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MEASURABLE SETS OF PARABOLAS IN THE GALILEAN PLANE

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ABSTRACT. The group H_5 of the general similitudes in the Galilean plane Γ_2 is considered. The measurable sets of parabolas in Γ_2 and the corresponding invariant densities with respect to H_5 and its subgroups are obtained.

1. Introduction. Let $E_n(x)$ be an n-dimensional space and $G_r(a)$ an r-parametric Lie group. Assume that $G_r(a)$ is defined by the equations

$$x'_i = f_i(x_1,\dots,x_n;a_1,\dots,a_r),\quad i=1,\dots,r,$$

where $a_1,\dots,a_r$ are independent parameters and the identity is determined by $a_1 = 0,\dots,a_n = 0$. On the other hand, the group $G_r(a)$ can be defined by the infinitesimal operators

$$X_k = \sum_{i=1}^{n} \xi^i_k(x_1,\dots,x_n)\frac{\partial}{\partial x_i},\quad k=1,\dots,r,$$

where

$$\xi^i_k(x_i,\dots,x_n) = \left(\frac{\partial x'_i}{\partial a_k}\right)_{a_1=0,\dots,a_r=0}.$$

A function $f(x_1,\dots,x_n)$ is called an integral invariant function of the group $G_r(a)$ if

$$\int_{X(x)} f(x_1,\dots,x_n)dx_1\dots dx_n = \int_{X(x')} f(x'_1,\dots,x'_n)dx'_1\dots dx'_n$$

for every domain X on which the integral can be defined [8; 28].

In [2; 28] (see also [6; 11] and [8; 29]) R. Deltheil has shown that any integral invariant function satisfies the system with total differentials

$$X_k(f) + \sigma_k f = 0,\quad k=1,\dots,r,$$

where

$$\sigma_k = \sum_{i=1}^{n} \frac{\partial \xi_i^k}{\partial x_i}.$$

The differential form

$$dx = |f(x_1, \ldots, x_n)| dx_1 \wedge \ldots \wedge dx_n$$

is called an invariant density under the group $G_r(a)$ of the elements $x(x_1, \ldots, x_n)$. If $G_r(a)$ acts transitively, then the invariant density, if it exists, is unique up to a constant factor.

Now let M_q be a q-parametric set of p-dimensional geometrical elements represented by the system

$$\varphi_j(x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_q) = 0, \quad j = 1, \ldots, n-p,$$

where $\alpha_1, \ldots, \alpha_q$ are independent parameters. If $G_r(a)$ leaves M_q invariant, then it generates the so-called associated group $H_r(\alpha)$ of $G_r(a)$ [6; 34] in the set $E_q(\alpha)$ of parameters. The group $H_r(\alpha)$ can be determined by a system of the form

$$\alpha'_j = \psi_j(\alpha_1, \ldots, \alpha_q; a_1, \ldots, a_r), \quad j = 1, \ldots, q,$$

and it is isomorphic to the group $G_r(a)$ [6; 33]. Then the invariant under $G_r(a)$ density of the elements of M_q, if it exists, coincides with the invariant under $H_r(\alpha)$ density of the points of $E_q(\alpha)$.

The densities are always considered in their absolute values; thus the sign is of no importance.

2. Preliminaries. In the affine version, the Galilean plane Γ_2 is an affine plane with a special direction which may be taken coincident with the y-axis of the basic affine coordinate system Oxy [3], [7], [9]. The affine transformations leaving the special direction Oy invariant can be written in the form

$$\begin{aligned} x' &= a_1 + a_2 x, \\ y' &= a_3 + a_4 x + a_5 y, \end{aligned} \tag{1}$$

where $a_1, \ldots, a_5 \in \mathbf{R}$ and $a_2 a_5 \neq 0$.

It is easy to verify that the transformations (1) map a line segment and an angle of Γ_2 into a proportional line segment and a proportional angle with the coefficients of proportionality a_2 and $a_2^{-1} a_5$, respectively. Thus they form the group H_5 of the general similitudes of Γ_2. The infinitesimal operators of H_5 are

$$X_1 = \frac{\partial}{\partial x}, \; X_2 = x\frac{\partial}{\partial x}, \; X_3 = \frac{\partial}{\partial y}, \; X_4 = x\frac{\partial}{\partial y}, \; X_5 = y\frac{\partial}{\partial y}.$$

In [1] we announced the following results:

I. *The four-parametric subgroups of* H_5 *can be reduced to one of the following subgroups:*

$$H_4^1 = (X_1, X_2, X_3, X_4), \quad H_4^2 = (X_1, X_2, X_3, X_5),$$
$$H_4^3 = (X_2, X_3, X_4, X_5), \quad H_4^4 = (X_1, X_3, X_4, \alpha X_2 + X_5).$$

