7. Cauchy’s integral theorem and Cauchy’s integral formula

7.1. Independence of the path of integration

Theorem 6.3. can be rewritten in the following form:

**Theorem 7.1:** Let $D$ be a domain in $\mathbb{C}$ and suppose that $f \in C(D)$. Suppose further that $F(z)$ is a continuous antiderivative of $f(z)$ through $D$. Let $z_0$ and $z_T$ be distinct points in $D$. Then the integral

$$\int_{z_0}^{z_T} f(z)dz$$

does not depend on the path of integration, e.g., for every smooth contour $\gamma \subset D$ which start at $z_0$ and terminates at $z_T$, we have

$$\int_{z_0}^{z_T} f(z)dz = \int_{\gamma} f(z)dz = F(z_T) - F(z_0).$$

Theorem 7.1. is called Theorem on the depend of the path of integration.

From this theorem we get the following obvious consequence:

**Corollary 7.2.**: Under the conditions on $f$ of Theorem 7.1., let $\gamma$ be a smooth closed contour which lies entirely in $D$. Then

$$\int_{\gamma} f(z)dz = 0$$

Our coming considerations are based on the following theorem:

**Theorem 7.3.** Let $D$ be a domain in $\mathbb{C}$ and $f \in C(D)$. Then the following statements are equivalent:

1. $f$ has a continuous antiderivative in $D$;
2. $\int_{\gamma} f(z)dz = 0$

for every loop $\gamma$ lying in $\subset D$.

\footnote{We call such contours \textit{loops}.}
(3) The integral
\[ \int_{z_1}^{z_2} f(z)dz \]
is independent of the path of integration; e.g., if \( \gamma_1 \) and \( \gamma_2 \) share the same initial and terminal points, then
\[ \int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz. \]

**Proof:** Since the implications 1) \( \rightarrow \) 2) and 2) \( \rightarrow \) 3) already established (Theorem 7.1 and Theorem 7.2), we will concentrate on the proof of 3) \( \rightarrow \) 1).

Select an arbitrary point \( z_0 \in \mathcal{D} \) and let \( z \in \mathcal{D} \).

Set
\[ F(z) := \int_{z_0}^{z} f(z)dz. \]

We claim that \( F(z) \) is an antiderivative of \( f \) in \( \mathcal{D} \). Before, we notice that the integral is well defined - because of the connectedness of the domain \( \mathcal{D} \) there is a contour which combines \( z_0 \) and \( z \).

We shall show that
\[ \frac{F(z + \Delta z) - F(z)}{\Delta z} \rightarrow f(z), \Delta z \rightarrow 0. \]

Indeed,
\[ \frac{F(z + \Delta z) - F(z)}{\Delta z} = \frac{\int_{z}^{z+\Delta z} f(w)dw}{\Delta z}, \]
where we integrate along a segment lying completely in the domain.

Regarding Theorem 6.4, we may write
\[ \left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| = \]
\[ = \left| \frac{\int_{z}^{z+\Delta z} f(w) dw}{\Delta z} \right| \leq \| f(w) - f(z) \|_{z, z+\Delta z} \rightarrow 0 \text{ as } \Delta z \rightarrow 0. \]

Thus \( F'(z) = f(z) \). This concludes the proof. \( \text{Q.E.D.} \)
7.2. Continuous deformations of loops.

**Definition:** The loop $\gamma_1$ is said to be continuously deformable to the loop $\gamma_2$ in the domain $D$, if there exists a function $z(s, t), (s, t) \in ([0, 1] \times [0, 1])$ that satisfies the conditions:

1. $z(s, t) \in C^2([0, 1] \times [0, 1])$;
2. For each fixed $s \in [0, 1]$ the function $z(s, t)$ parametrizes a loop in $D$;
3. The function $z(0, t)$ parametrizes $\gamma_1$;
4. The function $z(1, t)$ parametrizes $\gamma_2$.

**Example:** THE function $z(s, t) := (1 + s)e^{2\pi it}, 0 \leq s, t \leq 1$ deforms continuously the circle $C_0(1)$ into the circle $C_0(2)$.

7.3. Deformation Invariance Theorem.

We first recall the definition of a *simply connected domain*.

