PLISKA
STUDIA MATHEMATICA

ПЛИСКА
МАТЕМАТИЧЕСКИ
СТУДИИ

Pliska Stud. Math. **30** (2019), 85–98

PLISKA
STUDIA MATHEMATICA

THE BOSON STAR EQUATION WITH HARTREE TYPE NON-LINEARITY: GLOBAL EXISTENCE IN $H^{\frac{1}{2}}(\mathbb{R}^2)$

Vladimir Georgiev, Boris Shakarov

Local and global well-posedness for the Boson Star equation with Hartree type non linearity with initial data in the critical space $H^{\frac{1}{2}}(\mathbb{R}^2)$ with finite $L^2(\mathbb{R}^2)$ norm is established. The proof is based on Strichartz estimates, conservation laws, Coifman-Meyer theorem and Paley-Littlewood decomposition.

1. Introduction

In this paper we consider the Cauchy problem for the nonlinear Boson Star equations of the form

$$\begin{cases} (-i\partial_t + (1-\Delta)^{\frac{1}{2}})u = F(u), \\ u(0,x) = u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2), \end{cases} \tag{1}$$

where $u(t,x) : \mathbb{R}^{1+2} \to \mathbb{C}^2$, $m \geq 0$ is a mass parameter, $\langle D \rangle^s = (1-\Delta)^{\frac{s}{2}} u = \mathscr{F}^{-1}((1+|\xi|^2)^{\frac{s}{2}} \mathscr{F}(u))$ and $F(u) = ((b-\Delta)^{-1}|u|^2)u$ with $b > 0$ is the Hartree type nonlinearity. The nonlinearity could be also seen as

$$F(u) := (V_b * |u|^2)u \tag{2}$$

where V_b is a convolution kernel such that

$$V_b(x) \leq C e^{-b\frac{|x|}{2}}, \tag{3}$$

2010 *Mathematics Subject Classification:* 35Q55, 35Q85, 42B37.
Key words: Boson star equation, global existence.

when $|x| \geq 2$, and

$$M^{-1}h(x) \leq V(x) \leq Mh(x), \tag{4}$$

when $|x| \leq 2$, where $h(x) = 1 - \log(\frac{|x|}{2}) + O(|x^2|)$, and M, C are constants. For the rest of this work we will suppose that $b = 1$ and $m = 1$.

This paper is divided into two parts. In the first part, we proof the local existence of the equation using a contraction method and Strichartz estimates.

Theorem 1. (Local existence) *There exists a function* $T : H^{\frac{1}{2}}(\mathbb{R}^2) \to (0, \infty]$ *such that for any* $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2)$*, there exists a* $u \in C([0, T(u_0)); H^{\frac{1}{2}}(\mathbb{R}^2))$ *such that for all* $t \in (0, T(u_0))$*,* u *is the unique local solution to the equation* (1) *in the sense that, for all* $t < T(u_0)$ *it is true that*

$$\|u(t,x)\|_{H_x^{\frac{1}{2}}(\mathbb{R}^2)} < \infty. \tag{5}$$

In the second part we will proof that the solution is actually global, and so the time existence for all initial data u_0 is unbounded.

Theorem 2. (Global existence) *The local solution given by theorem 5 is actually global in* $C([0, \infty); H^{\frac{1}{2}}(\mathbb{R}^2))$ *for every* $\lambda \in \mathbb{R}$*, in the sense that, given an initial datum* $u_0(x) \in H^{\frac{1}{2}}(\mathbb{R}^2)$*, for any* $t \in [0, \infty)$*, there is a constant* C_{u_0} *such that*

$$\|u(t,x)\|_{H_x^{\frac{1}{2}}(\mathbb{R}^2)} \leq C_{u_0}. \tag{6}$$

For these two results, we will use tools coming from Harmonic Analysis, from general Partial Differential Equation theory and conservation laws. The key result, from which global existence will follow, is the bilinear estimate

Theorem 3. *There exists* $C > 0$ *such that for any* $f, g \in L^2(\mathbb{R}^2)$ *we have*

$$\|(1-\Delta)^{-1/2}\langle f, g\rangle_{\mathbb{C}^2}\|_{L^2(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}\|g\|_{L^2(\mathbb{R}^2)}. \tag{7}$$

2. Local existence

The main purpose of this section is to prove the local well-posedness of the solution to the equation (1). First of all, we show that the nonlinearity is locally Lipschitz continuous from $H^{\frac{1}{2}}(\mathbb{R}^2)$ into itself.

Lemma 1. *For all $\lambda \in \mathbb{R}$, the map $F(u)$ defined in (2) is locally Lipschitz continuous from $H^{\frac{1}{2}}(\mathbb{R}^2)$ into itself with*

$$\|F(u) - F(v)\|_{H^{\frac{1}{2}}(\mathbb{R}^2)} \lesssim L_M \|u - v\|_{H^{\frac{1}{2}}(\mathbb{R}^2)}, \tag{8}$$

for all $u, v \in H^{\frac{1}{2}}(\mathbb{R}^2)$ where L_M is a constant depending only on the $H^{\frac{1}{2}}(\mathbb{R}^2)$ norm of u and v, with $M = \max\{\|u\|_{H^{\frac{1}{2}}}, \|v\|_{H^{\frac{1}{2}}}\}$.

