EXAM in CA - 2009, June

Problem 1 Map conformally and univalently the upper half-plane $D := \{z, \Im z > 0\}$ onto the "cut" unit disk $G := \{w, |w| < 1\} \setminus \{z, w = x + iy, 0 \leq x \leq 1,\ y = 0\}$.

Solution:

$$z_1 = z^2,\ \text{D} \longleftrightarrow \mathcal{C} - \{0, +\infty\},$$

$$z_2 = z_1 - 1,\ \mathcal{C} - \{0, +\infty\} \longleftrightarrow \mathcal{C} - \{-1, +\infty\},$$

$$z_3 = z_2 + \sqrt{z_2^2 - 1} \text{ where } z_3(\infty) := 0.$$

Remark 1: Analogously, if $D := \{z, 2\pi > \Im z > 0\}$, we start at

$$z_1 = e^z.$$

Problem 2. Evaluate

$$I := \int_\Gamma f(z)dz,$$

where

$$f(z) := \frac{e^{1/z}}{(-z+2)\cos 2z}$$

and Γ is the unit circle positively oriented.

Solution: Apply residue theorem, namely

$$I = 2\pi i \text{Res}(f, \cdots),$$

where the sum is taken over all residue inside the unit disk. In our case, these are the points $z = 0$ (an essential singularity) and $z = \pi/4$, respectively (a simple pole.) How to calculate the residue in question?

$$\text{Res}(f, 0) = \frac{1}{2\pi i} \oint_{C_0(\varepsilon)} \frac{e^{1/z}}{\cos 2z(-z+2)}$$

with $\varepsilon-$ sufficiently small such that the other singularities do not lie inside. FUrther,

$$\frac{1}{2\pi i} \oint_{C_0(\varepsilon)} \frac{e^{1/z}}{\cos 2z(-z+2)} = \sum_{=1}^{\infty} \frac{1}{n!} \frac{1}{2\pi i} \oint_{C_0(\varepsilon)} \oint \frac{1}{z^n(-z+2)\cos 2z} =$$

$$\sum_{=1}^{\infty} \frac{1}{n!^2} \left(\frac{1}{(-z+2)\cos 2z}\right)^{(n)}(0).$$

In the last formula, we used the classical Cauchy's formula for the derivatives of analytic functions. As for $\mathrm{Res}(f, \pi/4)$, we notice that $\pi/4$ is a pole so that we may apply the known formula

$$\mathrm{Res}(f, \pi/4) = \lim z \to 0 \frac{2ze^{1/z}}{(2-z)\cos(2z)} = 0.$$

Problem 3 Evaluate

$$I := \int_\Gamma \frac{f'(z)}{f(z)} dz,$$

where

$$f(z) = \frac{e^{1/z}}{z}$$

and Γ is the unit circle positively oriented.

Solution: We may not apply argument principle with respect to f and the unit disk, since f has an essential singularity inside, that is the point of zero. That's why we first calculate the derivative of $\frac{e^{1/z}}{z}$, this is

$$(\frac{e^{1/z}}{z})' = -\frac{e^{1/z}}{z^3} - -\frac{e^{1/z}}{z^2}.$$

Then we may express f'/f in the form

$$\frac{f'(z)}{f(z)} = -\frac{1}{z^2} - \frac{1}{z}$$

and

$$I = -2\pi i.$$

Problem 4 Let D be a domain in in the complex plane $\mathbf{C}$ and f be analytic in D. Assume that

$$\Re f(z) = \text{ Const}, \ z \in D.$$

Show that $f(z) \equiv$ Const in D.

Solution: Set $\Re f(z) := u(z) = u(x, y)$ and $f(z) = u(x, y0 + iv(x, y)$. By the conditions,

$$u'_x = u'_y = 0$$

everywhere in D. Hence, by the equations of Cauchy-Riemann,

$$v'_x = v'_y = 0.$$

Then, f is a constant in D.

Problem 5 Let f be an entire function. Assume that

$$\Re f(z) = \text{ C}, \, z \in \mathbf{C}, \, C - \text{ a constant.}$$

Show that $f(z) \equiv$ Const.

Solution: Set $F(z) := e^{f(z)}$. Obviously, the function F is also entire. On the ether hand, by the conditions,

$$|F(z)| = e^{\Re f(z)} \equiv e^C.$$

Set $e^C = C_1$. Remark that C_1 is a finite number. Then, by Liouville's theorem, $F(z) \equiv C_1$. Then, $f(z)$ is also a constant.