II. *The three-parametric subgroups of* H_5 *can be reduced to one of the following subgroups:*

$$H_3^1 = (X_1, X_2, X_3), \quad H_3^2 = (X_1, X_2, X_5), \quad H_3^3 = (X_1, X_3, X_4),$$
$$H_3^4 = (X_2, X_3, X_4), \quad H_3^5 = (X_2, X_3, X_5), \quad H_3^6 = (X_2, X_4, X_5),$$
$$H_3^7 = (X_1, X_3, \alpha X_2 + \beta X_4 + X_5), \quad H_3^8 = (X_3, X_4, \alpha X_1 + X_5),$$
$$H_3^9 = (X_3, X_4, \alpha X_2 + X_5 | \alpha \neq 0), \quad H_3^{10} = (\alpha X_1 + X_4, X_2 + 2X_5, X_3 | \alpha \neq 0).$$

III. *The two-parametric subgroups of* H_5 *can be reduced to one of the following subgroups:*

$$H_2^1 = (X_1, X_2), \quad H_2^2 = (X_2, X_3), \quad H_2^3 = (X_2, X_4), \quad H_2^4 = (X_2, X_5),$$
$$H_2^5 = (X_1, \alpha X_2 + X_3), \quad H_2^6 = (X_1, \alpha X_2 + X_5), \quad H_2^7 = (X_3, \alpha X_1 + X_4 | \alpha \neq 0),$$
$$H_2^8 = (X_3, \alpha X_1 + X_5), \quad H_2^9 = (X_3, \alpha X_2 + \beta X_4 + X_5 | \alpha \neq 0),$$
$$H_2^{10} = (X_4, \alpha X_2 + X_3), \quad H_2^{11} = (X_4, \alpha X_2 + X_5),$$
$$H_2^{12} = (X_2 + 2X_5, \alpha X_1 + X_4 | \alpha \neq 0).$$

IV. *The one-parametric subgroups of* H_5 *can be reduced to one of the following subgroups:*

$$H_1^1 = (X_1), \quad H_1^2 = (X_2), \quad H_1^3 = (X_3), \quad H_1^4 = (X_4), \quad H_1^5 = (X_5),$$
$$H_1^6 = (\alpha X_1 + X_4 | \alpha \neq 0), \quad H_1^7 = (\alpha X_1 + X_5 | \alpha \neq 0), \quad H_1^8 = (\alpha X_2 + X_3 | \alpha \neq 0),$$
$$H_1^9 = (\alpha X_2 + X_5 | \alpha \neq 0), \quad H_1^{10} = (\alpha X_2 + \beta X_4 + X_5 | \alpha\beta \neq 0).$$

Here and everywhere in the text α and β are real constants.

Our purpose is to find the measurable sets of parabolas with respect to H_5, its subgroups and the corresponding densities.

3. Measurability under H_5 of a set of parabolas. We can assume, without loss of generality, that a parabola π in Γ_2 is determined by an equation of the form

$$(Bx + y)^2 + 2Dx + 2Ey + F = 0, \tag{2}$$

where B, D, E, F are real numbers and $D - BE \neq 0$. (See [4]). Then under the action of (1) the parabola $\pi(B,D,E,F)$ is transformed into the parabola $\pi'(B',D',E',F')$ as

$$
\begin{aligned}
&B' = a_2^{-1}(Ba_5 - a_4),\\
&D' = a_2^{-2}\left[-(Ba_5 - a_4)^2 a_1 - (Ba_5 - a_4)a_2 a_3 + (Da_5 - Ea_4)a_2 a_5\right],\\
(3)\qquad &E' = a_2^{-1}\left[-(Ba_5 - a_4)a_1 + (Ea_5 - a_3)a_2\right],\\
&F' = a_2^{-2}\left\{[(Ba_5 - a_4)a_1 + a_2 a_3]^2 + 2E(a_1 a_4 - a_2 a_3)a_2 a_5\right.\\
&\qquad\left. +(-2Da_1 + Fa_2)a_2 a_5^2\right\}.
\end{aligned}
$$

