsilon = 1$. We will demonstrate below examples when such reductions with $\epsilon = 1$ provide new types of NLEE.

2.3. Lax pair and reductions

Below we outline the formulation of the Lax pair for the D.III-type symmetric spaces:

$$
\begin{aligned}
&L(\lambda)\psi \equiv \left(i\frac{d}{dx} + Q(x,t) - \lambda J \right) \psi(x,t,\lambda) = 0,\\
&M(\lambda)\psi \equiv \left(i\frac{d}{dt} - \frac{1}{2}[Q, \mathrm{ad}\,_J^{-1}Q] + i\mathrm{ad}_J^{-1}\frac{\partial Q}{\partial x} + \lambda Q - \lambda^2 J \right) \psi(x,t,\lambda) = 0,
\end{aligned} \tag{16}
$$

where $Q = Q(x,t)$ and J are elements of the algebra $so(2r)$. In other words they are $2r \times 2r$ matrices with the following block-matrix form:

$$Q(x,t) = \begin{pmatrix} 0 & \boldsymbol{q}(x,t) \\ \boldsymbol{p}(x,t) & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{17}$$

The compatibility condition $[L(\lambda), M(\lambda)] = 0$ of the operators in (16) gives the general form of the D.III-type MNLS equations on symmetric spaces. It can be viewed as block-matrix generalization of the AKNS system [1]; see also [2]:

$$\frac{i}{2}\left[J, \frac{dQ}{dt}\right] + \frac{1}{2}\frac{d^2Q}{dx^2} - \left[\operatorname{ad}_J^{-1}Q, [\operatorname{ad}_J^{-1}Q, Q]\right] = 0. \tag{18}$$

Consider the Lax pair of the Zakharov - Shabat system

$$\begin{aligned} L\psi(x,t,\lambda) &= \left(i\frac{\partial}{\partial x} + U(x,t,\lambda)\right)\psi(x,t,\lambda) = 0, \qquad U = Q(x,t) - \lambda J, \\ M\psi(x,t,\lambda) &= \left(i\frac{\partial}{\partial t} + V(x,t,\lambda)\right)\psi(x,t,\lambda) = 0, \qquad V = V_0 + \lambda Q - \lambda^2 J. \end{aligned} \tag{19}$$

Let $\mathcal{M}_J$ be the co-adjoint orbit of $\mathfrak{g}$ passing through J. Then $Q(x,t) \in \mathcal{M}_J$.

As mentioned above, the choice of J determines the dimension of $\mathcal{M}_J$ which can be viewed as the phase space of the relevant nonlinear evolution equations (NLEE). It is equal to the number of roots of $\mathfrak{g}$ such that $\alpha(J) \neq 0$. Taking into account that if α is a root, then and $-\alpha$ is also a root of $\mathfrak{g}$ then $\dim \mathcal{M}_J$ is always even. Since all the examples are related to symmetric spaces of D.III-type it is natural to choose J as in (17). As a consequence $\mathfrak{g}^{(0)}$ which can be viewed as the kernel of the operator ad_J) is non-commutative and isomorphic to $so(r) \oplus so(r)$.

Below the automorphisms A_i and B_k used for reductions will be inner and will correspond to compositions of Weyl reflections. They will act as similarity transformations with $2r \times 2r$ matrices that belong to the $SO(2r)$ group and satisfy

$$A^r(X)A^{-r} \equiv X, \qquad B^r(X)B^{-r} \equiv X, \qquad \forall X \in \mathfrak{g}. \tag{20}$$

From eqs. (15) and (19) there follows that they must either preserve J or change its sign, i.e.

$$\begin{aligned} &AJA^{-1} = J, && BJB^{-1} = -J, \\ &A = \begin{pmatrix} \boldsymbol{a}_1 & 0 \\ 0 & -\boldsymbol{a}_1^T \end{pmatrix}, && B = \begin{pmatrix} 0 & \boldsymbol{b}_1 \\ \boldsymbol{b}_2 & 0 \end{pmatrix}, \\ &\hat{\boldsymbol{a}}_2 = \hat{\boldsymbol{s}}_1 \boldsymbol{a}_1^T \boldsymbol{s}_1, && \hat{\boldsymbol{b}}_2 = \boldsymbol{s}_1 \boldsymbol{b}_1^T \boldsymbol{s}_1. \end{aligned} \tag{21}$$

