7. Cauchy's integral theorem and Cauchy's integral formula

7.1. Independence of the path of integration

Theorem 6.3. can be rewritten in the following form:

Theorem 7.1 : *Let $\mathcal{D}$ be a domain in $\mathbb{C}$ and suppose that $f \in C(\mathcal{D})$. Suppose further that $F(z)$ is a continuous antiderivative of $f(z)$ through $\mathcal{D}$ $\mathcal{D}$. Let z_0 and z_T be distinct points in $\mathcal{D}$. Then the integral*

$$\int_{z_0}^{z_T} f(z)dz$$

does not depend on the path of integration, e.g., for every smooth contour $\gamma \subset \mathcal{D}$ which start at z_0 and terminates at z_T, we have

$$\int_{z_0}^{z_T} f(z)dz = \int_{\gamma} f(z)dz = F(z_T) - F(z_0).$$

Theorem 7.1. is called Theorem on the depend of the path of integration.
From this theorem we get the following obvious consequence:

Corollary 7.2. : *Under the conditions on f of Theorem 7.1., let γ be a smooth closed contour which lies entirely in $\mathcal{D}$.*[1] *Then*

$$\int_{\gamma} f(z)dz = 0$$

Our coming considerations are based on the following theorem:

Theorem 7.3. *Let $\mathcal{D}$ be a domain in $\mathbb{C}$ and $f \in C(\mathcal{D})$. Then the following statements are equivalent:*
(1) f has a continuous antiderivative in $\mathcal{D}$;
(2)

$$\int_{\gamma} f(z)dz = 0$$

for every loop γ lying in $\subset \mathcal{D}$.

[1] We call such contours *loops*.

(3) The integral

$$\int_{z_1}^{z_2} f(z)dz$$

is independent of the path of integration; e.g., if γ_1 and γ_2 share the same initial and terminal points, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof: Since the implications 1) $\to$ 2) and 2) $\to$ 3) already established (Theorem 7.1 and Theorem 7.2), we will concentrate on the proof of 3) $\to$ 1).

Select an arbitrary point $z_0 \in \mathcal{D}$ and let $z \in \mathcal{D}$.

Set

$$F(z) := \int_{z_0}^{z} f(z)dz.$$

We claim that $F(z)$ is an antiderivative of f in $\mathcal{D}$. Before, we notice that the integral is well defined - because of the connectedness of the domain $\mathcal{D}$ there ix a contour which combines z_0 and z.

We shall show that

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} \to f(z), \Delta z \to 0.$$

Indeed,

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{\int_z^{z+\Delta z}(w)dw}{\Delta z},$$

where we integrate along a segment lying completely in the domain.

Regarding Theorem 6.4, we may write

$$|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)| =$$

$$= |\frac{\int_z^{z+\Delta z}(f(w) - f(z)dw}{\Delta z}| \leq \|f(w) - f(z)\|_{[z,z+\Delta z]} \to 0 \text{ as } \Delta z \to 0.$$

Thus $F'(z) = f(z)$. This concludes the proof. **Q.E.D.**

7.2. Continuous deformations of loops.

Definition: The loop γ_1 is said to be continuously deformable to the loop γ_2 in the domain D, if there exists a function $z(s,t)$, $(s,t) \in ([0,1] \times [0,1])$ that satisfies the conditions:
1. $z(s,t) \in C^2([0,1] \times [0,1])$;
2. For each fixed $s \in [0,1]$ the function $z(s,t)$ parametrizes a loop in D;
3. The function $z(0,t)$ parametrizes γ_1;
4. The function $z(1,t)$ parametrizes γ_2.
Example: THE function

$$z(s,t) := (1+s)e^{2\pi it},\ 0 \leq s,t \leq 1$$

deforms continuously the circle $C_0(1)$ into the circle $C_0(2)$.

7.3. Deformation Invariance Theorem.

We first recall the definition of a *simply connected domain.*

Definition: Any domain D in the complex plane $\mathbb{C}$ possessing the property that every loop in D can be continuously deformed in D to a point is called simply connected. ℵ.

For example, any disk $D_a(r), r > 0$ is a simply connected domain.

