1. COMPLEX NUMBERS

Notations: **N**− the set of the natural numbers, **Z**− the set of the integers, **R**− the set of real numbers, **Q** := the set of the rational numbers.

Given a quadratic equation

$$ax^2 + bx + c = 0,$$

we know that it is not always solvable; for example, the simple equation

$$x^2 = -1 \tag{1}$$

cannot be satisfied for any real number. But we can expand our number system **R** by appending a symbol for a solution of (1); customary the symbol used is i, e.g.

$$i^2 = -1. \tag{2}$$

Definition: A complex number z is an expression of the form $z := a + ib$, where a, $b \in \mathbf{R}$. Two complex numbers $a+ib$ and $c+id$ are equal $(a+ib = c+id)$ if and only if $a = c$, $b = d$. ℵ

1.1. The algebra of the complex numbers

Set **C** for the set of complex numbers. Let $z_j = a_j + ib_j$. Following (2), we define
Addition by:

$$z_1 + z_2 := (a_1 + a_2) + i(b_1 + b_2);$$

Multiplication by;

$$z_1 z_2 := (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1). \tag{3}$$

The Division of the complex numbers $\frac{z_1}{z_2}$, $z_2 \neq 0$ is given by

$$\frac{a_1 + ib_1}{a_2 + ib_2} := \frac{a_1 + ib_1}{a_2 + ib_2}\frac{a_2 - ib_2}{a_2 - ib_2} = \frac{a_1 a_2 + b_1 b_2 + i(a_2 b_1 - a_1 b_2)}{a_2^2 + b_2^2}.$$

We easily prove that addition and multiplication are *commutative* and *distributive,* as well as that the *Distributive Law* takes place, that is:

$$(z_1 + z_2)z_3 = z_1z_3 + z_2z_3.$$

Definition: The **real part** $\Re z$ of the complex number $z = a + ib$ is the (real) number a, its **imaginary part** $\Im z$ is the (real) number b. If a is zero, the number is said to be a **pure imaginary number**. ℵ.

1.2. Point representation of complex numbers, absolute value and complex conjugate.

A convenient way to represent complex numbers as points in the $xy-$plane is suggested by the *Cartesian coordinate system;* namely, to each complex number $z = a + ib$ we associate that point in the $xy-$plane which has the coordinates (a, b) (the projection of the $0x-$ axis is a, and the one on the $Oy-$ axis is b. Obviously, the correspondence between the set of the complex numbers and the set of ordered pairs (x, y) is one-to -one.

When the $xy-$plane is used to describe complex numbers it is referred to as *complex plane or* **C** *plane.* The $Ox-$ axis is called the *real axis*, whereas the $oy-$ axis - the *imaginary axis.*

Definition: **The absolute value** or **the modulus** of the number $z = a+ib$ is denoted by $|z|$ and is given by

$$|z| := \sqrt{a^2 + b^2} \tag{4}$$

ℵ.

Remark: $|z|$ is always nonnegative; $|z| = 0$ iff $\Re z = \Im z = 0$. The *distance* between $z_i = a_i + ib_i$, $i = 1, 2$ is given by

$$|z_1 - z_2| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}.$$

Definition: The complex conjugate of the number $z = a + ib$ is denoted by $\bar{z}$ and is given by

$$\bar{z} := a - ib. \tag{5}$$

ℵ.

As we see, the complex conjugate of z is its reflection with respect to the real axis.

One easily can show that

$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$,

$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$,

$\overline{\frac{z_1}{z_2}} = \frac{\bar{z}_1}{\bar{z}_2}$,

$|z| = |\bar{z}|$, $|z|^2 = z\bar{z}$,

$\Re z = \frac{z+\bar{z}}{2}$, $\Im z = \frac{z-\bar{z}}{2}$.

1.3. Vectors and polar forms.

Definition: The Vector determined by the point z (the vector from the origin to the point z) in the complex plane $\mathbf{C}$ will be called **the vector z**. ℵ.

Addition: Let $z_i, i = 1, 2$ be two vectors, $z_i = 8a_i, b_i)$. Hereafter,the sum of z_1 and z_2 is presented by the vector sum of both vectors, e.g., by the *parallelogram law;* $z_1 + z_2 = (a_1 + a_2,\, b_1 + b_2)$.

Substraction: $z_1 - z_2 = (a_1 - a_2,\, b_1 - b_2)$.

Theorem1.1 **The Triangle Inequality.** *For any two complex numbers z_1 and z_2, the inequalities*

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2| \tag{6}$$

are valid.