**Definition:** Any domain $D$ in the complex plane $\mathbb{C}$ possessing the property that every loop in $D$ can be continuously deformed in $D$ to a point is called simply connected.

For example, any disk $D_a(r), r > 0$ is a simply connected domain.

Now we are in position to prove the Deformation Invariance Theorem.

**Theorem 7.3.** Let $D$ be a domain in $\mathbb{C}$ and suppose that $f \in \mathcal{A}(D)$. If $\gamma_1, \gamma_2$ are continuously deformable into each other closed curves, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$  

**Proof:**

Fix $s \in [0, 1]$ and set $\gamma(s) := z(s, t), t \in [0, 1]$. We shall show that the function $I(s) := \int_{\gamma(s)} f(z)dz$ equals a constant. Indeed,

$$\int_{\gamma(s)} f(z)dz = \int_{\gamma(s)} f(z(s, t)) \frac{\partial z(s, t)}{\partial t} dt.$$  

Look at the derivative of $I(s)$; we have

$$I'(s) = \int_{\gamma(s)} f(z(s, t)) \frac{\partial z(s, t)}{\partial t} dt =$$
\[ \int_{\gamma(s)} \left[ \frac{\partial f(z(s,t))}{\partial t} \frac{\partial z(s,t)}{\partial s} - \frac{\partial f(z(s,t))}{\partial s} \frac{\partial z(s,t)}{\partial t} \right] dt. \]

On the other hand,
\[ \frac{\partial}{\partial t} \left( f(z(s,t)) \frac{\partial z(s,t)}{\partial s} \right) = \frac{\partial f(z(s,t))}{\partial t} \frac{\partial z(s,t)}{\partial s} + f(z(s,t)) \frac{\partial^2 z(s,t)}{\partial s \partial t}. \]

The theorem by Weierstrass about the independence of second order derivatives of the order of differentiation guarantees that
\[ \frac{dI(s)}{ds} = \int_{0}^{1} \frac{\partial}{\partial t} \left[ f(z(s,t)) \frac{\partial z(s,t)}{\partial s} \right] dt = \]
\[ = f(z(s,1)) \frac{\partial z(s,t)}{\partial s}(s,1) - f(z(s,0)) \frac{\partial z(s,t)}{\partial s}(s,0). \]

As we know, the curves \( \gamma(s) \) are closed which means that for every \( s \in [0,1] \), \( z(s,0) = z(s,1) \).

Thus
\[ I(s) = \int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz. \quad \text{Q.E.D.} \]

**Cauchy’s integral theorem** An easy consequence of Theorem 7.3. is the following, familiarly known as *Cauchy’s integral theorem*.

**Theorem 7.4.** If \( D \) is a simply connected domain, \( f \in \mathcal{A}(D) \) and \( \Gamma \) is any loop in \( D \), then
\[ \int_{\Gamma} f(z) dz = 0. \]

**Proof:** The proof follows immediately from the fact that each closed curve in \( D \) can be shrunk to a point. \quad \text{Q.E.D.}

We conclude the following

**Theorem 7.5.** Let \( \mathcal{D} \) be a domain in \( \mathbb{C} \) and \( f \in \mathcal{A}(\mathcal{D}) \cap C(\overline{\mathcal{D}}) \). Set \( \partial \mathcal{D} := \Gamma \). Then
\[ \int_{\Gamma} f(z) dz = 0. \]
**Proof:** Without losing the generality, we may assume that all components of $\Gamma$ are smooth curves. It $D$ is simply connected, then we are done. Assume that $D$ is double connected and let $\Gamma = \Gamma_1 \cup \Gamma_2$. The domain is positively orientated with respect to $\Gamma$; let $\Gamma_1$ be the positive component (clockwise) and $\Gamma_2$ the negative (counterclockwise) ($\Gamma = \Gamma_1 \cup (-\Gamma_2)$.) Without losing the generality we suppose that $\Gamma_1$ and $\Gamma_2$ are continuously deformable into each other, and by Theorem 7.3.

\[ \int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz. \] (1)

On the other hand

\[ \int_{\Gamma} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{-\Gamma_2} f(z)dz = \int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz = 0. \]

Joining (1), we arrive at the statement.

The Cauchy’s integral theorem indicates the intimate relation between simply connectedness and existence of a continuous antiderivative.

