2. Topology in $\mathbf{C}$ and in $\overline{C}$ and Convergence Theory

Notations: We introduce following notations which will be actual along the course.

Given a complex point a and a positive number r, we set $D_a(R)$ for the open disk of radius r and centered at a; the boundary circle will be denoted by $C_a(r)$.

$$D_a(r) := \{z, |z-a| < r\}, C_a(r) := \partial D_a(r) = \{z, |z-a| = r\}.$$

In what follows we will call any disk $D_a(r)$ *a neighborhood* of a.

2.1. Topology

Topology in $\mathbf{C}$. Let M be a set in $\mathbf{C}$. We say that M is open, if any point $a \in M$ belongs to M together with some disk $D_a(r)$. Further, the set N is closed , if its complement $N^c := \mathbf{C} \setminus N$ with respect to $\mathbf{C}$ is open. The set K is compact, if it is closed and bounded. The the set $D \subset \mathbf{C}$ is a domain in $\mathbf{C}$, if it is open and connected.

Topology in $\overline{\mathbf{C}}$. In the same way, as in $\mathbf{C}$, we define open sets on the Riemann sphere $\mathcal{S}_f$. A set N on $\mathcal{S}_f$ is closed, if its complement with respect to $\mathcal{S}_f$ is open. Defining compact sets on the sphere as before, we remark that each closed set on $\mathcal{S}_f$ is necessarily a compact set in $\mathcal{S}_f$.

We recall well known identities:
Given the sets A and B, we have

$$A \bigcup B \equiv A^c \bigcap B^c, \, A \bigcap B \equiv A^c \bigcup B^c.$$

From here, we derive
a) Let M_i, $i = 1, 2, \cdots$ be open sets. Then $\bigcup_{i=1}^{\infty} M_i$ and $\bigcap_{i=1}^{k} M_i$ are open; $k-$ any integer.
b) Let M_i, $i = 1, 2, \cdots$ be closed sets. Then $\bigcap_{i=1}^{\infty} M_i$ and $\bigcup_{i=1}^{k} M_i$ are open; $k-$ any integer.

2.2. Convergence theory.

Definition: Given an infinite sequence of complex numbers $\{a_n\}$, we say that a is a concentration point of the sequence, if any neighborhood contains infinitely many numbers a_n, i.e., if for any $r > 0$, there is an infinite sequence $\Lambda \subset \mathbb{N}$ of integers such that $|a_n - a| < r$ for all $n \in \Lambda$. For instance, the sequence

$$a_n = \begin{cases} \frac{n}{n-1}, & n = 2k, \\ \frac{1}{n}, & n = 2k+1. \end{cases}$$

has two points of concentration: $a = 0, a = 1$.

Definition: The sequence $\{a_n\}$ point is said to **converge to** a **as** $n \to \infty$, if the point $a \in \mathbf{C}$ is the only concentration point. We write

$$\lim_{n\to\infty} a_n = a$$

or, equivalently,

$$a_n \to a, \text{ as } n \to \infty.$$

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For instance, the sequence

$$a_n := \frac{i^n}{2^n}$$

converges to zero.

Theorem 2.1, **(a necessary and sufficient condition for a convergence):**

$$a_n \to a,\, n \to \infty$$

iff for every $\varepsilon > 0$ *there exists a number* $n_0 \in \mathbb{N}$ *such that*

$$|a_n - a| < \varepsilon$$

every time when $n \geq n_0$.[1]

The convergence could be extended to the complex point of infinity (i.e. $z = \infty$), namely:

$$a_n \to \infty,\, n \to \infty$$

iff for every $R > 0$ the inequality

$$|a_n| > R$$

for all n sufficiently large. We say that a_n *diverges to infinity.*

Suppose that the sequence $\{a_n\}$ converges to $a \in \mathbf{C}$. We easily can prove

[1] or, as we use to say, *for all n sufficiently large.*

Theorem 2.2. *Suppose that*

$$a_n \to a,\ n \to \infty.$$

Then

$$\Re a_n \to \Re a,\ \Im a_n \to \Im a, n \to \infty$$

and

$$|a_n| \to |a|,\ n \to \infty.$$

Further, a_n diverges to infinity iff the sequence $1/a_n$ tends to zero.