The compatibility condition $[L, M] = 0$ leads to the MNLS of the form:

$$i\frac{\partial Q}{\partial t} + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2} + [Q, V_0] = 0. \tag{22}$$

We will consider further reductions of the above system. Namely, we will consider reduction of type $\mathbb{Z}_r$ of the form

$$\begin{aligned} AU(x,t,\lambda)A^{-1} &= U(x,t,\lambda), & AV(x,t,\lambda)A^{-1} &= V(x,t,\lambda), \\ BU(x,t,-\lambda)B^{-1} &= U(x,t,\lambda), & BV(x,t,-\lambda)B^{-1} &= V(x,t,\lambda), \end{aligned} \tag{23}$$

where A and B are automorphisms of order r. This will restrict the number of the independent variables in Q to twice the number of orbits of A. This means that the potential of L takes the form

$$\begin{aligned} Q_A &= \sum_{\alpha\in\delta_A^+} q_\alpha(x,t)\mathcal{E}_\alpha^A + p_\alpha(x,t)\mathcal{E}_{-\alpha}^A, & \mathcal{E}_\alpha^A &= \sum_{s=0}^{r-1} A^s E_\alpha A^{-s}, \\ Q_B &= \sum_{\alpha\in\delta_B^+} q_\alpha(x,t)\mathcal{E}_\alpha^B + p_\alpha(x,t)\mathcal{E}_{-\alpha}^B, & \mathcal{E}_\alpha^B &= \sum_{s=0}^{r-1} B^s E_\alpha B^{-s}, \end{aligned} \tag{24}$$

and δ_A^+ and δ_B^+ contains only one root α from each orbit of the corresponding automorphism.

In addition we will impose also a $\mathbb{Z}_2$ reductions of the type

$$U^\dagger(x,t,\lambda^*) = U(x,t,\lambda), \qquad V^\dagger(x,t,\lambda^*) = V(x,t,\lambda). \tag{25}$$

This reduction will restrict the form of Q to $p = q^\dagger$.

3. MNLS with $\mathbb{Z}_4$-reductions related to D_4

We will consider $\mathfrak{g} = D_4$ and $J = \operatorname{diag}(1,1,1,1,-1,-1,-1,-1)$. There are two realizations of the reduction (23) that will give two-component MNLS:

$$A = S_{e_1-e_2} \circ S_{e_2-e_3} \circ S_{e_3-e_4}, \qquad B = S_{e_1+e_2} \circ S_{e_2+e_3} \circ S_{e_3+e_4} \tag{26}$$

In the first case A acts in the root space by $A : e_1 \to e_2 \to e_3 \to e_4 \to e_1$ and splits Δ into 8 orbits

$$\begin{aligned} &\mathcal{O}_1^\pm : \pm(e_1+e_2) \to \pm(e_2+e_3) \to \pm(e_3+e_4) \to \pm(e_1+e_4), \\ &\mathcal{O}_2^\pm : \pm(e_1+e_3) \to \pm(e_2+e_4), \\ &\mathcal{O}_3^\pm : \pm(e_1-e_2) \to \pm(e_2-e_3) \to \pm(e_3-e_4) \to \pm(e_4-e_1), \\ &\mathcal{O}_4^\pm : \pm(e_1-e_3) \to \pm(e_2-e_4) \to \pm(e_3-e_1) \to \pm(e_4-e_2). \end{aligned} \tag{27}$$

four of which $\mathcal{O}_1^{\pm} \cup \mathcal{O}_2^{\pm}$ span the set of roots $\Delta_1^+ \cup \Delta_1^-$; the other four orbits $\mathcal{O}_3^{\pm} \cup \mathcal{O}_4^{\pm}$ span the set of roots $\Delta_0^+ \cup \Delta_0^-$.

