6. Integration in the Complex Plane

6.1. A smooth curve in $\mathbb{C}$.

Definition: Let $z = z(t)$, $t \in [\alpha, \beta]$ be a continuous complex valued function with following properties:

$$(1)$$

a) The mapping $z(t)$ is one-to- one on the domain of definition $[\alpha, \beta;]$
b) $z(t) \in C^1([\alpha, \beta])$;
c) $z'(t) \neq 0$, $t \in [\alpha, \beta]$.
$(z'(a) := z'(a^+)$, $z'(b) := z'(b^-))$.

The curve γ is the image of $[\alpha, \beta]$ under the mapping $z(t)$. ℵ.

Definition: The curve γ is closed, if $z(\alpha) = z(\beta)$ and $z'(\alpha) = z'(\beta)$. A curve that satisfies the first conditions a) and b) and additionally
c')

$$z'(\alpha) = z'(\beta)$$

is called *A Jordan curve* Also, we will use the term *an arc.*

Given a function $z(4)$ as above, we say that $z(t)$ is an *admissible parametrization of* γ.

Jordan's curve theorem: *Any closed Jordan curve separates the complex plane into two disjoint simply connected domains.*

The proof will be omitted.

The bounded domain is called *the interior of* γ, *the unbounded - the exterior.*

Example:

$$z_1(t) = \cos t + i \sin t,\ t \in [0, 2\pi], \tag{2}$$

and

$$z_2(t) = \sin t + i \cos t,\ t \in [0, 2\pi] \tag{3}$$

$C_0(1)$

Both z_1, z_2 are admissible parametrization of the unit circle.

Directed arcs. Given an arc γ with endpoints Z_I and Z_{II}, we see that there are two ways of of ordering the points on γ; to start at Z_I and to terminate at Z_{II}, or conversely, to start at Z_{II} and to terminate at Z_I. Declaring the initial and the terminal points among Z_I and Z_{II}, we declare a direction on γ.

Definition: A smooth arc together with a specific ordering of its points, is called *directed smooth arc.*

In Examples (2) the unit circle is directed clockwise, whereas in Example (3) - counterclockwise. For the first case, we write $C_0(1)$. The unit circle in (3) will be denoted by $-C_0(1)$; e.g. the opposite of $C_0(1)$.

Let now γ be a closed directed curve. Let D be the interior of γ.

Definition: If D lies to the left with respect to the direction of γ, then it is called *positively orientated.* Otherwise, it is *negative orientated.* $\aleph$.

Definition: A contour Γ is either a single point z_0 or a finite sequence of directed smooth curves $(\gamma_1, \cdots, \gamma_m)$such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for each $k = 1, \cdots, m-1$. $\aleph$.

In analogy with simple curves, one can introduce the terms directed contours.

Consider, as an example, the annulus $\{z, 1 \le |z| \le 2\}$. The interior is positively orientated with respect to $\gamma_1 := C_0(2)$ and $\gamma_2 := -C_0(1)$. Under this orientation, the open unit disk is negatively orientated with respect to γ_2, whereas the open disk $D_0(2)$ is positively orientated with respect to γ_1.

We recall that if γ is a smooth curve and $z = z(t), t \in [\alpha, \beta]-$ an admissible parametrization, then its length $l(\gamma)$ is given by

$$l(\gamma) = \int_\gamma ds = \int_\alpha^\beta \frac{ds}{dt} dt = \int_\alpha^\beta |z'(t)| dt. \qquad (4)$$

6.2. Contours integrals.

Definition: Let γ is a smooth directed curve and $z = z(t) = x(t) + iy(t), t \in [\alpha, \beta]-$ an admissible parametrization, and suppose that $f \in C(\gamma)$, $f(z) = u(x, y) + iv(x, y)$. Then

$$\int_\gamma f(z) dz := \int_\alpha^\beta f(z(t)) z'(t) dt =$$

$$= \int_\alpha^\beta (u(x(t), y(t)) + iu(x(t), y(t))(x'(t) + iy'(t)) dt, \qquad (5)$$

where the integral is an integral of Riemann. $\aleph$.

It is a natural question whether the integral does exist. A positive answer gives the theorem of Riemann, saying that every function, continuous on an interval $[a, b]$ is integrable in the sense of Riemann. We leave to the reader the answer of the question whether the function $z(t)z'(t)$ is continuous on the interval of paramatrization $\alpha, \beta]$.

The following properties result from the definition.