We remark that the statement $\mathrm{Arg} a_n \to \mathrm{Arg} a$, $n \to \infty$ is, in general, not correct. Indeed, consider the sequence

$$a_n := \frac{(i)^n}{n},\ n = 1, 2, \cdots.$$

which tends to zero. At the same time, the sequence of the arguments has four concentration points $(-\pi/2, 0, \pi/2, \pi.)$ This expresses the circumstance that the numbers a_n can approach the limit a from from any direction in the plane.

The latter statement is true if $a \neq 0$.

2.3. Functions of a complex variable.

Recall that a function is a rule that assigns to each element in a set $A \subset \mathbf{C}$ one and only one element in the set $B \subset \mathbf{C}$. if f assigns the value of b to the value of a, we write

$$f(a) = b.$$

The set A is the domain of definition (even if A is not a domain in the sense of P.2.1, and the set of all images $f(a)$ is the range of f. We sometimes refer to f as a *mapping* of A into B. $\aleph$

If f is expressed by a formula such as

$$f(z) := \frac{z^2+1}{z^2-1},$$

then, unless stated otherwise, we take the domain of f to be the set of all z for which the formula is well defined (in this case $\mathbf{C} \setminus 1$. If we agree that $f(\infty) = 1$, then the domain of definition coincides with the extended complex plane $\overline{\mathbf{C}}$), and the range with $\overline{\mathbf{C}}$.

Let

$$w = f(z).$$

Just as z decomposes into real and imaginary part as $z = x + iy$, the real and imaginary part of w are real valued function of z, or, equivalently, of x and y, and so we customary write

$$f(z) = u(x, y) + iv(x, y).$$

Example: Let $f(z) := z^2 + 1$. Then

$$f(z) = x^2 - y^2 + 1 + 2ixy.$$

A fundamental concept in the function theory is the continuity. In what follows we will get acquainted with.

2.4 Continuous functions.

Definition: Convergence of f at the point $z = z_0$. Let f be defined in a neighborhood of $z = z_0$ with possible exception at $z = z_0$. We say that the **limit of $f(z)$ as z goes to z_0 is the number** w_0 and write

$$\lim_{z \to z_0} f(z) = w_0,$$

or equivalently,

$$f(z) \to w_0,\ z \to z_0,$$

of for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(z) - w_0| \leq \varepsilon \text{ whenever } |z - z_0| < \delta.$$

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Example:: Show that $\lim_{z \to i} f(z) = 0$, where

$$f(z) := \frac{z^2 + 1}{z + i}.$$

We note the obvious statement:

Theorem 2.3. *Let $f(z) = u(x, y) + iv(x, y)$ be defined in a neighborhood of $z_0 = (x_0, y_0)$. Then $f(z) \to w_0 = w_1 + iw_2$, $z \to z_0$, iff*

$$u(x, y) \to w_1,\ z \to z_0,$$

and

$$v(x,y) \to w_2,\ z \to z_0.$$

Definition: Continuity of a function f at the point $z = z_0$. Suppose that f is defined in a neighborhood of $z = z_0$. Then f is continuous at $z = z_0$, if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

A function is continuous in a set A (we write $f \in C(A)$), if it is continuous at every point of A. ℵ

Because of the analogy to real analysis, many of familiar theorems on real sequences, limits and continuity remain valid in the complex case. A theorem is stated here:

Theorem 2.4. *If the functions f and g are continuous at z_0, then so are $f(z) \pm g(z)$, and $f(z)g(z)$. If $g(z_0) \neq 0$, then so does the quotient $f(z)/g(z)$.*

Consider the definition on continuity. If $f \in C(A)$, then the number δ depends in general on the number z_0. Look for instance at the function $f(z) = z^2$. This fact can lead to essential difficulties. So, it is of interest for us when δ does not depend on z. This is the case of *uniform continuity.*

Definition: Suppose that f is well defined in the set E. We say that f is continuous on E, if for every $\varepsilon >$ there is a number δ such that $|f(z_1) - f(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta, z_1, z_2 \in E$. ℵ

The classical result of Weierstraß provides a sufficient condition for A uniform continuity of a function.