In the second case B acts in the root space by $B : e_1 \to -e_2 \to e_3 \to -e_4 \to e_1$ and splits Δ into 8 orbits

$$
\begin{aligned}
&\tilde{\mathcal{O}}_1^{\pm} : \pm(e_1 + e_2) \to \mp(e_2 + e_3) \to \pm(e_3 + e_4) \to \mp(e_1 + e_4),\\
&\tilde{\mathcal{O}}_2^{\pm} : \pm(e_1 + e_3) \to \mp(e_2 + e_4),\\
&\tilde{\mathcal{O}}_3^{\pm} : \pm(e_1 - e_2) \to \mp(e_2 - e_3) \to \pm(e_3 - e_4) \to \mp(e_4 - e_1),\\
&\tilde{\mathcal{O}}_4^{\pm} : \pm(e_1 - e_3) \to \mp(e_2 - e_4) \to \pm(e_3 - e_1) \to \mp(e_4 - e_2).
\end{aligned} \tag{28}
$$

four of which $\tilde{\mathcal{O}}_1^{\pm} \cup \tilde{\mathcal{O}}_2^{\pm}$ span the set of roots $\Delta_1^+ \cup \Delta_1^-$; the other four orbits $\tilde{\mathcal{O}}_3^{\pm} \cup \tilde{\mathcal{O}}_4^{\pm}$ span the set of roots $\Delta_0^+ \cup \Delta_0^-$.

We will consider the first case. The potential of the Lax operator $Q(x,t)$ (see (8)) is given by

$$
\boldsymbol{q}(x,t) = \begin{pmatrix} q_1 & \sqrt{2}q_2 & q_1 & 0 \\ \sqrt{2}q_2 & q_1 & 0 & q_1 \\ -q_1 & 0 & q_1 & -\sqrt{2}q_2 \\ 0 & -q_1 & -\sqrt{2}q_2 & q_1 \end{pmatrix}, \qquad \boldsymbol{p}(x,t) = \boldsymbol{q}^{\dagger}(x,t). \tag{29}
$$

Imposing also the second reduction (25) ($p_i = q_i^*$) the equations become

$$
\begin{aligned}
&i\frac{\partial q_1}{\partial t} + \frac{1}{2}\frac{\partial q_1}{\partial x^2} + 2q_1(|q_1|^2 + 2|q_2|^2) + 2q_2^2 q_1^* = 0,\\
&i\frac{\partial q_2}{\partial t} + \frac{1}{2}\frac{\partial q_2}{\partial x^2} + 2q_2(2|q_1|^2 + |q_2|^2) + 2q_1^2 q_2^* = 0.
\end{aligned} \tag{30}
$$

They admit a Hamiltonian formulation, with a Hamiltonian density given by

$$
\mathcal{H} = \frac{1}{2}\left|\frac{\partial q_1}{\partial x}\right|^2 + \frac{1}{2}\left|\frac{\partial q_2}{\partial x}\right|^2 - (|q_1|^2 + |q_2|^2)^2 - (q_1 q_2^* + q_1^* q_2)^2 \tag{31}
$$

4. MNLS with $\mathbb{Z}_5$-reductions related to D_5

Let $\mathfrak{g} = D_4$ and $J = \operatorname{diag}(1,1,1,1,1,-1,-1,-1,-1,-1)$. Again, we will also impose (23) with $A = S_{e_1-e_2} \circ S_{e_2-e_3} \circ S_{e_3-e_4} \circ S_{e_4-e_5}$, that is $A : e_1 \to e_2 \to e_3 \to e_4 \to e_5$.

