3. Analytic functions

3.1. Differentiability and analycity.

Definition: Let the function $f(z)$ be well defined in a neighborhood $\mathcal{G}$ of a point z_0. We say that f *is differentiable at* z_o, if the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

does exists whenever

$$\Delta z \to 0.$$

The expression $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$, provided the limit exists, is called *the derivative at* z_0 and is denoted by $f'(z_0)$: e.g.

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}; = f'(z_0) := \frac{df}{dz}. \tag{1}$$

ℵ

As we see, the definition is just the same as for the real-valued functions in real-analysis. Similarly to the real analysis, we have

Theorem 3.1. *If f and g are differentiable at z_0, then so are $f \pm g$ and fg, and*

$$(f + g)'(z) = f'(z) + g'(z),\ (fg)'(z) = f'(z)g(z) + f(z)g'(z).$$

The function $\frac{f}{g}$ is differentiable if $g'(z_0) \neq 0$ and

$$(\frac{f}{g})'(z_0) = \frac{f'g - fg'}{g^2}(z_0).$$

Definition: The complex valued function f is **analytic** in the open set $\mathcal{D}$, if it is differentiableat any point in $\mathcal{D}$. We will use the notation $f \in \mathcal{A}(\mathcal{D})$. ℵ

2.2. Geometric interpretation of the derivative. Let f be differentiableat z_0 and suppose that $f'(z_0) \neq 0$. We set $\Delta z := z - z_0$. From (1) we deduce that

$$\frac{|f(z) - f(z_0)|}{|z - z_0|} \to |f'(z_0)|$$

and

$$\text{Arg}(f(z) - f(z_0) - \ \text{Arg}(z - z_0) \to \ \text{Arg}f'(z_0).$$

We rewrite as

$$\text{Arg}(f(z) - f(z_0) - \ \text{Arg}(z - z_0) \approx \ \text{Arg}f'(z_0).$$

Setting $w := f(z)$, we see thanks to the condition $f'(z_0) \neq 0$ that in "the closure of z_0" the mapping $f(z)$ is "similar" to the linear transformation

$$w = f(z_0) + f'(z_0)(z - z_0).$$

This mapping preserves the angles, and is, as it is easy to see, one-to-one mapping. Such mappings are called *conformal.*

Definition: The function f is called to be **entire,** if it is analytic in the entire complex plane **C**. We write $f \in \mathcal{E}$.

3.3. Cauchy-Reimann equations.

Let $(\mathcal{D})$ be an open set in $\mathbb{C}$ and $f \in \mathcal{A}(\mathcal{D})$.

We write down

$$f(z) = u(x, y) + iv(x, y),\ z = x + iy, (x, y) \in \mathcal{G}$$

and

$$\Delta z = \Delta x + i\Delta y.$$

Let first $\Delta z \to 0$ horizontally, e.g. $\Delta y = 0$. The $\Delta z = \Delta x$ and by (1),

$$f'(z_0) = \frac{\partial u(x_0, y_0)}{\partial x} + i\frac{\partial v(x_0, y_0)}{\partial x}. \tag{2}$$

On the pother hand, if the approach is vertical, e,g, if $\Delta z = i\Delta y$, then

$$f'(z_0) = -i\frac{\partial u(x_0, y_0)}{\partial y} + \frac{\partial v(x_0, y_0)}{\partial y}. \tag{3}$$

Since the limits are just the derivative $f'(z_0)$, we deduce that

$$u'_x(x_o, y_0) = v'_y(x_o, y_0),\ u'_y(x_o, y_0) = -v'_x(x_o, y_0) \tag{4}$$

Equations (4) are called *Cauchy.Riemann equations.*

Theorem 3.2 *A necessary condition for a function $f(z) = u(x, y) + iv(x, y)$ to be differentiableat z_0 is that the Cauchy-Riemann equations hold at z_0.*

Consequently, if $f \in \mathcal{A}(\mathcal{D})$ then the Cauchy-Riemann equations hold at every point of $\mathcal{D}$.

Definition: The functions $u(x, y)$ and $v(x, y)$ are called *harmonic conjugate*.$\aleph$.

We now are going to establish the sufficient conditions for a function f to be analytic at some point z_0. The story is given by the following theorem

Theorem 3.3. *Let $f(z)\,, f(z) = u(x, y) + iv(x, y)$, be defined in an neighborhood $\mathcal{U}$ of z_0, suppose that the real and imaginary components $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations and are continuous in $\mathcal{U}$. Then f is differentiableat z_0.*

Proof: Set as before $\Delta z := \Delta x + i\Delta y$ and consider the quotient

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} =$$

$$\frac{u(x_0 + \Delta x, y_o + \Delta y) - u(x_0, y_0) + i(v(x_0 + \Delta x, y_o + \Delta y) - v(x_0, y_0))}{\Delta x + i\Delta y} := \mathcal{L}_\Delta.$$

We write the diference

$$u(x_0 + \Delta x, y_o + \Delta y) - u(x_0, y_0)$$

as

$$[u(x_0 + \Delta x, y_o + \Delta y) - u(x_0, y_0 + \Delta y)] + [u(x_0, y_0 + \Delta y) - u(x_0, y_0)]\,.$$