Properties:

1.
$$\int_{\gamma} f(z)dz = -\int_{-\gamma} f(z)dz.$$

2.
$$\int_{\gamma} (af(z) + bg(z))dz = a\int_{\gamma} f(z)dz + b\int_{\gamma} f(z)dz,\ a, b \in \mathbb{C}.$$

3. Let Γ is a directed contour in $\mathbb{C}, \gamma = \bigcup_{i=1}^{k} \gamma_i$, then

$$\int_{\gamma} f(z)dz = \sum_{i=1}^{k} \int_{\gamma_i} f(z)dz.$$

Since equation (5) is valid for all suitable parametrizations of γ and since the integral of f along γ is defined independently on any parametzization, we immediately deduce the following

Theorem 6.1. Let $z_1(t), t \in [a, b]$ and $z_2(t), t \in [c, d]$ be two admissible parametrizations of γ, preserving the direction. Then

$$\int_{\gamma} f(z)dz = \int_a^b f(z_1(t))z_1'(t)dt = \int_c^d f(z_2(t))z_2'(t)dt.$$

6.3. Independence on the path of integration.

We start by establishing Theorem 6.2. Before introducing it, we recall that a function F is an *antiderivative*of f through a domain D, if

$$F'(z) = f(z)$$

for each $z \in D$.

Theorem 6.2. Suppose that $f \in C[a, b]$, $[a, b]$- a real segment and let $F(t)$ be a n antiderivative. Then

$$\int_{[a,b]} f(z)dz = F(b) - F(a).$$

Proof: Indeed, let $f(t) = u(t) + iv(t)$, $t \in [a, b]$ and $F(t) = U(t) + iV(t)$, $t \in [a, b]$. In view of the conditions,

$$U'(t) = u(t),\ V'(t) = v(t)\, t \in [a, b].$$

Joining now (1), we may write

$$\int_\gamma f(z)dz = \int_a^b f(t)dt = \int_a^b (u(t) + iv(t))dt =$$

$$\int_a^b (U'(t) + iV'(t))dt = \int_a^b \frac{\partial(u(t) + iv(t))}{\partial t} dt = F(b) - F(a).$$

Q.E.D.

Theorem 6.2 is a particular case of the main basic result given by

Theorem 6.3 Given $\gamma-$ a directed curve with Z_I and Z_T an initial and a terminal point, and $f \in C(\gamma)$, let $F(z)$ be an antiderivative through γ. Then

$$\int_\gamma f(z)dz = F(Z_T) - F(Z_I).$$

Proof: INdeed, by definition

$$\frac{d}{dz}F(z) = f(z),$$

which implies that

$$\frac{dF(z)}{dt} = f(z)\frac{dz}{dt} = f(z(t))z'(t).$$

Let $[a, b]$ be the definition interval of the parametrization $z(t)$; the function $f(z(t))z'(t)$ is defined on this interval, and at the same time because of the last equality, the function $F(z(t))$ is an antiderivative of its. Applying Theorem 6.2, we arrive at

$$\int_{[a,b]} f(z(t))z'(t)de + t = F(z(b)) - F(z(a)) = F(Z_T) - F(Z_I).$$

Q.E.D.

In our further considerations, we often will use the following

Theorem 6.4. Let Γ be a contour and $f \in C(\Gamma)$. Then

$$| \int_\Gamma f(z)dz| \leq \|f\|_\Gamma l(\Gamma).$$

The proof is left to the reader.

Exercises:

1. Parametrize the triangle with vertices at $= (-1, 0)$, $(1, 0)$ and $(0, i)$.
2. Using an appropriate parametrization, find the length of $[z_1,\, z_2]$ and of $C_a(\rho)$.
3. Find an upper estimate of $\int_\Gamma \frac{e^z}{z^2+1} dz$, where $\Gamma = C_0(2)$, traversed one time in the positive direction.

Is it true that

4. $| \int_\Gamma \frac{dz}{z^2-i}| \leq \frac{3\pi}{4}$, $\Gamma := C_0(3)$;
5. $| \int_\Gamma \frac{e^{3z}}{1+e^z} dz| \leq \frac{2\pi}{e^R-1}$, $\Gamma :=$ the segment$[R, R + 2i\pi]$;
6. $| \int_\Gamma e^{\sin z} dz| \leq 1$, with Γ being the segment with endpoints at $z = 0$ and $z = i$.
7. Let $f \in C[a, b], \infty < a \leq b < \infty$. Prove that

$$| \int_a^b f(t)dt| \leq \int_a^b |f(t)|dt.$$