This splits Δ into orbits

$$
\begin{aligned}
&\mathcal{O}_1^{\pm} : \pm(e_1 + e_2) \to \pm(e_2 + e_3) \to \pm(e_3 + e_4) \to \pm(e_4 + e_5) \to \pm(e_1 + e_5),\\
&\mathcal{O}_2^{\pm} : \pm(e_1 + e_3) \to \pm(e_2 + e_4) \to \pm(e_3 + e_5) \to \pm(e_1 + e_4) \to \pm(e_2 + e_5),\\
&\mathcal{O}_3^{\pm} : \pm(e_1 - e_2) \to \pm(e_2 - e_3) \to \pm(e_3 - e_4) \to \pm(e_4 - e_5) \to \mp(e_1 - e_5),\\
&\mathcal{O}_4^{\pm} : \pm(e_1 - e_3) \to \pm(e_2 - e_4) \to \pm(e_3 - e_5) \to \mp(e_1 - e_4) \to \mp(e_2 - e_5).
\end{aligned} \tag{32}
$$

The $\mathbb{Z}_5$ reduction is realized by a type-A automorphism as in (21) with

$$\boldsymbol{a}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{33}$$

The reduced potential $Q(x,t)$ as in (17) with

$$\boldsymbol{q}(x,t) = \begin{pmatrix} q_1 & q_2 & q_2 & q_1 & 0 \\ q_2 & -q_2 & q_1 & 0 & q_1 \\ q_2 & q_1 & 0 & q_1 & -q_2 \\ q_1 & 0 & q_1 & q_2 & q_2 \\ 0 & q_1 & -q_2 & q_2 & -q_1 \end{pmatrix}, \quad \boldsymbol{p}(x,t) = \begin{pmatrix} p_1 & p_2 & p_2 & p_1 & 0 \\ p_2 & -p_2 & p_1 & 0 & p_1 \\ p_2 & p_1 & 0 & p_1 & -p_2 \\ p_1 & 0 & p_1 & p_2 & p_2 \\ 0 & p_1 & -p_2 & p_2 & -p_1 \end{pmatrix}. \tag{34}$$

After imposing the second reduction $\boldsymbol{p} = \boldsymbol{q}^\dagger$ the equations become

$$\begin{aligned} i\frac{\partial q_1}{\partial t} + \frac{1}{2}\frac{\partial^2 q_1}{\partial x^2} + q_1(3|q_1|^2 + 4|q_2|^2) + q_2(2|q_1|^2 - |q_2|^2) + q_1^2 q_2^* + 2q_1^* q_2^2 = 0, \\ i\frac{\partial q_2}{\partial t} + \frac{1}{2}\frac{\partial^2 q_2}{\partial x^2} + q_1(|q_1|^2 - 2|q_2|^2) + q_2(4|q_1|^2 + 3|q_2|^2) - q_1^* q_2^2 + 2q_1^2 q_2^* = 0. \end{aligned} \tag{35}$$

The above equations admit a Hamiltonian formulation, with a Hamiltonian density given by

$$\begin{aligned} \mathcal{H} = \frac{1}{2}\left|\frac{\partial q_1}{\partial x}\right|^2 + \frac{1}{2}\left|\frac{\partial q_2}{\partial x}\right|^2 - (|q_1|^2 + |q_2|^2)^2 \\ - \frac{1}{2}(|q_1|^2 + q_1^* q_2 + q_1 q_2^*)^2 - \frac{1}{2}(|q_2|^2 - q_1^* q_2 - q_1 q_2^*)^2 \end{aligned} \tag{36}$$

5. Direct and inverse scattering problems

Basic tools in this analysis are the Jost solutions (we will avoid writing explicit time dependence, to avoid cluttering the notation)

$$\lim_{x\to-\infty} \phi(x,\lambda)e^{iJ\lambda x} = \mathbb{1}, \qquad \lim_{x\to\infty} \psi(x,\lambda)e^{iJ\lambda x} = \mathbb{1}. \tag{37}$$

Formally the Jost solutions must satisfy Volterra type integral equations. If we introduce

$$\xi(x,\lambda) = \psi(x,\lambda)e^{i\lambda Jx}, \quad \eta(x,\lambda) = \phi(x,\lambda)e^{i\lambda Jx}, \tag{38}$$

then $\xi_{\pm}(x,\lambda)$ must satisfy

$$\begin{aligned}\xi(x,t) &= \mathbb{1} + i\int_{\infty}^{x} dy\, e^{-i\lambda J(x-y)} Q(y,t)\xi(y,\lambda) e^{i\lambda J(x-y)}, \\ \eta(x,t) &= \mathbb{1} + i\int_{-\infty}^{x} dy\, e^{-i\lambda J(x-y)} Q(y,t)\eta(y,\lambda) e^{i\lambda J(x-y)}.\end{aligned} \tag{39}$$