The transformations (3) form the associated group $\bar{H}_5$ of H_5. The infinitesimal operators of $\bar{H}_5$ are

$$
\begin{aligned}
&Y_1 = -B^2\frac{\partial}{\partial D} - B\frac{\partial}{\partial E} - 2D\frac{\partial}{\partial F}, &&Y_2 = -B\frac{\partial}{\partial B} - D\frac{\partial}{\partial D},\\
&Y_3 = -B\frac{\partial}{\partial D} - \frac{\partial}{\partial E} - 2E\frac{\partial}{\partial F}, &&Y_4 = -\frac{\partial}{\partial B} - E\frac{\partial}{\partial D},\\
&Y_5 = B\frac{\partial}{\partial B} + 2D\frac{\partial}{\partial D} + E\frac{\partial}{\partial E} + 2F\frac{\partial}{\partial F}.
\end{aligned}
$$

The system of Deltheil

$$
\begin{aligned}
&Y_1(f) = 0, \quad Y_2(f) - 2f = 0, \quad Y_3(f) = 0,\\
&Y_4(f) = 0, \quad Y_5(f) + 6f = 0
\end{aligned}
$$

has the unique solution $f(B,D,E,F) = 0$ and therefore we can state:

Theorem 1. *A set of parabolas in Γ_2 is not measurable with respect to group H_5 of the general similitudes.*

4. Invariant densities of the parabolas in Γ_2 under the four-parametric subgroups of H_5. Consider the subgroup $H_4^1 = (X_1, X_2, X_3, X_4)$ of H_5. The corresponding associated group $\bar{H}_4^1 = (Y_1, Y_2, Y_3, Y_4)$ acts simply transitively on the set of parabolas (2) and therefore it is measurable. The integral invariant function $f = f(B,D,E,F)$ satisfies the system of R. Deltheil

$$Y_1(f) = 0, \quad Y_2(f) - 2f = 0, \quad Y_3(f) = 0, \quad Y_4(f) = 0$$

and is of the form $f = (D - BE)^{-2}$. From here it follows that the invariant under H_4^1 density of parabolas (2) is

$$d\pi = (D - BE)^{-2} dB \wedge dD \wedge dE \wedge dF. \tag{4}$$

Using similar arguments as above, we obtain that the invariant densities of the parabolas (2) under H_4^2 and H_4^3 are

$$d\pi = B^2(D - BE)^{-4} dB \wedge dD \wedge dE \wedge dF \tag{5}$$

and

$$d\pi = (D - BE)^{-4}|F - E^2|^{-1} dB \wedge dD \wedge dE \wedge dF, \tag{6}$$

respectively.

Now let us examine the group $\bar{H}_4^4 = (Y_1, Y_3, Y_4, \alpha Y_2 + Y_5)$. If $\alpha = 2$, we find

$$2Y_2 + Y_5 = -\frac{F - E^2}{D - BE} Y_1 + \frac{BF - DE}{D - BE} Y_3 + BY_4$$

and therefore the infinitesimal operators Y_1, Y_3, Y_4 and $2Y_2 + Y_5$ are arcwise connected. Then the associated group $\bar{H}_4^{4\prime} = (Y_1, Y_3, Y_4, 2Y_2 + Y_5)$ is intransitive and consequently the set (2) of parabolas is not measurable under the group $H_4^{4\prime} = (X_1, X_3, X_4, 2X_2 + X_5)$.

If $\alpha = 3$, the parabolas (2) have the invariant density

$$d\pi = dB \wedge dD \wedge dE \wedge dF. \tag{7}$$

For $\alpha \neq 2, 3$ we obtain that the set (2) of parabolas is measurable under H_4^4 and the invariant density is of the form

$$d\pi = (D - BE)^{\frac{2(\alpha-3)}{2-\alpha}} dB \wedge dD \wedge dE \wedge dF. \tag{8}$$

We are now in a position to state the result:

Theorem 2. *The invariant densities of the parabolas* (2) *with respect to the four-parametric subgroups of* H_5 *are* (4) - (8).

5. Measurable subsets of parabolas under the three-parametric subgroups of H_5. The following results are motivated by M. I. Stoka's paper [5]. The associated group $\bar{H}_3^1 = (Y_1, Y_2, Y_3)$, corresponding to the group $H_3^1 = (X_1, X_2, X_3)$, acts intransitively on the set of parabolas (2) and therefore the parabolas have not an invariant under H_3^1 density. The system $Y_1(f) = 0$, $Y_2(f) = 0$, $Y_3(f) = 0$ has an independent integral

$$f = B(D - BE)^{-1}$$

and it is an absolute invariant of $\bar{H}_3^1$. Consider the subset of parabolas satisfying the condition

$$D - BE = hB, \tag{9}$$

where $hB \neq 0$, $h = \text{const}$. The group $\bar{H}_3^1$ induces on the invariant variety (9) the group $\tilde{H}_3^1$ with the infinitesimal operators

$$Z_1 = -B\frac{\partial}{\partial E} - 2B(h+E)\frac{\partial}{\partial F},$$

$$Z_2 = -B\frac{\partial}{\partial B}, \quad Z_3 = -\frac{\partial}{\partial E} - 2E\frac{\partial}{\partial F}.$$