Theorem 2.5. (**Weierstraß:**) *Let K be a compact set in* $\mathbf{C}$ *and $f \in C(K)$. Then f is uniformly continuous on E.*

The proof proceeds along the same argumentation as in the real case.

Before continuing, we recall another classical result by Weierstraß.

Theorem 2.6. **(Weierstraß:)** *In the conditions of Theorem 2.5, there is a point $z_0 \in K$ such that*

$$\max_{z \in K} |f(z)| = |f(z_0)|.$$

In what follows we will write $\|f\|_K$ instead of $\max_{z\in K} |f(z)|$. The expression $\|f\|_K$ will be called *Chebyshev or* $\max -$ *norm of f on K.*.

2.5 Convergence of sequences of functions.

Definition: Let the functions $\{f_n\}$ be continuous in the set A. We say that the sequence $\{f_n\}$ converges uniformly to a function f in A, if

$$\|f_n\|_A \to ||f||_A \text{ as } n \to \infty. \quad (1)$$

The following important theorem is due (again) to Weierstraß.

Theorem 2.7 (**Weierstraß:**) *Let K be a compact set and $f_n(z) \in C(K.)$ Suppose that $\{f_n\}$ converges uniformly to a function f. Then $f \in C(K)$.*
Proof: Select an arbitrary positive number ε. If we find a number $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ every time when $|z - w| < \delta$ and $z, w \in K$, then we are done.

Indeed, in the conditions of the theorem,

$$\|f_n - f\|_K \leq \frac{\varepsilon}{3} \quad (2)$$

for all n great enough. Take such a number m. By Theorem 2.5 each function f_n is uniformly continuous on K, and so does f_m. Hence, there is a positive number δ such that

$$|f_m(z) - f_m(w)| < \frac{\varepsilon}{3} \text{ whenever } |z - w| < \delta. \quad (3)$$

Let now $|z - w| < \delta$. Applying successively (2) and (3), we get

$$|f(z) - f(w)| <$$

$$|f(z) - f_m(z)| + |f_m(z) - f_m(w)| + |f_m(w) - f(w)| \leq$$
$$\leq 2\|f_m - f\|_K + |f_m(z) - f_m(w)| < \varepsilon.$$

This completes the proof. **Q.E.D.**

Exercises:
1. Given the sets A, B, show that $A \bigcup B = (\mathbf{C} \setminus A) \bigcap (\mathbf{C} \setminus A)$.

2. Let $\{M_i\}_{i=1}^{\infty}$ be open sets in $\mathbf{C}$. Show that
a) $\bigcup_{i=1}^{\infty} M_i$ is open;
b) $\bigcap_{i=1}^{m} M_i$ is open for every $m \in \mathbb{N}$.

3. Let $\{N_i\}_{i=1}^{\infty}$ be closed sets in $\mathbf{C}$. Show that
a) $\bigcap_{i=1}^{\infty} M_i$ is closed ;
b) $\bigcup_{i=1}^{m} M_i$ is closed for every $m \in \mathbb{N}$.

4. Let K be a compact set in $\mathbf{C}$. Show that

$$L(f) := \|f\|_K, f \in C(K)$$

is a Norm, that is:
a) $L(f) \geq 0$ and $L(f) = 0$ iff $f \equiv 0$.
b) $L(\alpha f) = |\alpha| L(f)$ for every real number α.
c) $L(f + g) \leq L(f) + L(g)$.
5. Show that $f(z) := \bar{z}$ is continuous everywhere in $\mathbf{C}$.
6. Suppose that f is continuous at z_0. Show that the functions $|f(z)|$, $\text{Re } f(z)$, $\text{Im } f(z)$ do so.
7. Prove that $\lim z_n = 0$ iff $|z_n| \to 0$.
8. Prove that

$$z^n \to \begin{cases} 0, & \text{if } |z| < 1, \\ \infty, & \text{if } |z| > 1. \end{cases}$$

9. Show that the function Arg is continuous at each point on the nonpositive real axis.