The Jost solutions can not be extended for $\mathrm{Im}\lambda \neq 0$. However some of their columns can be extended for $\lambda \in \mathbb{C}_+$ or $\lambda \in \mathbb{C}_-$. The Jost solutions can be written in the following block-matrix form

$$\psi(x,\lambda) = \begin{pmatrix} \boldsymbol{\psi}_1^-(x,\lambda) & \boldsymbol{\psi}_1^+(x,\lambda) \\ \boldsymbol{\psi}_2^-(x,\lambda) & \boldsymbol{\psi}_2^+(x,\lambda) \end{pmatrix}, \quad \phi(x,\lambda) = \begin{pmatrix} \boldsymbol{\phi}_1^+(x,\lambda) & \boldsymbol{\phi}_1^-(x,\lambda) \\ \boldsymbol{\phi}_2^+(x,\lambda) & \boldsymbol{\phi}_2^-(x,\lambda) \end{pmatrix}, \tag{40}$$

where the superscript $\pm$ shows that the corresponding $r\times r$ block allows analytic extension for $\lambda \in \mathbb{C}_{\pm}$. Then the scattering matrix is introduced by

$$T(\lambda) = \hat{\psi}(x,\lambda)\phi(x,\lambda), \quad T(\lambda) = \begin{pmatrix} \boldsymbol{a}^+(\lambda) & -\boldsymbol{b}^-(\lambda) \\ \boldsymbol{b}^+(\lambda) & \boldsymbol{a}^-(\lambda) \end{pmatrix} \tag{41}$$

where by "hat" we denote matrix inverse. Since without loss of generality the Jost solutions are group elements the scattering matrix will also be a group element. Note that if we impose one of the $\mathbb{Z}_r$ reduction (A or B) on the Lax pair then $T(\lambda,t)$ must satisfy one of the relations below.

$$AT(\lambda,t)A^{-1} = T(\lambda), \qquad BT(-\lambda,t)B^{-1} = T(\lambda). \tag{42}$$

This imposes the following constraints on the blocks $\boldsymbol{a}^{\pm}(\lambda)$ and $\boldsymbol{b}^{\pm}(\lambda,t)$:

$$\begin{aligned}\boldsymbol{a}^+(\lambda) &= \boldsymbol{a}_1\boldsymbol{a}^+(\lambda)\hat{\boldsymbol{a}}_1, & \boldsymbol{b}^-(\lambda,t) &= \boldsymbol{a}_1\boldsymbol{b}^-(\lambda,t)\hat{\boldsymbol{a}}_2, \\ \boldsymbol{b}^+(\lambda,t) &= \boldsymbol{a}_2\boldsymbol{b}^+(\lambda,t)\hat{\boldsymbol{a}}_1, & \boldsymbol{a}^-(\lambda) &= \boldsymbol{a}_2\boldsymbol{a}^-(\lambda)\hat{\boldsymbol{a}}_2,\end{aligned} \tag{43}$$

for the type-A reductions and

$$\begin{aligned}\boldsymbol{a}^+(\lambda) &= \boldsymbol{b}_1\boldsymbol{a}^-(-\lambda)\hat{\boldsymbol{b}}_1, & \boldsymbol{b}^-(\lambda,t) &= -\boldsymbol{b}_1\boldsymbol{b}^+(-\lambda,t)\hat{\boldsymbol{b}}_2, \\ \boldsymbol{b}^+(\lambda,t) &= -\boldsymbol{b}_2\boldsymbol{b}^-(-\lambda,t)\hat{\boldsymbol{b}}_1, & \boldsymbol{a}^-(\lambda) &= \boldsymbol{b}_2\boldsymbol{a}^+(-\lambda)\hat{\boldsymbol{b}}_2.\end{aligned} \tag{44}$$

where the constant matrices $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$, $k=1,2$ are given in eq. (21).