For the proof, we used the Kato-Ponce inequality (For a proof, see e.g. [1]) and, since the proof uses Sobolev Embedding theorem, it relies on the dimension of the space. The same result for the space $\mathbb{R}^3$ was proven in [2].

P r o o f. Take $u, v \in H^{\frac{1}{2}}(\mathbb{R}^2)$. We have that

$$\|F(u) - F(v)\|_{H^{\frac{1}{2}}} = \|\langle D\rangle^{\frac{1}{2}}((\langle D\rangle^{-2}|u|^2)u) - \langle D\rangle^{\frac{1}{2}}((\langle D\rangle^{-2}|v|^2)v)\|_{L^2} \tag{9}$$

$$= \|\frac{1}{2}\langle D\rangle^{\frac{1}{2}}(\langle D\rangle^{-2}(|u|^2 - |v|^2)(u+v)) + \frac{1}{2}\langle D\rangle^{\frac{1}{2}}(\langle D\rangle^{-2}(|u|^2 + |v|^2)(u-v))\|_{L^2}$$

$$\lesssim \|\langle D\rangle^{\frac{1}{2}}(\langle D\rangle^{-2}(|u|^2 - |v|^2)(u+v))\|_{L^2} + \|\langle D\rangle^{\frac{1}{2}}(\langle D\rangle^{-2}(|u|^2 + |v|^2)(u-v))\|_{L^2}.$$

Now we want to bound the two parts separately. Starting from

$$B := \|\langle D\rangle^{\frac{1}{2}}(\langle D\rangle^{-2}(|u|^2 + |v|^2)(u-v))\|_{L^2}, \tag{10}$$

and using the Kato-Ponce inequality, we have that

$$B \lesssim \|\langle D\rangle^{-\frac{3}{2}}(|u|^2 + |v|^2)\|_{L^4}\|u - v\|_{L^4} + \|\langle D\rangle^{-2}(|u|^2 + |v|^2)\|_{L^\infty}\|\langle D\rangle^{\frac{1}{2}}(u-v)\|_{L^2}. \tag{11}$$

Using the Sobolev Embedding theorem in $\mathbb{R}^2$ we have that $\|u-v\|_{L^4} \leq \|u-v\|_{H^{\frac{1}{2}}}$. For the same reason

$$\|\langle D\rangle^{-\frac{3}{2}}(|u|^2 + |v|^2)\|_{L^4} \leq \|\langle D\rangle^{-\frac{3}{2}}(|u|^2 + |v|^2)\|_{H^{\frac{1}{2}}} = \|\langle D\rangle^{-1}(|u|^2 + |v|^2)\|_{L^2}. \tag{12}$$

Use the fact that Bessel operators are L^p Fourier multipliers to obtain

$$\|\langle D\rangle^{-1}(|u|^2 + |v|^2)\|_{L^2} \leq \|(|u|^2 + |v|^2)\|_{L^2} \leq \||u|^2\|_{L^2} + \||v|^2\|_{L^2} \tag{13}$$

$$= \|u\|^2_{L^4} + \|v\|^2_{L^4} \leq \|u\|^2_{H^{\frac{1}{2}}} + \|v\|^2_{H^{\frac{1}{2}}}.$$

The last term is bounded due to the Sobolev Embedding theorem

$$\|\langle D\rangle^{-2}(|u|^2 + |v|^2)\|_{L^\infty} \lesssim \||u|^2 + |v|^2\|_{L^2} \lesssim \|u\|^2_{H^{\frac{1}{2}}} + \|v\|^2_{H^{\frac{1}{2}}}. \tag{14}$$

Putting all together, we have that

$$B \lesssim (\|u\|^2_{H^{\frac{1}{2}}} + \|v\|^2_{H^{\frac{1}{2}}})\|u-v\|_{H^{\frac{1}{2}}}. \tag{15}$$

Now let bound

$$A := \|\langle D\rangle^{\frac{1}{2}}(\langle D\rangle^{-2}(|u|^2-|v|^2)(u+v))\|_{L^2}. \tag{16}$$

As before,

$$A \lesssim \|\langle D\rangle^{-\frac{3}{2}}(|u|^2-|v|^2)\|_{L^4}\|u+v\|_{L^4} + \|\langle D\rangle^{-2}(|u|^2-|v|^2)\|_{L^\infty}\|\langle D\rangle^{\frac{1}{2}}(u+v)\|_{L^2},$$

and

$$\|\langle D\rangle^{-\frac{3}{2}}(|u|^2-|v|^2)\|_{L^4} \lesssim \||u|^2-|v|^2\|_{L^2},$$

Using Hölder inequality and Sobolev Embedding theorem, we can arrive to

$$\||u|^2-|v|^2\|_{L^2} \le \|u-v\|_{L^4}(\|u\|_{L^4}+\|v\|_{L^4}) \lesssim \|u-v\|_{H^{\frac{1}{2}}}(\|u\|_{H^{\frac{1}{2}}}+\|v\|_{H^{\frac{1}{2}}}). \tag{17}$$

