4. Elementary functions

4.1. The exponential function e^z.

We repeat the definition from CHp. 1:

Definition: Given $z = x + iy \to e^z = e^x e^{iy} := e^x(\cos y + i \sin y.)$

Properties:

1. e^z is one-to-one in any horizontal strip of length $\leq 2\pi$, e.g. in $\{z, -\infty < x < \infty,\ y \in [a, a+l], a \in \mathbb{R}, a - \text{fixed},\ 0 \leq l \leq 2\pi\}$;
2. e^z is differentiable everywhere in $\mathbb{C}$ and

$$\frac{d}{dz} e^z = e^z,$$

3. The mapping by e^z is conformal, since the derivative is $\neq 0$.
4. e^z is periodic with complex period $2\pi i$, e.g.

$$e^z = e^{z+2\pi i};$$

further,

$$e^z = 1 \Longleftrightarrow z = 2k\pi i, k \in \mathbb{Z},$$

5. e^z maps

$$\begin{array}{ll} \{z, z = C + iy, \alpha \leq y \leq \beta\} \text{ on } \{w = e^C e^{i\phi}, \alpha \leq \phi \leq \beta\}, & C \text{ real constant} \\ \{z, z = x + iC, a \leq x \leq b\} \text{ on } \{e^x e^{iC}\}, & C \text{ real constant} \end{array}.$$

Definition: 4.2. Trigonometric functions:

Definition: Given any complex number z, we define

$$\cos z := \tfrac{e^{iz}+e^{-iz}}{2}, \quad \sin z := \tfrac{e^{iz}-e^{-iz}}{2i}$$

$$\tan z = \tfrac{\sin z}{\cos z}, \quad \cot z = \tfrac{\cos z}{\sin z}$$

$$\csc z = \tfrac{1}{\sin}, \quad \sec z = \tfrac{1}{\cos z}$$

$$\sinh = \tfrac{e^z - e^{-z}}{2}, \quad \cosh = \tfrac{e^z + e^{-z}}{2}.$$

It is left to the reader to deduce the known properties of the trigonometric functions.

4.3. The logarithmic function:

Definition: If $z \neq 0$, then we define $w = \log z$ to be any of the solutions of the equation

$$z = e^w,$$

e.g.

$$\text{if } z \neq 0 \rightarrow \log z = \ln|z| + i\text{Arg } z + 2k\pi i, k \in \mathbb{Z} \tag{1}$$

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Remark 1;

$$z = e^{\log z},$$

but

$$\log e^z = z + 2k\pi i, \text{ any } k \in \mathbb{Z}. \tag{2}$$

From (1), we get

$$\log z_1 z_2 = \log z_1 + \log z_2 \tag{3}$$

and

$$\log(z_1/z_2) = \log z_1 - \log z_2. \tag{4}$$

Pay attention to the fact that (3) and (4) must be interpreted as equality among classes. If (3) and (4) are assigned to particular values then there exists a value of the third term such that an equality holds. For example, if $z_1 = z_2 = -1$ and we select πi to be the value of both $\log z_1$ and $\log z_2$, then (3) is satisfied if we use the particular value $2\pi i$ for $\log z_1 z_2$.

Definition: The *principal value* of the logarithmic function is the valued inherited by the principal part of the argument, that is

$$\text{Log}\, z := \ln|z| + i\text{Arg } z.$$

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Properties:

Theorem 4.1. *The function Log z is analytic in* $D* := \{z, -\pi < Arg\ z < \pi\}$ *and*

$$\frac{d}{dz}\text{Log}\, z = \frac{1}{z}.$$

From Theorem 4.1 we deduce

Corollary 4.1: *The functions* $\ln|z|$ *and Arg z are harmonic in* $D*$.

4.4. Single-valued branches of $\log z$**.**

$$\mathcal{L}_\tau(z) = \ln|z| + i \text{ arg }_\tau z,$$

where

$$\text{arg}_{\tau} z \in [\tau, \tau + 2\pi].$$

4.5. Complex Powers.

Definition: Given $\alpha \in \mathbb{C}$ and $z \neq 0$, we define

$$z^\alpha := e^{\alpha \log z} \tag{5}$$

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Theorem 4.2 *The function z^α is differentiable and*

$$\frac{d}{dz} z^\alpha = \alpha z^{\alpha - 1}.$$

Remark 2: Let $\alpha = m \in \mathbb{Z}$. Then

$$(e^{i\Theta})^m = \cos(m\Theta) + i \sin(m\Theta).$$

The last equality is the famous *De Moivre's formula:*.

Exercises:

1. Show that

$$\text{Log}\, e^z = z \Longleftrightarrow -\pi \leq \text{arg} z < \pi.$$

2. Find the "error" in the following proof that $z = -z$: $z^2 = (-z)^2 \to \text{Log}\,(z) = \text{Log}\,(-z) \to \text{Log}\, z = e^{\text{Log}\, z} = e^{\text{Log}\,(-z)} = -z$.
3. Show that if $z_1 = i$ and $z_2 = -1 + i$, then $\text{Log}\, z_1 z_2 \neq \text{Log}\, i + \text{Log}\,(i - 1)$.
4. Show that for any $m \in \mathbb{Z}$, $m > 0$

$$z^{1/m} = z^{1/m} e^{\frac{\text{Arg}\ z + 2k\pi i}{m}},\ k = 0, 1, \cdots m - 1.$$

5. Find i^i and 1^z, $z \in \mathbb{C}$.