Because of the continuity of $u'x, u'y$ we may apply the mean valued theorem which yields

$$u(x_0 + \Delta x, y_o + \Delta y) - u(x_0, y_0 + \Delta y) = \Delta x \frac{\partial u}{\partial x}(x*, y_0 + \Delta y),$$

where the point $x* \in [x, x + \Delta x)]$ is appropriate. Again by continuity, we may write

$$\frac{\partial u}{\partial x}(x*, y_0 + \Delta y) = \frac{\partial u}{\partial x}(x_0, y_0) + \varepsilon_1,$$

where $\varepsilon_1 \to 0$, $x* \to x_0$ and $\Delta y \to 0$. Summarizing, we write

$$u(x_0 + \Delta x, y_o + \Delta y) - u(x_0, y_0 + \Delta y) = \Delta x \left[\frac{\partial u}{\partial x}(x_0, y_0) + \varepsilon_1\right].$$

Treating the expression $\mathcal{L}_\Delta$ similarly, we get

$$\mathcal{L}_\Delta := \frac{\Delta x\left[\frac{\partial u}{\partial x}+\varepsilon_1+i\frac{\partial v}{\partial x}+i\varepsilon_3\right]+\Delta y\left[\frac{\partial u}{\partial y}+\varepsilon_2+i\frac{\partial v}{\partial y}+i\varepsilon_4\right]}{\Delta x+i\Delta y}, \tag{5}$$

where the partial derivatives are taken at the point $z_0=(x_0,y_0)$. Now we use the equations of Cauchy-Riemann:

$$\mathcal{L}_\Delta = \frac{\Delta x\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]+i\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]}{\Delta x+i\Delta y}+\frac{\lambda}{\Delta x+i\Delta y},$$

with $\lambda := \Delta x(\varepsilon_1+i\varepsilon_3)+\Delta y(\varepsilon_2+i\varepsilon_4)$. Since

$$\mathcal{L}_\Delta = \frac{\Delta x\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]+i\left[\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right]}{\Delta x+i\Delta y}+\frac{\lambda}{\Delta x+i\Delta y},$$

with $\lambda := \Delta x(\varepsilon_1+i\varepsilon_3)+\Delta y(\varepsilon_2+i\varepsilon_4)$, we see that (5) approaches the zero if $\Delta z\to 0$. Thus, f is differentiableat z_0 and

$$f'(z_0)=\lim\frac{f(z+\Delta z)-f(z)}{\Delta z}=(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial y})(x_0,y_0).$$

Q.E.D.

As a further applications of these techniques, let us prove the following theorem

Theorem 5.4 *Let $\mathcal{U}$ be a domain and let $f\in\mathcal{A}()(\mathcal{U})$. If $f'(z)=$ for every point of $\mathcal{U}$, then $f\equiv Const$.*

Before proceeding with the proof, we observe that the connectedness of the domain $\mathcal{U}$ of essential. We illustrate this by an example. Let

$$f(z)=\begin{cases}1, & |z|<1\\ 0, & |z|>2\end{cases}$$

Here $f'(z)=0$ at every point of the domain of definition (which is not a domain), yet f is not constant.

Proof: From (2) and from (3) we get

$$\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=\frac{\partial u}{\partial y}=0.$$

Thus, f is constant. **Q.E.D.**

Using previous theorems and the Cauchy-Riemann equations, one can show that $f \in \mathcal{A}()(\mathcal{U})$ is constant if

$$\begin{aligned} u(x,y) &\equiv Const, \\ v(x,y) &\equiv Const, \\ |f(z)| &\equiv Const. \end{aligned}$$

Definition: The function h is said to be harmonic in $\mathcal{D}$, if $h \in C^2(\mathcal{D})$ and $\Delta h := h_{x,x} + h_{y,y} = 0$ in $\mathcal{D}$. The operator Δ is called *the Laplacian* $\aleph$.

Going back to our considerations, we see that se have established the following theorem

Theorem 5.5. *If $f(z) \in \mathcal{A}(D), D-$ an open set, then both u and v are harmonic and harmonic conjugate to each other.*

*Exercises:*1. Show that the function

$$f(z) = \frac{1}{z}$$

is nowhere differentiable.

2. Do the same for $f(z) = \sqrt{|z^2 + z|}$. (or $f(z) \notin \mathcal{A}(\mathcal{C})$. $\clubsuit$.)

3. Suppose that $f \in \mathcal{A}(D)$ and $\bar{f} \in \mathcal{A}(D)$. Prove that $f \equiv Const$.

4. Given

$$f(z) := \begin{cases} \frac{x^{4/3}y^{5/3} + ix^{5/3}y^{4/3}}{x^2y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

show that Cauchy-Riemann equations are satisfied at $z = 0$, but is not differentiablethere.

5. 5. $f(z) \in \mathcal{A}(\mathcal{D})$, $f(z) = u(x,y) + iv(x,y)$. Write the functions u and v in polar coordinates (r, Θ). Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \Theta}, \ -\frac{1}{r}\frac{\partial u}{\partial \Theta} = \frac{\partial v}{\partial r}.$$