Moreover

$$\|\langle D\rangle^{-2}(|u|^2-|v|^2)\|_{L^\infty} \le \||u|^2-|v|^2\|_{L^2} \lesssim \|u-v\|_{H^{\frac{1}{2}}}(\|u\|_{H^{\frac{1}{2}}}+\|v\|_{H^{\frac{1}{2}}}).$$

The last two terms are bounded in this way

$$\|u+v\|_{L^4} \lesssim \|u+v\|_{H^{\frac{1}{2}}} \le \|u\|_{H^{\frac{1}{2}}}+\|v\|_{H^{\frac{1}{2}}}, \tag{18}$$

and

$$\|\langle D\rangle^{\frac{1}{2}}(u+v)\|_{L^2} = \|u+v\|_{H^{\frac{1}{2}}} \le \|u\|_{H^{\frac{1}{2}}}+\|v\|_{H^{\frac{1}{2}}}. \tag{19}$$

Putting all together, we have

$$A \lesssim (\|u\|_{H^{\frac{1}{2}}}+\|v\|_{H^{\frac{1}{2}}})^2\|u-v\|_{H^{\frac{1}{2}}}. \tag{20}$$

Then the final estimate is

$$\|F(u)-F(v)\|_{H^{\frac{1}{2}}} \lesssim (\|u\|^2_{H^{\frac{1}{2}}}+\|v\|^2_{H^{\frac{1}{2}}}+\|u\|_{H^{\frac{1}{2}}}\|v\|_{H^{\frac{1}{2}}})\|u-v\|_{H^{\frac{1}{2}}}. \tag{21}$$

Finally, let's prove (17). If we define the following function $f(s) := |v+s(u-v)|^2$, then it is clear that $f(1) = |u|^2$, $f(0) = |v|^2$ and

$$|u|^2-|v|^2 = f(1)-f(0) = \int_0^1 f'(s)ds. \tag{22}$$

With some calculations,

$$f'(s) = (u-v)(\bar{v}+s(\bar{u}-\bar{v})) + (\bar{u}-\bar{v})(v+s(u-v)). \tag{23}$$

Then, we have that

(24) $|u|^2 - |v|^2 = (u-v)\bar{v} + (u-v)(\bar{u}-\bar{v}) + (\bar{u}-\bar{v})v = (u-v)\bar{u} + (\bar{u}-\bar{v})v.$

It is sufficient to take the L^2 norm and use Hölder inequality to conclude. □

The integral form of the solution to (1) is

$$u(t,x) = e^{-it\langle D\rangle}u(0,x) + i\int_0^t e^{i(s-t)\langle D\rangle}\lambda F(u(s,x))ds. \tag{25}$$

We want to prove that our problem is locally well-posed with a fixed $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2)$ using a contraction principle and Strichartz estimates.

Theorem 4. *Given the equation* (25) *with the linear part of the solution* $u_{lin}(t,x) = e^{-it\langle D\rangle}u_0(x)$*, the following Strichartz estimate is true:*

$$\|u_{lin}(t,x)\|_{L_t^\infty(\mathbb{R}),L_x^2(\mathbb{R}^2)} \lesssim \|u_0(x)\|_{L_x^2(\mathbb{R}^2)}.$$

Proof. We apply the result proved in [3]. In our case, we have that $h(\rho) = \sqrt{1+|\rho|^2}$, and

1. $h'(\rho) = \dfrac{\rho}{\sqrt{1+|\rho|^2}} > 0.$
2. $h''(\rho) = \dfrac{1}{\sqrt{1+|\rho|^2}^3} > 0.$
3. $h^{(3)}(\rho) = \dfrac{-3\rho}{\sqrt{1+|\rho|^2}^5}.$

In particular the hypothesis are verified and choosing $p = 2$, $q = \infty$ $s_1 = s_2 = s = 0$, we have the estimate above. □

Lemma 2. *Let* $T > 0$, $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2)$ *and let* $u, v \in C([0,T]; H_x^{\frac{1}{2}}(\mathbb{R}^2)$ *two solutions to integral form* (25)*. Then* $u = v$.

Proof. We set $M = \sup\limits_{t\in[0,T]} \max\{\|u(t,x)\|_{H_x^{\frac{1}{2}}}, \|v(t,x)\|_{H_x^{\frac{1}{2}}}\}$. Then

$$w(t) := \|u(t,x) - v(t,x)\|_{H_x^{\frac{1}{2}}} \leq \int_0^t \|F(u(s,x)) - F(v(s,x))\|_{H_x^{\frac{1}{2}}} ds$$

$$\leq L_M \int_0^t \|u(s,x) - v(s,x)\|_{H_x^{\frac{1}{2}}} ds = L_M \int_0^t w(s) ds.$$

We conclude, using the Gronwall's lemma, that $w(t) \leq 0$ a.e. □

Theorem 5. (Local Existence) *Let $M > 0$ and let $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2)$ be such that $\|u_0\|_{H^{\frac{1}{2}}} \leq M$. Then there exists a unique solution $u \in C([0, T_M]; H^{\frac{1}{2}}(\mathbb{R}^2))$ of the equation* (25) *with*

$$T_M := \frac{1}{2L_{2M}} > 0. \tag{26}$$

Proof. For this prove we use the notation $H^{\frac{1}{2}} := H^{\frac{1}{2}}(\mathbb{R}^2)$.
The uniqueness is proven in the previous lemma. Let $u_0 \in H^{\frac{1}{2}}$ and define

$$E = \{u(t,x) \in C([0, T_M]; H^{\frac{1}{2}});\ \|u(t,x)\|_{H_x^{\frac{1}{2}}} \leq 2M,\ \forall t \in [0, T_M]\}.$$