There is another set of parameters that characterize the vector from the origin to the point z. This is the set of polar coordinates $r-$ the modulus, and $\Theta-$ the argument of z. The coordinate r is the distance from the origin to the point $z; r := |z|$. *Theta* is an angle of inclination of the vector z measured positively in a counterclockwise sense from the positive real axis (and thus measured negative when clockwise). Let x, y be the Cartesian (rectangular) coordinates of z; then

$$x = r\cos\Theta \tag{7}$$

and

$$y = r\sin\Theta \tag{8}$$

(recall that $r = |z|$.) Further,

$$\cos\Theta = \frac{x}{r},\ \sin\Theta = \frac{y}{r}. \tag{9}$$

Remark: *Although it is certainly true that* $\tan\Theta = x/y$, *the natural conclusion*

$$\Theta = \arctan(y/x)$$

is not true in the second and the third quadrants.

On the "uniqueness" of Θ_0. Let Θ satisfy (9). Then so does each

$$\Theta_0 + 2k\pi,\ k \in \mathbf{Z}.$$

We shall call the value of any of these angles an *argument* and denote it by $\arg z$. That value of the argument which belongs to the interval $(-\pi, \pi]$ will be called *the Principal Part of the argument* and denoted by $\operatorname{Arg} z$.

For instance,

$$\begin{array}{lll} \arg 1 = 0, & 2\pi, & -2\pi, \cdots, \\ \arg i = \frac{\pi}{2}, & \frac{5\pi}{2}, & \frac{-3\pi}{2}, \cdots \\ \arg(1-i) = \frac{-\pi}{4}, & \frac{7\pi}{4}, & \frac{-9\pi}{4}, \cdots \end{array}.$$

With these convention in hand, one can now write $z = x + iy$ in a *polar form*

$$z = x + iy = r(\cos\Theta + \sin\Theta). \tag{10}$$

Let now $z_i = r_i(\cos\Theta_i + i\sin\Theta_i), i = 1, 2$. Applying (3), we get

$$z_1 z_2 = r_1 r_2(\cos(\Theta_1 + \Theta_2) + i\sin(\Theta_1 + \Theta_2)).$$

So we conclude that

$$|z_1 z_2| = |z_1||z_2| \tag{11}$$

and

$$\arg\,(z_1 z_2) = \arg\, z_1 + \arg\, z_2. \tag{12}$$

Also,

$$\arg(1/z) = -\arg z \tag{13}$$

$$\arg\bar{z} = -\arg z. \tag{14}$$

1.4. The complex exponential - Euler's equation.

Definition: Euler's equation:

$$e^{iy} := \cos y + i \sin y.$$

This enables us to define the exponential function e^z, $z \in \mathbf{C}$:

Definition: if $z = x + iy$, then e^z is defined to be the complex number

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

EXERCISES:
1. Write in the form $a + ib$.
a): $-3(i/2)$; $2/i$; $(-1+i)^2$, $i^3(i+1)^2$.
b) Show that $\Re(iz) = -\Im(z)$ for every complex number z.
2.
a) Let $z = 3 - 2i$. Plot the points $z, -z, \bar{z}$ and $1/z$ in the complex plane.
b) Describe the set of points z in $\mathbf{C}$ that satisfy $\Im z = -2$, $|z-2| \le 1$, $\Re z > 2$, $|z| = \Re z - 2$.
c) Prove that $|\Re z| \le |z|$, $|\Im z| \le |z|$.
d) Let $a_i, i = 1, \cdots, n$ are real numbers. Show that if z_0 is a root of the polynomial $z^n + a_1 z^{n-1} + \cdots + a_n = 0$, then so is $\bar{z}_0$.

3.
a) Write down in a polar form $1, -1, i, -i, 1 \pm i, \frac{\pm 1 + i \pm \sqrt{3}}{2}, \frac{\pm 1 + i \pm 1}{2}$.
b) Is it true or not: $\text{Arg}\,(z_1 z_2) = \text{Arg}\, z_1 + \text{Arg}\, z_2$; $\text{Arg}\, z = -\text{Arg}\,(-z)$, $\arg z = \text{Arg}\, z + 2k\pi, k \in \mathbf{Z}$
4.
Prove Moivre's formula

$$(\cos \Theta + i \sin \Theta)^n = \cos n\Theta + i \sin n\Theta.$$

Show that $|e^{i\alpha}| = 1$, $e^{2k\pi i} = 1, k \in \mathbf{Z}$, $e^{\pi i/2} = i$, $e^{(2k+1)\pi/2} = -1$.

$$\sum_{k=0}^{n} z^k = \frac{z^{n+1} - 1}{z - 1},\ z \neq 1.$$