The second reduction $Q(x,t) = Q^{\dagger}(x,t)$ imposes on the Jost solutions and on the scattering matrix the constraints (below "hat" denotes matrix inverse) :

$$\begin{aligned}\psi^{\dagger}(x,t,\lambda^*) &= \hat{\psi}(x,t,\lambda), & \phi^{\dagger}(x,t,\lambda^*) &= \hat{\phi}(x,t,\lambda), \\ T^{\dagger}(\lambda^*,t) &= \hat{T}(\lambda,t), & \hat{T}(\lambda,t) &= \begin{pmatrix} \boldsymbol{c}^- & \boldsymbol{d}^- \\ -\boldsymbol{d}^+ & \boldsymbol{c}^+ \end{pmatrix}.\end{aligned} \tag{45}$$

The corresponding blocks of $T(\lambda,t)$ and its inverse $\hat{T}(\lambda,t)$ must satisfy

$$
\begin{aligned}
(\boldsymbol{a}^+)^\dagger(\lambda^*) &= \boldsymbol{c}^-(\lambda), & (\boldsymbol{b}^-)^\dagger(\lambda^*,t) &= \boldsymbol{d}^-(\lambda,t),\\
(\boldsymbol{b}^+)^\dagger(\lambda^*,t) &= -\boldsymbol{d}^+(\lambda,t), & (\boldsymbol{a}^-)^\dagger(\lambda^*) &= \boldsymbol{c}^+(\lambda).
\end{aligned} \tag{46}
$$

We end this section by formulating the time-dependence of the scattering matrix which follows naturally from the Lax representation (16):

$$
i\frac{\partial T}{\partial t} - \lambda^2[J,T(\lambda,t)] = 0, \qquad i\frac{\partial \hat{T}}{\partial t} - \lambda^2[J,\hat{T}(\lambda,t)] = 0, \tag{47}
$$

i.e.

$$
\begin{aligned}
\frac{\partial \boldsymbol{a}^\pm}{\partial t} &= 0, & i\frac{\partial \boldsymbol{b}^\pm}{\partial t} \mp b^\pm(\lambda,t) &= 0,\\
\frac{\partial \boldsymbol{c}^\pm}{\partial t} &= 0, & i\frac{\partial \boldsymbol{d}^\pm}{\partial t} \pm b^\pm(\lambda,t) &= 0.
\end{aligned} \tag{48}
$$

In particular, the diagonal blocks can be viewed as generating functionals of the integrals of motion of the MNLS.

6. Discussion and conclusions

The inverse scattering method, applied to the scalar NLS equation has all the properties of a Generalized Fourier Transform (GFT). The derivation of these properties is based on the Wronskian relations [1, 3, 4]. These results allow natural generalizations to the MNLS equations, see [11] and references therein.

So the mapping $\mathfrak{F}\colon \mathcal{M}_J \to T(\lambda,t)$ is directly related to the GFT which instead of the usual exponentials $e^{\pm i\lambda J}$ uses the so-called 'squared solutions' of L, $\boldsymbol{e}_\alpha^\pm(x,\lambda) = \pi_{0,J}(\chi^\pm E_\alpha \hat{\chi}^\pm(x,\lambda)$. Here $\chi^\pm(x,\lambda)$ are the FAS of L and $\pi_{0,J}X = \mathrm{ad}_J^{-1}\mathrm{ad}_J X$ is a projector onto the image of the operator ad_J. Next, one can: i) prove that the system of 'squared solution' $\boldsymbol{e}_\alpha^\pm(x,\lambda), \alpha\in\Delta_1$ are complete set of functions on $\mathcal{M}_J$; ii) the minimal set of scattering data can be viewed as expansion coefficients of $Q(x,t)$ over $\boldsymbol{e}_\alpha^\pm(x,\lambda)$; ii) the variations of the minimal set of scattering data can be viewed as expansion coefficients of $\mathrm{ad}_J^{-1}\delta Q(x,t)$ over $\boldsymbol{e}_\alpha^\pm(x,\lambda)$. Finally these expansions can be used to derive the fundamental properties of the whole class of multi-component NLEE, for more details and proofs see. [11]. In particular this means that the MNLS equations (22) admit a hierarchy of hamiltonian formulations, see [11]. The simplest of them has as Hamiltonian