We equip E with the distance generated by the norm of $C([0, T_M]; H^{\frac{1}{2}})$:

$$d(u,v) := \max_{t \in [0, T_M]} \|u(t,x) - v(t,x)\|_{H_x^{\frac{1}{2}}}, \tag{27}$$

which makes E a complete metric space since $C([0, T_M]; H^{\frac{1}{2}})$ is a Banach space. For all $u \in E$, we define $\phi_u \in C([0, T_M]; H^{\frac{1}{2}})$ by

$$\phi_u(t,x) = e^{-it\langle D \rangle} u(0,x) + i \int_0^t e^{i(s-t)\langle D \rangle} F(u(s,x)) ds. \tag{28}$$

We have that $F(0) = 0$ and so $\|F(u(s,x))\|_{H_x^{\frac{1}{2}}} \leq 2ML_{2M} = \dfrac{M}{T_M}$. It follows that

$$\|\phi_u(t,x)\|_{H_x^{\frac{1}{2}}} \leq \|u_0\|_{H^{\frac{1}{2}}} + \int_0^t \|F(u(s,x))\|_{H_x^{\frac{1}{2}}} ds \leq M + t\frac{M}{T_M} \leq 2M. \tag{29}$$

Consequently $\phi : E \to E$ and for all u, v in E,

$$\|\phi_v(t,x) - \phi_u(t,x)\|_{H_x^{\frac{1}{2}}} \tag{30}$$

$$\leq L_{2M} \int_0^t \|v(s,x) - u(s,x)\|_{H_x^{\frac{1}{2}}} \leq T_M L_{2M} d(u,v) \leq \frac{1}{2} d(u,v).$$

Therefore, ϕ is a contraction in E and so ϕ has a fixed point $u \in E$, which solves the integral solution (25). □

Corollary 1. *There exists a function* $T : H^{\frac{1}{2}}(\mathbb{R}^2) \to (0, \infty]$ *such that for every* $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2)$, $T : u_0 \to T_{u_0}$ *and there exists a* $u \in C([0, T_{u_0}); H^{\frac{1}{2}}(\mathbb{R}^2))$ *such that for all* $T \in (0, T_{u_0})$, *u is the unique solution to the equation* (25) *in* $C([0, T]; H^{\frac{1}{2}}(\mathbb{R}^2))$. *In addition,*

$$2L_{2\|u(t,x)\|_{H_x^{\frac{1}{2}}}} \geq (T_{u_0} - t)^{-1}, \tag{31}$$

for all $t \in [0, T_{u_0})$. *In particular there are two alternatives:*

1. $T_{u_0} = \infty$;
2. $T_{u_0} < \infty$ *and* $\lim_{t \to T_{u_0}^-} \|u(t,x)\|_{H_x^{\frac{1}{2}}(\mathbb{R}^2)} = \infty$.

The proof is a standard argue by contradiction. In particular, there can be some initial data for which the solution is only local and other for which the solution is global ($T_{u_0} = \infty$).

3. Global Existence

In the previous section, we proved the local existence and uniqueness of a solution to the equation (25) in the space $C([0, T); H^{\frac{1}{2}}(\mathbb{R}^2))$, showing also the minimum guaranteed time of existence for all initial data $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2)$ and showing a persistence of regularity for small time. The next step is proving that the time of existence is $T_{u_0} = \infty$. In other words, we want to show that, given $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2)$, the solution will have a finite norm $\|u(t,x)\|_{H_x^{\frac{1}{2}}(\mathbb{R}^2)}$ for any $t > 0$.

The first step for proving the global existence is obtaining these conservation laws:

$$E[u] := \frac{1}{2} \int_{\mathbb{R}^2} \bar{u} \langle D \rangle u dx + \frac{1}{4} \lambda \int_{\mathbb{R}^2} (\langle D \rangle^{-2} |u|^2) |u|^2 dx, \tag{32}$$

$$N[u] := \int_{\mathbb{R}^2} |u|^2 dx. \tag{33}$$

Lemma 3. *Given* $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^2)$, *the local solution given by the theorem 5 obeys to these conservation laws and in particular*

$$E[u(t,x)] = E[u_0(x)] \text{ and } N[u(t,x)] = N[u_0(x)], \tag{34}$$

for all $t \in [0, T_{u_0})$.

