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SOME SEQUENCE SPACES AND ALMOST CONVERGENCE

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ABSTRACT. The purpose of this paper is to introduce and investigate some new sequence spaces. Also some inclusion theorems have been established.

1. Introduction. Let s be the set of all real or complex sequences and let us denote by ℓ_∞ the Banach space of bounded sequence $x = (x_k)$ normed by $\|x\| = \sup|x_k|$. Let D be the shift operator on s, that is,

$$D((x_k)) = (x_{k+1}).$$

It may be recalled that (see, Banach [1]) Banach limit L is a non-negative linear functional on ℓ_∞ such that L is invariant under the shift operator (that is, $L(Dx) = L(x)$, $\forall x \in \ell_\infty$) and $L(e) = 1$ where $e = (1, 1, 1, \ldots)$. A sequence $x \in \ell_\infty$ is called almost convergent (Lorentz [4]) if all Banach limits of x coincide. Let $\hat{c}$ be the set of all almost convergent sequences. Lorentz proved that

$$\hat{c} = \left\{ x : \lim_m \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i} \quad \text{exists uniformly in} \quad n \right\}.$$

Several authors including Duran [2], King [3] and Schaefer [8] have studied almost convergent sequences.

It is natural to expect that almost convergence must be related to some concept $\widehat{\mathrm{BV}}$ in the same vein as convergence is related to the concept of BV. BV denotes the set of all sequences of bounded variation and a sequence in $\widehat{\mathrm{BV}}$ will mean a sequence of almost bounded variation. $\widehat{\mathrm{BV}}$ was introduced and discussed in [6]. Also $\widehat{\widehat{\mathrm{BV}}}$ naturally comes up for investigation and is considered along with $\widehat{\mathrm{BV}}$. To define $\widehat{\mathrm{BV}}$, let

$$d_{mn} = d_{mn}(x) = \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i} \tag{1}$$

with $D^0 = 1$. It is evident that

$$d_{0n}(x) = x_n = D^0 x_n \tag{2}$$

Now define

$$d_{-1,n}(x) = D^{-1}x_n = x_{n-1} \tag{3}$$

and then write for $m, n \geq 0$,

$$t_{mn}(x) = d_{mn}(x) - d_{m-1,n}(x). \tag{4}$$

So that by (2), (3) and (4) we have

$$t_{0,n}(x) = D^0 x_n - D^{-1}x_n = x_n - x_{n-1}.$$

When $m \geq 1$ a straightforward calculation shows that

$$t_{mn}(x) = \frac{1}{m(m+1)}\sum_{i=0}^{m}(x_{n+i} - x_{n+i-1}).$$

We define (see, [6])

$$\widehat{\mathrm{BV}} = \left\{x : \sum_m |t_{mn}(x) \text{ converges uniformly in } n\right\}$$

$$\widehat{\widehat{\mathrm{BV}}} = \left\{x : \sup_n \sum_m |t_{mn}(x)| < \infty\right\}.$$

Note that BV= $\left\{x : \sum_k |x_k - x_{k-1}| < \infty\right\}$, where we define $x_{-1} = 0$. BV is Banach space normed by

$$\|x\| = \sum_k |x_k - x_{k-1}| < \infty.$$

Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_k a_{nk}x_k$ converges for each n. Let X and Y be any two nonempty subsets of s. If $x = (x_k) \in X$ implies that $Ax = (A_n(x)) \in Y$, we say that A define a matrix transformation from X into Y and we denote it by $A : X \to Y$. By (X, Y) we mean the class of matrices A such that $A : X \to Y$.

In this paper we introduce and study, using the idea of infinite matrices, some new sequence spaces which generalize the spaces $\widehat{\mathrm{BV}}$ and $\widehat{\widehat{\mathrm{BV}}}$ studied by Nanda and Nayak. We establish some inclusion relations. Our results include the corresponding results of Nanda and Nayak.

The spaces $|\widehat{B},p|$ and $|\widehat{\widehat{B}},p|$. Let $B = (b_{nk})$ be an infinite matrix of real or complex numbers and let (p_m) be a sequence of real numbers such that $p_m > 0$, $\sup p_m < \infty$. We define

$$|B,p| = \left\{x : \sum_n |B_n(x) - B_{n-1}(x)|^{p_n} < \infty\right\}$$

$$(B,p)_\infty = \left\{X : \sup_n |B_n(x) - B_{n-1}(x)|^{p_n} < \infty\right\}$$

$$\left|\widehat{B},p\right| = \left\{x : \sum_m |t_{mn}(Bx)|^{p_m} \text{ converges uniformly in } n\right\}$$

$$\left|\widehat{\widehat{B}},p\right| = \left\{x : \sup_n \sum_m |t_{mn}(Bx)|^{p_m} < \infty\right\},$$

where

$$t_{mn}(Bx) = \sum_k b(n,k,m)x_k$$

is such that

$$b(n,k,m) = \frac{1}{m(m+1)} \sum_{i=0}^{m} i(b_{n+i,k} - b_{n+i-1,k}) \quad m \geq 1,$$

$$b(n,k,0) = b_{nk} - b_{n-1,k}.$$

If $p_n = p$ for all n, then we write $|B|_p$ for $|B,p|$. Similarly if $p_m = p$ for all m, we write $|\widehat{B}|_p$ and $|\widehat{\widehat{B}}|_p$ for $|\widehat{B},p|$ and $|\widehat{\widehat{B}},p|$ respectively. If $p_m = p$ for all m, we write $(B)_\infty$ for $(B,p)_\infty$.