$$
H^{(0)} = \frac{1}{2r}\int_{-\infty}^{\infty} dx\, \left(\left\langle \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial x} \right\rangle - \left\langle \left[\mathrm{ad}_J^{-1}Q, Q\right], \left[\mathrm{ad}_J^{-1}Q, Q\right] \right\rangle \right). \tag{49}
$$

The relevant symplectic form is given by

$$\Omega^{(0)} = \frac{i}{2r}\int_{x=-\infty}^{\infty} dx \, \left\langle \mathrm{ad}_J^{-1}\delta Q \mathop{\wedge}_{,} \mathrm{ad}_J^{-1}\delta Q \right\rangle . \tag{50}$$

The other members of the hierarchy are generated by the recursion operators $\Lambda_\pm$. The proof of their compatibility is based on the completeness relations for the 'squared solutions' that are eigenfunctions of $\Lambda_\pm$. Thus it is natural to expect that the new 2-component NLS will also possess hierarchies of Hamiltonian structures. These type of results can be viewed also as more strict proofs, that he been formally derived by Dickey and Gelfand [8, 9], by Drinfeld and Sokolov [5] and by Lombardo and Mikhailov [24, 25], see also [17, 18].

Our last remark is that one can apply similar ideas also to the other types of symmetric spaces. Thus one can find other 2-component MNLS related to the other symmetric spaces: BD.I, C.I etc. These and other natural problems, such as deriving their soliton solutions, the construction of their integrals of motion, derivation of the fundamental properties etc . which will be published elsewhere.

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Appendix

The simple Lie algebras $D_r \equiv \mathfrak{so}(2r)$ are usually represented by a $2r \times 2r$ antisymmetric matrices. In this realization the Cartan subalgebra is not diagonal, so we will use a realization for which every $X \in D_r$ satisfies

$$SX + X^T S = 0, \qquad S = \begin{pmatrix} 0 & \boldsymbol{s} \\ \hat{\boldsymbol{s}} & 0 \end{pmatrix}, \qquad S^2 = \mathbb{1}. \tag{51}$$

Here the block $\boldsymbol{s}$ is a $r \times r$ matrix given by $\boldsymbol{s} = \sum_{k=1}^{r} (-1)^{k+1} E_{k,r+1-k}$, where E_{kj} are $r \times r$ matrix given by $(E_{kj})_{mn} = \delta_{km}\delta_{jn}$. This definition is convenient, because with it the Cartan subalgebra is given by diagonal matrices.

The root system Δ of D_r consists of positive roots Δ_+ and negative roots Δ_-. If $\alpha \in \Delta_+$ then $-\alpha \in \Delta_-$. If e_i is an orthonormal basis in $\mathbb{R}^r$ then the set of positive roots Δ_+ consists of $e_i - e_j$, $e_i + e_j$ with $1 \leq i < j \leq r$.

The Cartan-Weyl basis of D_r is given by

$$\begin{aligned} &H_i = E_{ii} - E_{2r+1-i,2r+1-i}, && 1 \leq i \leq r, \\ &E_{e_i-e_j} = E_{ij} - (-1)^{i+j} E_{2r+1-j,2r+1-i}, && 1 \leq i < j \leq r, \\ &E_{e_i+e_j} = E_{i,2r+1-j} + (-1)^{i+j} E_{j,2r+1-i}, && 1 \leq i < j \leq r, \quad E_{-\alpha} = (E_\alpha)^T, \end{aligned} \tag{52}$$

where by E_{ij} we denote a $2r \times 2r$ matrix that has a one at the $i - th$ row and $j - th$ column and is zero everywhere else.

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