P r o o f. This proof follows the traces in [2]. Fixed a initial datum u_0, let first multiply the equation (1) for $\bar{u}(t,x)$ and then integrate it in x. Taking the imaginary part we have that

$$\partial_t \|u\|^2_{L^2(\mathbb{R}^2)} = 0 \tag{35}$$

for all $t \in [0, T_{u_0})$. Consequently $N[u(t,x)] = N[u_0(x)]$.
The conservation of energy is more delicate. Formally, it is sufficient to multiply the equation (1) by $\partial_t \bar{u}(t,x) \in H^{-\frac{1}{2}}(\mathbb{R}^2)$ and then integrate over $\mathbb{R}^2$. Taking the real part we have that

$$0 = \partial_t \left(\frac{1}{2} \int_{\mathbb{R}^2} \bar{u} \langle D \rangle u dx - \frac{1}{4} \int_{\mathbb{R}^2} (\langle D \rangle^{-2} |u|^2) |u|^2 dx \right). \tag{36}$$

In particular $E[u(t,x)] = E[u_0(x)]$. The problem is pairing two elements of the space $H^{-\frac{1}{2}}(\mathbb{R}^2)$ and then integrating them is generally not well defined. In this case, we need to introduce a regularization procedure (see [2] and other regularization methods in [5], [4]). The idea is that we can approximate the operator $\langle D \rangle = (1 - \Delta)^{\frac{1}{2}}$ with the family of operators

$$M_\varepsilon := (\varepsilon \langle D \rangle + 1)^{-1}, \ \ for \ \ \varepsilon > 0. \tag{37}$$

When using the fact that for all $u \in H^s$ and $s \in \mathbb{R}$, $M_\varepsilon u \to u$ strongly, we can approximate the difference

$$E[u(t_2,x)] - E[u(t_1,x)] = \lim_{\varepsilon \to 0^+} \left(E[M_\varepsilon u(t_2,x)] - E[M_\varepsilon u(t_1,x)] \right). \tag{38}$$

Now it can be seen that, whenever $\varepsilon > 0$, there are not two $H^{-\frac{1}{2}}$ elements paired, in contrast to the case $\varepsilon = 0$. Then, using the dominated convergence theorem, it can be proven that $\lim_{\varepsilon \to 0^+} (E[M_\varepsilon u(t_2,x)] - E[M_\varepsilon u(t_1,x)) = 0$. □

Definition 1. *A solution $u(t,x)$ to the equation* (25) *exists globally in $H^{\frac{1}{2}}_x(\mathbb{R}^2)$ if and only if for any finite time $t > 0$ the norm $\|u(t,x)\|_{H^{\frac{1}{2}}_x(\mathbb{R}^2)}$ is finite.*

The idea to gain the global solution is to proof a a priori bound of the $H^{\frac{1}{2}}$ norm of the solution is such a way that

$$\|u(t,x)\|_{H^{\frac{1}{2}}_x(\mathbb{R}^2)} \le C_{u_0}, \tag{39}$$

for all $t > 0$ and for all initial data $u_0(x) \in H^{\frac{1}{2}}$, where C_{u_0} is a constant, depending on u_0. In the case of a defocusing nonlinearity we have that $\lambda = 1$

and so

$$E[u] := \frac{1}{2}\int_{\mathbb{R}^2_x} \bar{u}\langle D\rangle u dx + \frac{1}{4}\int_{\mathbb{R}^2_x} (\langle D\rangle^{-2}|u|^2)|u|^2 dx. \tag{40}$$

This leads to the simple bound

$$\|u(t,x)\|^2_{H^{\frac{1}{2}}_x(\mathbb{R}^2)} \leq 2E[u] = 2E[u_0]. \tag{41}$$

In the case of the focusing nonlinearity ($\lambda = -1$) we obtain

$$\|u(t,x)\|^2_{H^{\frac{1}{2}}_x(\mathbb{R}^2)} = \frac{1}{2}\|\,\langle D\rangle^{-1}\,|u(t,x)|^2\|^2_{L^2_x(\mathbb{R}^2)} + 2E[u_0(x)], \tag{42}$$

And, using theorem 3, we arrive to

$$\|u(t,x)\|^2_{H^{\frac{1}{2}}_x(\mathbb{R}^2)} \lesssim 2E[u_0(x)] + \|u(t,x)\|^4_{L^2_x(\mathbb{R}^2)} = 2E[u_0(x)] + \|u_0(x)\|^4_{L^2_x(\mathbb{R}^2)} \tag{43}$$

4. Proof of the Theorem 3

We take two functions $f, g \in L^2(\mathbb{R}^2)$. Using Gagliardo–Nirenberg inequality, we have that

$$\|(1-\Delta)^{-1/2}\langle f,g\rangle\|^2_{L^2(\mathbb{R}^2)} \lesssim \|\nabla(1-\Delta)^{-1/2}\langle f,g\rangle\|^2_{L^1(\mathbb{R}^2)}. \tag{44}$$

So, if we define

$$T_\sigma(f_1,f_2)(x) = \int_{\mathbb{R}^n} \sigma(\xi_1,\xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{ix(\xi_1+\xi_2)}d\xi_1 d\xi_2, \tag{45}$$

where $\sigma : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is

$$\sigma(\xi_1,\xi_2) = \frac{1}{(1+|\xi_1+\xi_2|^2)^{\frac{1}{2}}}(\xi_1+\xi_2). \tag{46}$$

it is clear that theorem 3 is equivalent to proving that

$$\|T_\sigma(f,\bar{g})(x)\|_{L^1(\mathbb{R}^2)} \leq C\|f(x)\|_{L^2(\mathbb{R}^2)}\|g(x)\|_{L^2(\mathbb{R}^2)}. \tag{47}$$

