8. Cauchy's integral theorem and its consequences

We first provide the theorem of Morera which is an inverse result to Cauchy's theorem:

Theorem 8.1. *Let the domain - $\mathcal{D}$ be simply connected and suppose that $f \in C(\overline{\mathcal{D}})$. Assume that*

$$\oint_\gamma f(z)dz = 0$$

along each closed contour $\gamma \subset \mathcal{A}(\mathcal{D})$. Then $f \in \mathcal{A}(\mathcal{D})$

The proof will be omitted.

Given a domain $\mathcal{D}$ in $\mathbb{C}$ and $f \in C(\overline{\mathcal{D}})$ we know by the classical theorem by Weierstrass that the function $|f(z)|$ attains its absolute maximum valued on $\overline{\mathcal{D}}$. Where does it lie? In the case of analytic function the answer is given by the following theorem known as *a maximum principle for analytic functions.*

Theorem 8.2. *Let $\mathcal{D}$ be a domain in $\mathbb{C}$ and suppose that $f \in \mathcal{A}(\mathcal{D}) \bigcap C(\overline{\mathcal{D}})$. Then $|f(z)|$ attains its maximal valued in $\overline{\mathcal{D}}$ on the boundary $\partial\mathcal{D}$, unless f is a constant.* [1]

Proof: If $f \equiv Const$, the theorem is trivial. THat's why we will consider the case when $f \not\equiv Const$. Suppose that the statement of the theorem is wrong. Let $\max_{z\in\overline{\mathcal{D}}} |f(z)| := |f(z_0)|$ with z_0 being an inner point in the domain $\mathcal{D}$.

For z_0 is an inner point, we may apply Cauchy's formula, namely,

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_{z_0}(\rho)} \frac{f(z)}{z - z_0} dz, \tag{1}$$

with some ρ small enough and a circle $C_{z_0}(\rho)$ traversed once in a positive direction. Since $f \not\equiv Const$, there is a curve $\gamma \in C_{z_0}(\rho)$ on the circle of positive length $l(\gamma)$, such that $|f(z)| < |f(z_0)|$ on γ. Let, for definiteness,

$$|f(z)| \leq |f(z_0)| - \delta, \delta > 0, \; z \in \gamma \text{ for some } \delta > 0.$$

[1] *Obviously* $\mathcal{A}(\mathcal{D}) \bigcap C(\overline{\mathcal{D}}) \supset \mathcal{A}(\overline{\mathcal{D}})$.

We estimate (1) by using this inequality:

$$|f(z_0)| \leq \frac{1}{2\pi}\frac{(|f(z_0)|-\delta)l(\gamma)}{\rho} + |f(z_0)|\frac{(2\pi\rho - l(\gamma))}{2\pi\rho}.$$

Since δ, $l(\gamma) > 0$, we conclude that

$$|f(z_0)| < |f(z_0)|$$

which is impossible. Hence our assumption is not correct and $z_0 \in \mathcal{D}$.**Q.E.D.**

Another important consequence of Cauchy's integral formula is that every analytic function is infinitely many times differentiable.

Theorem 8.3. *Let $\mathcal{D}$ be a domain and $f \in \mathcal{A}(\mathcal{D})$. Let $a \in \mathcal{A}(\mathcal{D})$ be an arbitrary point. Then f is infinitely many times differentiable at a and*

$$f^n(a) = \frac{n!}{2\pi i}\oint_{C_{z_0}(\rho)} \frac{f(z)}{(z-a)^{n+1}}dz; \tag{2}$$

the number ρ is small enough and we integrate counterclockwise along the circle $C_a(\rho)$.

Proof: Indeed, the expression $\oint_{C_{z_0}(\rho)} \frac{f(z)}{(z-a)}dz$ is a differentiable function at a. We have

$$\frac{d}{da}\oint_{C_{z_0}(\rho)} \frac{f(z)}{(z-a)}dz = \oint_{C_{z_0}(\rho)} d\frac{\frac{f(z)}{(z-a)}}{da}dz = \oint_{C_{z_0}(\rho)} \frac{f(z)}{(z-a)^2}dz,$$

which implies (2) for $n = 1$. Using mathematical induction we prove (2) for every n. The further proof is left to the reader. **Q.E.D.**

From Theorem 8.3. we deduce the theorem of *Loiuville*:

Theorem 8.4. *Let f be entire and bounded in $\mathbb{C}$. Then $f \equiv Const$.*