We now have

Theorem 1. $|\widehat{B},p| \subset |\widehat{\widehat{B}},p|$.

Proof. Let $x \in |\widehat{B},p|$. Then, there is an integer M such that

$$\sum_{m \geq M} \left|\sum_k b(n,k,m)x_k\right|^{p_m} \leq 1. \tag{5}$$

Hence it is enough to show that for fixed m, $\sum_k b(n,k,m)x_k$ is bounded. It follows from (5) that

$$\left|\sum_k b(n,k,m)x_k\right| \leq 1 \text{ for } m \geq M \text{ and all } n.$$

If $m \geq 1$, then

$$(m+1)\sum_k b(n,k,m)x_k - (m-1)\sum_k b(n,k,m-1)x_k$$

$$= \sum_k b_{n+m,k}x_k - \sum_k b_{n+m-1,k}x_k.$$

Hence for any fixed $m \geq M+1$, $\sum_k b_{n+m,k}x_k - \sum_k b_{n+m-1,k}x_k$ is bounded. Therefore $\sum_k b(n,k,m)x_k$ is bounded for all m, n and this completes the proof. □

Theorem 2. *$|B,p|$ is a linear topological space paranormed by*

$$f(x) = \left(\sum_n \left|\sum_k (b_{nk} - b_{n-1,k})x_k\right|^{p_n}\right)^{1/M},$$

where $M = \max(1, \sup p_n)$.$|\widehat{B},p|$ is paranormed by

$$g(x) = \sup_n \left(\sum_m \left|\sum_k b(n,k,m)x_k\right|^{p_m}.\right)^{1/M} \tag{6}$$

$|\widehat{\widehat{B}},p|$ is paranormed by (6) if $\inf p_m > 0$. Also if $\inf p_n > 0$, then $(B,p)_\infty$ is paranormed by

$$h(x) = \sup_n \left|\sum_k (b_{nk} - b_{n-1,k})x_k\right|^{p_n/M}$$

Proof. Because of Theorem 1, (6) is meaningful for $x \in |\widehat{B},p|$. The proof is a routine verification by using standard techniques and therefore we omit it.

Theorem 3. *If $p \geq 1$, then $|B|_p \subset |\widehat{B}|_p$.*

In order to prove Theorem 3 we require the following Lemma.

Lemma 1. *Suppose that*

(i) $\sum_m |a_{mn}|$ *converges for each* n,

(ii) $\sum_m |a_{mn}| \to 0$ *as* $n \to \infty$.

Then $\sum_m |a_{mn}|$ *converges uniformly in* n.

see, for example, Maddox [5], p. 168.

Proof of Theorem 3. Let $x \in |B|_p$ and $m \geq 1$. We have by Hölder's inequality when $p > 1$ and trivially when $p = 1$

$$\left|\frac{1}{m(m+1)}\sum_{i=1}^{m}\sum_k i(b_{n+i,k} - b_{n+i-1,k})x_k\right|^p \leq$$

$$\leq \frac{1}{m(m+1)^p}\sum_{i=1}^{m} i^p \left|\sum_k (b_{n+i,k} - b_{n+i-1,k})x_k\right|^p .$$

Hence,

$$\sum_{i=1}^{\infty}\left|\sum_k b(n,k,m)x_k\right|^p$$

$$\leq \sum_{i=1}^{\infty} i^p \left|\sum_k (b_{n+i,k} - b_{n+i-1,k})x_k\right|^p \sum_{m=i}^{\infty}\frac{1}{m(m+1)^p}$$

$$\leq \sum_{i=1}^{\infty}\left|\sum_k (b_{n+i,k} - b_{n+i-1,k})x_k\right|^p .$$

Then since $b(n,k,0) = b_{nk} - b_{n-1,k}$, we have

$$\sum_m \left|\sum_k b(n,k,m)x_k\right|^p \leq \sum_{i=n+1}^{\infty}\left|\sum_k (b_{ik} - b_{i-1,k})x_k\right|^p .$$

Hence the hypotheses of Lemma 1 are satisfied for

$$b_{mn} = \left|\sum_k b(n,k,m)x_k\right|^p .$$

This completes the proof. □

Theorem 4. $|\widehat{\widehat{B}}|_p \subset (B)_\infty$.

Proof. Note that

$$\sup_{m,n}\left|\sum_k b(n,k,m)x_k\right| \le \left(\sup_n \sum_m \left|\sum_k b(n,k,m)x_k\right|^p\right)^{1/p} \tag{7}$$

$$\sup_{m,n}\left|\sum_k b(n,k,m)x_k\right| \ge \sup_n \left|\sum_k b(n,k,0)x_k\right| = \sup_n \left|\sum_k (b_{nk} - b_{n-1,k})x_k\right| \tag{8}$$

$$\sup_{m,n}\left|\sum_k b(n,k,m)x_k\right| = \sup_{n,m}\left|\frac{1}{m(m+1)}\sum_{i=0}^{\infty} i \sum_k (b_{n+i,k} - b_{n+i-1,k})x_k\right|$$

$$\le \sup_{n,i}\left|\sum_k (b_{n+1,k} - b_{n+i-1,k})x_k\right| \sup_m \left(\frac{1}{m(m+1)}\sum_{i=1}^{m} i\right) \tag{9}$$

$$= \frac{1}{2}\sup\left|\sum_k (b_{nk} - b_{n-1,k})x_k\right|$$

Now the result follows from (7), (8) and (9). This completes the proof. □

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