Now we are in position to prove the *Deformation Invariance Theorem.*

Theorem 7.3. *Let $\mathcal{D}$ be a domain in $\mathbb{C}$ and suppose that $f \in \mathcal{A}(\mathcal{D})$. If γ_1, γ_2 are continuously deformable into each other closed curves, then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof:

Fix $s \in [0,1]$ and set $\gamma(s) := z(s,t), t \in [0,1]$. We shall show that the function $I(s) := \int_{\gamma(s)} f(z)dz$ equals a constant. Indeed,

$$\int_{\gamma(s)} f(z)dz = \int_{\gamma(s)} f(z(s,t))\frac{\partial z(s,t)}{\partial t}dt.$$

Look at the derivative of $I(s)$; we have

$$I'(s) = \int_{\gamma(s)} f(z(s,t))\frac{\partial z(s,t)}{\partial t}dt =$$

$$= \int_{\gamma(s)} [\frac{\partial f(z(s,t))}{dt} \frac{\partial z(s,t)}{\partial s} \frac{\partial z(s,t)}{\partial t} + f(z(s,t)) \frac{\partial^2 z(s,t)}{\partial s \partial t}] dt.$$

On the other hand,

$$\frac{\partial}{dt}(f(z(s,t)) \frac{\partial z(s,t)}{\partial s}) = \frac{\partial f(z(s,t))}{dt} \frac{\partial z(s,t)}{\partial t} \frac{\partial z(s,t)}{\partial s} + f(z(s,t)) \frac{\partial^2 z(s,t)}{\partial t \partial s}.$$

The theorem by Weierstrass about the independence of second order derivatives of the order of differentiation guarantees that

$$\frac{dI(s)}{ds} = \int_0^1 \frac{\partial}{dt} [f(z(s,t)) \frac{\partial z(s,t)}{\partial s}] dt =$$

$$= f(z(s,1)) \frac{\partial z(s,t)}{\partial s}(s,1) - f(z(s,0)) \frac{\partial z(s,t)}{\partial s}(s,0).$$

As we know, the curves $\gamma(s)$ are closed which means that for every $s \in [0,1]$ $z(s,0) = z(s,1)$.

Thus

$$I(s) = \int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz.$$

Q.E.D.

Cauchy's integral theorem An easy consequence of Theorem 7.3. is the following, familiarly known as *Cauchy's integral theorem.*

Theorem 7.4. *If D is a simply connected domain, $f \in \mathcal{A}(D)$ and Γ is any loop in D, then*

$$\int_\Gamma f(z) dz = 0.$$

Proof: The proof follows immediately from the fact that each closed curve in D can be shrunk to a point. **Q.E.D.**

We conclude the following

Theorem 7.5. *Let $\mathcal{D}$ be a domain in $\mathbb{C}$ and $f \in \mathcal{A}(\mathcal{D}) \bigcap C(\overline{\mathcal{D}})$. Set $\partial \mathcal{D} := \Gamma$. Then*

$$\int_\Gamma f(z) dz = 0.$$

Proof: Without losing the generality, we may assume that all components of Γ are smooth curves. It $\mathcal{D}$ is simply connected, then we are done. Assume that $\mathcal{D}$ is double connected and let $\Gamma = \Gamma_1 \bigcup \Gamma_2$. The domain is positively orientated with respect to Γ; let Γ_1 be the positive component (clockwise) and $\Gamma_2 -$ the negative (counterclockwise) ($\Gamma = \Gamma_1 \bigcup (-\Gamma_2)$.) Without loosing the generality we suppose that Γ_1 and Γ_2 are continuously deformable into each other, and by Theorem 7.3.

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz. \tag{1}$$

On the other hand

$$\int_{\Gamma} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{-\Gamma_2} f(z)dz = \int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz = 0.$$

Joining (1), we arrive at the statement.

The Cauchy's integral theorem indicates the intimate relation between simply connectedness and existence of a continuous antiderivative.

Theorem 7.6. *Let $\mathcal{D}$ be simply connected in $\mathbb{C}$ and $f \in \mathcal{A}(\mathcal{D})$.*

Then f possesses a continuous antiderivative and its contour integral does not depend on the path of integration.

The proof follows from Theorem 7.3.