One of the fundamental theorem to arrive to the desired estimate is the Coifman–Meyer theorem, (see e.g. [6]). We introduce the Paley–Littlewood decomposition:

Let $\psi(\xi)$ be a real-valued radial and symmetric bump function with support $\text{supp}(\psi(\xi)) = \{\xi \in \mathbb{R}^n : \|\xi\| \leq 2\}$ which is equals 1 in the ball $B = \{\xi \in \mathbb{R}^n : \|\xi\| \leq 1\}$. Now, for $j \in \mathbb{Z}$, let $\phi_j(\xi) = \psi(2^{-j}\xi) - \psi(2^{-j+1}\xi)$ be a bump function supported in the annulus $\left\{\left(\frac{1}{2}\right)^{-j+1} \leq \|\xi\| \leq (2)^{j+1}\right\}$ whose derivatives satisfy the inequality $2^{j|\alpha|}\,|\partial^\alpha \phi_j(\xi)| \le c_\alpha$ for some positive number c_α and for all multi-indices $\alpha \in \mathbb{Z}^n$. By construction, the bump functions ϕ_j satisfy $\sum_{j\in\mathbb{Z}} \phi_j(\xi) = 1$ for all $\xi \neq 0$, thus they provide a specific partition of unity which allows to decompose an arbitrary function u as $u = \sum_{j\in\mathbb{Z}} P_j u = \sum_{j\in\mathbb{Z}} u_j$, where P_j is a projection operator defined by $P_j(u) = (\phi_j \hat{u})^\vee$.

This proof will be divided in two step: the first one deals with a non problematic subset of the frequencies space $\mathbb{R}^2_{\xi_1} \times \mathbb{R}^2_{\xi_2}$, in which we will gain some uniform bounds using Coifman–Meyer theorem; and the second one in which this theorem cannot be used and it will be required to study accurately high and low frequencies. A useful way to see this separation is through Paley-Littlewood decomposition. We now that for every f Schwartz we have that

$$f = \sum_{j\in\mathbb{Z}} P_j(f) = \sum_{j\in\mathbb{Z}} f_j. \tag{48}$$

Then, we can decompose the product $f\cdot g$ as $\sum_{j\in\mathbb{Z}} f_j \sum_{k\in\mathbb{Z}} \bar{g}_k$. Then the two parts we consider in this proof are $\sum_{j\in\mathbb{Z}} f_j \sum_{|k-j|>M} \bar{g}_k$, and $\sum_{j\in\mathbb{Z}} f_j \sum_{|k-j|\le M} \bar{g}_k$. In this way, we divides the phase space $(\xi_1, \xi_2) \in R^4$ into two parts in such a way that the first part is the set in which the phases are not similar. In this part, it is possible to use Coifman-Meyer theorem due to the fact that the symbol σ verifies proper bounds of the derivatives. To prove that, let's think the symbol σ to be $\sigma : \mathbb{R}^4 \to \mathbb{R}^2$, $\xi = (\xi^1, \xi^2, \xi^3, \xi^4)$,

$$\sigma(\xi) := \frac{1}{(1 + (\xi^1+\xi^3)^2 + (\xi^2+\xi^4)^2)^{\frac{1}{2}}}(\xi^1+\xi^3, \xi^2+\xi^4). \tag{49}$$

We define $y := (\xi^1+\xi^3)^2 + (\xi^2+\xi^4)^2$. We have that

$$\partial_{\xi^i}\sigma(\xi) = (1,0)\frac{1}{(1+y)^{\frac{1}{2}}} - (\xi^1+\xi^3, \xi^2+\xi^4)\frac{\xi^i + \xi^{i\pm 2}}{(1+y)^{\frac{3}{2}}} \tag{50}$$

Computing the norm, it could be seen that

$$|\partial_{\xi^i}\sigma(\xi)|^2 \le C\frac{1}{|\xi|^2}. \tag{51}$$

We can go further and arrive to the fourth order derivative, but the arguments will be the same. The key observation is that each differentiation gives two orders of infinite rate. This fact matches perfectly with the required bound $|\xi|^{-2\alpha}$, where

α is the derivative order.
So the hypothesis of the Coifman-Meyer are satisfied when the frequencies are near to each other. Choosing $r = 1$, $p = q = 2$, for every $f, g \in L^2(\mathbb{R}^2)$, there exists a constant C such that we have the uniform bound

$$\|T_\sigma(f, \bar{g})\|_{L^1(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}\|g\|_{L^2(\mathbb{R}^2)}, \tag{52}$$

which is exactly what we wanted to prove.