Proof: Take an arbitrary $a \in \mathbb{C}$ and fix $\in \mathbf{N}$. Regarding (2), we may write

$$f^{(n)}(a) = n!\frac{1}{2\pi i}\oint_{C_a(r)} \frac{f(z)}{(z-a)^{n+1}}dz.$$

Since $f \in \mathcal{E}$, the last equality is valid for every $r > 0$. Applying Theorem 6.4, we get

$$|f^{(n)}(a)| \leq n!\frac{M}{r^n}.$$

Letting $r \to \infty$, we see that

$$f^{(n)}(a) = 0$$

everywhere in $\mathbb{C}$. Thus, $f \equiv Const$. **Q.E.D.**

We provide a result known as *Schwartz's Lemma:*

Theorem 8.5. *Suppose that $f \in \mathcal{A}(D_0(1))$, $f(0) = 1$ and $\|f\|_{\overline{D}_0(1)} := M$. Then for every $z \in \overline{D}_0(1)$ the inequality*

$$|f(z)| \leq M|z| \tag{3}$$

holds. If for some $z_0, |z_0| < 1$

$$|f(z_0)| = M|z_0|,$$

then $f(z) \equiv Mze^{i\alpha}$ for some $\alpha \in \mathbb{R}$.

Proof: We introduce the function $g(z) := \frac{f(z)}{z}$. From the definition, $g \in \mathcal{A}(\overline{D}_0(1))$, $\|g\|_{\overline{D}_0(1)} = M$ and, by the maximum principle,

$$|g(z)| \leq \|g\|_{\overline{D}_0(1)} = M.$$

Estimation (3) follows immediately from here. On the other hand, if

$$|g(z_0)| = M$$

for some $z_0 \in D_0(1)$, then necessarily $g \equiv Const = Me^{i\alpha}$ for some α, so that

$$f(z) = zMe^{i\alpha}.$$

Q.E.D.

Theorem 8.6. *$\mathcal{D}$-a domain in $\mathbb{C}$, $\{f_n\} \in \mathcal{A}(\mathcal{D})$ and suppose that $\{f_n\}$ converges to a function f uniformly on compact subsets of $\mathcal{D}$. Then $f \in \mathcal{A}(\mathcal{D})$.*

Proof: Let K be a compact subset of $\mathcal{D}$. By Theorem 2.7, $f \in C(K)$. Since K is arbitrary, it follows that $f \in C\mathcal{D}$.

Take now γ an arbitrary loop in $\mathcal{D}$. Cauchy's theorem yields

$$\int_\gamma f_n(z)dz = 0, \; n = 1, 2, \cdots.$$

On the other hand, by the uniform convergence on γ,

$$\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz.$$

Hence,

$$\int_\gamma f(z)dz = 0$$

along every closed curve in $\mathcal{D}$. Thus, by Morera's theorem, $f \in \mathcal{A}(\mathcal{D}.)$
Remark: Using the mean-value theorem for harmonic functions and proceeding along the same way of considerations, one can prove *the maximum principle for harmonic functions.* Even more, in case of harmonic function one can show *the minimum principle.* The proof is left to the reader.
Exercises:

1. Let $f \in \mathcal{A}(\mathcal{D})\bigcap C(\overline{\mathcal{D}})$, and suppose that $f(z) \neq 0, z \in \overline{\mathcal{D}}$. Prove that $\min_{z\in\overline{\mathcal{D}}}|f|(z)$ is attained at boundary point $\mathcal{D}$, unless f is a constant.♠

2. Using the maximum principle for analytic functions show that each polynomial which is not a constant, has at least one zero in $\mathbb{C}$. ♣
3. Let $f \in \mathcal{A}(D_0(1)$, and suppose that $|f(z)| \leq 1/(1-|z|)$. Prove the inequality

$$|f^{(n)}(0)| \leq \frac{n!}{r^n(1-r)}, \ 0, r > 1.$$

♠

4. Let $f \in \mathcal{A}(D_0(r))$ be bounded from above by M when $|z| \leq r$. Prove that

$$|f^{(n)}(z)| \leq \frac{Mn!}{(r-|z|)^n}, \ |z| < r.$$

♠

5. Let $f \in \mathcal{E}$ and Re f be bounded in $\mathbb{C}$. Show that $f \equiv Const.$♠
Hint. Consider the function $e^{f(z)}$.

6. Let $f \in \mathcal{A}(\mathcal{D})\bigcap C(\overline{\mathcal{D}})$ and suppose that $|f(z)| \equiv Const$, $z \in \partial\mathcal{D}$. Show that there exists at least one inner point z_0 such that $f(z_0) = 0$.♠

7. Let $f \in \mathcal{E}$ and suppose that Re $f(z)$ is bounded in $\mathbb{C}$. Prove that $f \equiv Const.$
Hint: Show that $e^{f(z)} \in \mathcal{E}$; then apply Loiuville's theorem.♠