7.4. Cauchy's integral formula

Theorem 7.7. *Let $\mathcal{D}$ be a domain in $\mathbb{C}$, $\Gamma := \partial\mathcal{D}$ and $f \in \mathcal{A}(\mathcal{D}) \bigcap C(\overline{\mathcal{D}})$. Then, for every point $a \in \mathcal{D}$ the representation*

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-a} dz \tag{2}$$

holds.

Proof:

Take r sufficiently small (e.g. $\overline{D}_a(r) \subset \mathcal{D}$) and consider $\oint_{|z-a|=r} \frac{f(z)}{z-a} dz$. (the circle is traversed once in the positive direction). We have

$$\frac{1}{2\pi i} \oint_{|z-a|=r} \frac{f(z)}{z-a} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\Theta})}{re^{i\Theta}} ire^{i\Theta} d\Theta.$$

Letting now $r \to 0$ we obtain that

$$\frac{1}{2\pi i}\oint_{|z-a|=r} f(z)dz = f(a).$$

To complete the proof, we apply Theorem 7.5. with respect to the function $\frac{f(z)}{z-a}$ and to the domain $\mathcal{D} \setminus \overline{\mathcal{D}}_a(r)$. **Q.E.D.**

As an application, we provide the *mean value theorem for harmonic functions.*

Theorem 7.7. *Let h be harmonic in the disk $D_a(R), R > 0$. Then*

$$h(a) = \frac{1}{2\pi}\int_0^{2\pi} h(a + Re^{i\Theta})d\Theta.$$

Proof: : We recall that the real and the imaginary components of an analytic function are complex conjugate harmonic functions. Let $f \in \mathcal{A}(D_a(R))$ be such that $h(z) := \Re f(z)$. Denote the imaginary component by $k(z)$.

$$f(f) = h(z) + ik(z),\ z \in K_a(R).$$

Using (2), we get

$$h(a) + ik(a) = \frac{1}{2\pi i}\int_{C_a(R)} \frac{h(\zeta) + ik(\zeta)}{\zeta - a}d\zeta.$$

Hence,

$$h(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{h(a + Re^{i\Theta})}{Re^{i\Theta}} iRe^{i\Theta}d\Theta.$$

The statement follows after completing the needed cancellations.

Exercises:

1. Prove that

$$\int_{C_a(\rho)} \frac{dz}{(z-a)^m} = \begin{cases} 0, & m \neq 1 \\ 2\pi i, & m = 1 \end{cases} \qquad \clubsuit$$

2. Prove that

$$\int_{C_0(\rho)} \frac{dz}{(z-a)} = \begin{cases} 0, & |a| > \rho \\ 2\pi i, & |a| < \rho \end{cases} \qquad \clubsuit$$

3. Which of the following domains are simply connected?
a) $\{z, |\text{ Im } z| < 1\}$;
b) $\{z, 1 < |z| < 2\}$;
c)$\{z, |z| < 1\}$;
d) $\{z, |z| > 1\}$;
e) $\{z, |z| < 1\} \setminus \{z, 0 < \text{ Re } z < 1\}$. $\clubsuit$
3. Calculate

$$\int_{\mathcal{S}} \frac{1}{1+z^2} dz,$$

with $\mathcal{S}$ being the interval $[1, 1+i]$. $\clubsuit$
4. Show that if $f(z)$ is of the form

$$f(z) = \sum_{k=0}^{n} \frac{A_k}{z^k} + g(z),$$

where $g(z)$ is analytic outside $C_0(1)$, then

$$\oint_{|z|=1} f(z) dz = 2\pi i A_1.$$

$\clubsuit$
(By definition, $\oint_{|z|=1} := \int_{C_0(1)}$, $C_0(1)$ traversed once in positive direction.) $\clubsuit$
5. Let P be a polynomial of degree ≥ 2, such that all zeros lie in $\mathcal{D}_\prime(\mathcal{R})$, $R > 0$. Show that

$$\oint_{|z|=R} \frac{1}{P(z)} dz = 0.\clubsuit$$

Hint Apply Theorem 7.5. with respect to the annulus $\{z, R < |z| < R + r\}$ and then let r increase to infinity. $\clubsuit$