Note that Coifman-Meyer theorem cannot be used near zero. For this reason, we supposed that $|\xi_i| \leq \delta|\xi_j|$. Now we have to deal with this case in which the two frequencies are similar. So, we want to find a bound for the L^2 norm of the sum $\sum_{j\in\mathbb{Z}} f_j \sum_{|k-j|\leq M} \bar{g}_k$. Let's rewrite it in another equivalent form:

$$\sum_{k\in\mathbb{Z}} f_k \sum_{|k-m|\leq M} \bar{g}_m = \sum_{\substack{|k-m|\leq M \\ k,m\in\mathbb{Z}}} f_k \bar{g}_m. \tag{53}$$

Let's start with fixed $k, m \in \mathbb{Z}$. We have that

$$\|(1-\Delta)^{-\frac{1}{2}}\nabla(f_k\bar{g}_m)\|_{L^1(\mathbb{R}^2)} \tag{54}$$

$$\leq \|(1-\Delta)^{-\frac{1}{2}}(\nabla(f_k)\bar{g}_m)\|_{L^1(\mathbb{R}^2)} + \|(1-\Delta)^{-\frac{1}{2}}(f_k\nabla(\bar{g}_m))\|_{L^1(\mathbb{R}^2)}.$$

On the right hand side, the two elements are similar, so it is sufficient to show how to bound one of them. Using the fact that $(1-\Delta)^{-\frac{1}{2}}$ is a Fourier multiplier in all L^p spaces with $p \geq 1$, we have that

$$\|(1-\Delta)^{-\frac{1}{2}}(\nabla(f_k)\bar{g}_m)\|_{L^1(\mathbb{R}^2)} \leq \|\nabla(f_k)\bar{g}_m\|_{L^1(\mathbb{R}^2)} \leq \|\nabla(f_k)\|_{L^2(\mathbb{R}^2)}\|\bar{g}_m\|_{L^2(\mathbb{R}^2)}.$$

To bound the term with the gradient, we need to use Hardy spaces.

$$\|\nabla(f_k)\|_{L^2(\mathbb{R}^2)} = C\|\nabla(-\Delta)^{-\frac{1}{2}}((-\Delta)^{\frac{1}{2}}(f_k))\|_{L^2(\mathbb{R}^2)} \tag{55}$$

$$\lesssim \sum_{j=1}^{2} \|R_j(-\Delta)^{\frac{1}{2}}(f_k)\|_{L^2(\mathbb{R}^2)} \leq \|(-\Delta)^{\frac{1}{2}}(f_k)\|_{h^2(\mathbb{R}^2)},$$

where R_j are Riesz transforms and h^2 is the Hardy space, which is equivalent to the L^2 space. So we have that

$$\|(-\Delta)^{\frac{1}{2}}(f_k)\|_{h^2(\mathbb{R}^2)} \lesssim \|(-\Delta)^{\frac{1}{2}}(f_k)\|_{L^2(\mathbb{R}^2)}.$$

Using lemma 6.2.1. of the book [7], we arrive to

$$\|(-\Delta)^{\frac{1}{2}}(f_k)\|_{L^2(\mathbb{R}^2)} \leq C2^k\|f_k\|_{L^2(\mathbb{R}^2)}.$$

Applying the same passages to the second term we have that

$$\|(1-\Delta)^{-\frac{1}{2}}\nabla(f_k\bar{g}_m)\|_{L^1(\mathbb{R}^2)} \tag{56}$$

$$\lesssim 2^k\|f_k\|_{L^2(\mathbb{R}^2)}\|g_m\|_{L^2(\mathbb{R}^2)} + 2^m\|f_k\|_{L^2(\mathbb{R}^2)}\|g_m\|_{L^2(\mathbb{R}^2)}.$$

We will use (56) when $\min(k,m) < 1$. In this case 2^k and 2^m are bounded by a constant.
When $\min(k,m) \geq 1$, we can get another estimate. Let's for a moment suppose that $m = k$ (as we will see below this supposition is irrelevant until $|k-m| \leq M$). Then,

$$\|(1-\Delta)^{-\frac{1}{2}}(\nabla(f_k)\bar{g}_k)\|_{L^1(\mathbb{R}^2)} = \|(1-\Delta)^{-\frac{1}{2}}P_k((\nabla(f_k)\bar{g}_k))\|_{L^1(\mathbb{R}^2)},$$

where P_k is, as before, the kth Paley-Littlewood projection. Then we have that

$$\|(1-\Delta)^{-\frac{1}{2}}P_k((\nabla(f_k)\bar{g}_k))\|_{L^1(\mathbb{R}^2)} \leq 2^{-k}\|P_k((\nabla(f_k)\bar{g}_k))\|_{L^1(\mathbb{R}^2)}.$$

Thanks to Bernstein inequality, if we take $p = q = 1$, then $s = 0$ and

$$2^{-k}\|P_k((\nabla(f_k)\bar{g}_k))\|_{L^1(\mathbb{R}^2)} \lesssim 2^{-k}\|(\nabla(f_k)\bar{g}_k)\|_{L^1(\mathbb{R}^2)}.$$

From here, making the same passages as in the first case, it is clear that

$$\|(1-\Delta)^{-\frac{1}{2}}(\nabla(f_k)\bar{g}_m)\|_{L^1(\mathbb{R}^2)}$$

$$\lesssim 2^{-k}2^k\|f_k\|_{L^2(\mathbb{R}^2)}\|\bar{g}_m\|_{L^2(\mathbb{R}^2)} + 2^k 2^{-m}\|f_k\|_{L^2(\mathbb{R}^2)}\|\bar{g}_m\|_{L^2(\mathbb{R}^2)},$$

and we have the bound

$$\|(1-\Delta)^{-\frac{1}{2}}\nabla(f_k\bar{g}_m)\|_{L^1(\mathbb{R}^2)} \tag{57}$$

$$\lesssim \|f_k\|_{L^2(\mathbb{R}^2)}\|\bar{g}_m\|_{L^2(\mathbb{R}^2)}(2 + 2^{k-m} + 2^{m-k}).$$