**Theorem 7.6.** Let $D$ be simply connected in $\mathbb{C}$ and $f \in \mathcal{A}(D)$.

Then $f$ possesses a continuous antiderivative and its contour integral does not depend on the path of integration.

The proof follows from Theorem 7.3.

**7.4. Cauchy’s integral formula**

**Theorem 7.7.** Let $D$ be a domain in $\mathbb{C}$, $\Gamma := \partial D$ and $f \in \mathcal{A}(D) \cap C(\overline{D})$. Then, for every point $a \in D$ the representation

\[ f(a) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - a}dz \] (2)

holds.

**Proof:**

Take $r$ sufficiently small (e.g. $\overline{D_a}(r) \subset D$) and consider $\int_{|z-a|=r} \frac{f(z)}{z-a}dz$.

(the circle is traversed once in the positive direction). We have

\[ \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{z-a}dz = \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}}ire^{i\theta}d\Theta. \]
Letting now $r \to 0$ we obtain that

$$\frac{1}{2\pi i} \oint_{|z-a|=r} f(z)dz = f(a).$$

To complete the proof, we apply Theorem 7.5. with respect to the function $\frac{f(z)}{z-a}$ and to the domain $\mathcal{D} \setminus \mathcal{D}_a(r)$. Q.E.D.

As an application, we provide the mean value theorem for harmonic functions.

**Theorem 7.7.** Let $h$ be harmonic in the disk $D_a(R)$, $R > 0$. Then

$$h(a) = \frac{1}{2\pi} \int_0^{2\pi} h(a + Re^{i\Theta})d\Theta.$$

**Proof:** We recall that the real and the imaginary components of an analytic function are complex conjugate harmonic functions. Let $f \in \mathcal{A}(D_a(R))$ be such that $h(z) := \mathfrak{Re} f(z)$. Denote the imaginary component by $k(z)$.

$$f(f) = h(z) + ik(z), \quad z \in K_a(R).$$

Using (2), we get

$$h(a) + ik(a) = \frac{1}{2\pi i} \int_{C_a(R)} \frac{h(\zeta) + ik(\zeta)}{\zeta - a} d\zeta.$$  

Hence,

$$h(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{h(a + Re^{i\Theta})}{Re^{i\Theta}}iRe^{i\Theta} d\Theta.$$  

The statement follows after completing the needed cancellations.

**Exercises:**

1. Prove that

$$\int_{C_a(\rho)} \frac{dz}{(z-a)^m} = \left\{ \begin{array}{ll} 0, & m \neq 1 \\ 2\pi i, & m = 1 \end{array} \right.$$
2. Prove that
\[ \int_{C_0(\rho)} \frac{dz}{(z-a)} = \begin{cases} 0, & |a| > \rho \\ 2\pi i, & |a| < \rho \end{cases} \]

3. Which of the following domains are simply connected?
   a) \( \{z, |\text{Im } z| < 1\} \)
   b) \( \{z, 1 < |z| < 2\} \)
   c) \( \{z, |z| < 1\} \)
   d) \( \{z, |z| > 1\} \)
   e) \( \{z, |z| < 1\} \setminus \{z, 0 < \text{Re } z < 1\} \)

3. Calculate
\[ \int_{S} \frac{1}{1 + z^2} \, dz, \]
with \( S \) being the interval \([1, 1 + i]\).

4. Show that if \( f(z) \) is of the form
\[ f(z) = \sum_{k=0}^{n} \frac{A_k}{z^k} + g(z), \]
where \( g(z) \) is analytic outside \( C_0(1) \), then
\[ \oint_{|z|=1} f(z) \, dz = 2\pi i A_1. \]

(By definition, \( \oint_{|z|=1} := \oint_{C_0(1)} \) \( C_0(1) \) traversed once in positive direction.)

5. Let \( P \) be a polynomial of degree \( \geq 2 \), such that all zeros lie in \( \mathcal{D}_R(\mathcal{R}) \), \( \mathcal{R} > 0 \).
Show that
\[ \oint_{|z|=R} \frac{1}{P(z)} \, dz = 0. \]

**Hint** Apply Theorem 7.5. with respect to the annulus \( \{z, R < |z| < R + r\} \)
and then let \( r \) increase to infinity.