Note that in the right hand side the element $(2+2^{k-m}+2^{m-k})$ can be bounded by 2^{M+1}. Then, putting together (56), (57), we have that

$$\|\nabla(1-\Delta)^{-\frac{1}{2}}(\sum_{\substack{|k-m|\leq M\\ k,m\in\mathbb{Z}}} f_k\bar{g}_m)\|_{L^1} \leq$$

$$\|\nabla(1-\Delta)^{-\frac{1}{2}}(\sum_{\substack{|k-m|\leq M\\ \min(k,m)\leq 0}} f_k\bar{g}_m)\|_{L^1} + \|\nabla(1-\Delta)^{-\frac{1}{2}}(\sum_{\substack{|k-m|\leq M\\ \min(k,m)\geq 1}} f_k\bar{g}_m)\|_{L^1}$$

$$\lesssim \sum_{\substack{|k-m|\le M\\ \min(k,m)\le 0}} (2^k+2^m)\|f_k\|_{L^2}\|g_m\|_{L^2} + \sum_{\substack{|k-m|\le M\\ \min(k,m)\ge 1}} 2^{M+1}\|f_k\|_{L^2}\|g_m\|_{L^2}$$

$$\lesssim 2^{M+1} \sum_{\substack{|k-m|\le M\\ k,m\in\mathbb{Z}}} \|f_k\|_{L^2}\|g_m\|_{L^2}.$$

From here we conclude observing that

$$\sum_{\substack{|k-m|\le M\\ k,m\in\mathbb{Z}}} \|f_k\|_{L^2}\|g_m\|_{L^2} \le \|f_k\|_{\ell^2L^2(\mathbb{R}^2)}\|g_m\|_{\ell^2L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}\|g\|_{L^2(\mathbb{R}^2)},$$

where we use Hölder inequality and discrete Young inequality, that is, in general, for any sequence $(a_i)_{i\in\mathbb{N}}$,

$$\sum_{|i-m|\le M} a_i a_m = \sum_i (b*a)_i a_i \le \|a_i\|_{\ell^2}\|(b*a)_i\|_{\ell^2} \le \|a_i\|_{\ell^2}\|a_i\|_{\ell^2}\|b\|_{\ell^1},$$

where b is the indicator function $\mathbb{1}_{(k\le M)}$ and so $\|b\|_{\ell^1}$ is finite. The last inequality is a well know property of the Besov space $B^0_{2,2}$, which norm is equivalent to the norm of L^2.

Acknowledgement

V. Georgiev was supported in part by Project 2017 "Problemi stazionari e di evoluzione nelle equazioni di campo nonlineari" of INDAM, GNAMPA – Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni, by Institute of Mathematics and Informatics, Bulgarian Academy of Sciences and Top Global University Project, Waseda University, by the University of Pisa, Project PRA 2018 49 and project "Dinamica di equazioni nonlineari dispersive", "Fondazione di Sardegna", 2016.

REFERENCES

[1] L. Grafakos, S. Oh. The Kato-Ponce inequality. *Comm. Partial Differential Equations* **39**, *6* (2014), 1128–1157.

[2] E. Lenzmann Well-posedness for semi-relativistic Hartree equations of critical type. *Math. Phys. Anal. Geom.* **10**, *1* (2007), 43–64.

[3] Y. Cho, S. Lee. Strichartz estimates in spherocal coordinates. *Indiana Univ. Math. J.* **62**, *3* (2013), 991–1020.

[4] Jean Ginibre and Giorgio Velo Scattering theory in the energy space for a class of Hartree equations. Nonlinear wave equations (Providence, RI, 1998), 29–60, *Contemp. Math.* **263**. Providence, RI, Amer. Math. Soc., 2000.

[5] T. Cazenave Semilinear Schrödinger Equations. Courant Lecture Notes in Mathematics vol. **10**. New York, New York University, Courant Institute of Mathematical Sciences; Providence, RI, American Mathematical Society, 2003.

[6] L. Grafakos. Modern Fourier analysis, 3rd edition. Graduate Texts in Mathematics vol. **250**.New York, Springer, 2014.

[7] J. Bergh, J. Löfström. Interpolation Spaces. An Introduction. Grundlehren der Mathematischen Wissenschaften No. **223**. Berlin-New York, Springer-Verlag, 1976.

V. Georgiev
Dipartimento di Matematica, Università di Pisa
Largo B. Pontecorvo 5, 56100 Pisa, Italy
and
Faculty of Science and Engineering, Waseda University
3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
and
Institute of Mathematics and Informatics–BAS
Acad. G. Bonchev Str., Block 8, 1113 Sofia, Bulgaria
e-mail: georgiev@dm.unipi.it

Boris Shakarov
Dipartimento di Matematica Università di Pisa
Largo B. Pontecorvo 5, 56100 Pisa, Italy
e-mail: shabor993@gmail.com