$\tilde{H}_3^1$ is a simply transitive group and therefore it is measurable. The integral invariant function $f = f(B, E, F)$ satisfying the system of R. Deltheil

$$Z_1(f) = 0, \quad Z_2(f) - f = 0, \quad Z_3(f) = 0$$

is of the form $f = B^{-1}$. Consequently the subset of parabolas

$$(10) \qquad \begin{gathered}(Bx+y)^2 + 2B(h+E)x + 2Ey + F = 0,\\ (hB \neq 0;\ h = \text{const})\end{gathered}$$

is measurable and the invariant under H_3^1 density is $d\pi = |B|^{-1} dB \wedge dE \wedge dF$.

Further we shall omit details of the proofs when they are similar to those of analogous results given above. The group $\bar{H}_3^2 = (Y_1, Y_2, Y_5)$, corresponding to the group $H_3^2 = (X_1, X_2, X_5)$, is intransitive on the set of parabolas (2) and has the invariant variety $D^2 - B^2F = h(D - BE)^2$, where $h = \text{const}$. Then the subset of parabolas

$$(11) \qquad \begin{gathered}(Bx+y)^2 + 2Dx + 2Ey + B^{-2}\left[D^2 - h(D-BE)^2\right] = 0,\\ (B \neq 0;\ h = \text{const})\end{gathered}$$

is measurable under H_3^2 and has the invariant density $d\pi = (D - BE)^{-2} dB \wedge dD \wedge dE$.

The group $\bar{H}_3^3 = (Y_1, Y_3, Y_4)$ has the invariant variety $D - BE = h$, where $h = \text{const} \neq 0$. The subset

$$(12) \qquad \begin{gathered}(Bx+y)^2 + 2(h+BE)x + 2Ey + F = 0,\\ (h \neq 0;\ h = \text{const})\end{gathered}$$

is measurable under H_3^3 and the invariant density is $d\pi = dB \wedge dE \wedge dF$.

Remark. H_3^3 coincides with the group H_3 of motions in Γ_2.

Now let us consider the group $\bar{H}_3^4 = (Y_2, Y_3, Y_4)$. The measurable subset is determined by the equation

$$(13) \qquad (Bx+y)^2 + 2Dx + 2Ey + E^2 + h = 0, \quad (h = \text{const})$$

and has the invariant density $d\pi = (D - BE)^{-2} dB \wedge dD \wedge dE$.

Examining the group $\bar{H}_3^5 = (Y_2, Y_3, Y_5)$, we get the measurable subset

$$(Bx+y)^2 + 2Dx + 2Ey + hB^{-2}(D - BE)^2 + E^2 = 0, \quad (B \neq 0;\ h = \text{const}) \tag{14}$$

and the invariant density $d\pi = (D - BE)^{-2} dB \wedge dD \wedge dE$.

For the group $\bar{H}_3^6 = (Y_2, Y_4, Y_5)$ we obtain the measurable subset

$$(Bx+y)^2 + 2Dx + 2Ey + E^2 + h = 0, \ (h = \text{const}) \tag{15}$$

and the invariant density is of the form $d\pi = (D - BE)^{-2} dB \wedge dD \wedge dE$.

Consider the group $\bar{H}_3^7 = (Y_1, Y_3, \alpha Y_2 + \beta Y_4 + Y_5)$. If $\alpha = 1$, $\beta \neq 0$, then we have the measurable subset

$$(Bx+y)^2 + 2(he^{-\frac{B}{\beta}} + BE)x + 2Ey + F = 0, \quad (\beta h \neq 0;\ h = \text{const}) \tag{16}$$

and the invariant density is $d\pi = e^{\frac{3B}{\beta}} dB \wedge dE \wedge dF$. For $\alpha = 2$, $B + \beta \neq 0$ we get the measurable subset

$$(Bx+y)^2 + 2(BE + h)x + 2Ey + F = 0, \quad (B + \beta \neq 0,\ h \neq 0;\ h = \text{const}) \tag{17}$$

and it has the invariant density $d\pi = (B + \beta)^2 dB \wedge dE \wedge dF$. If $\alpha = 4$, $3B + \beta \neq 0$, then we find the measurable subset

$$(Bx+y)^2 + 2\left[-h(3B+\beta)^{\frac{2}{3}} + BE\right]x + 2Ey + F = 0, \quad (3B + \beta \neq 0,\ h \neq 0;\ h = \text{const}) \tag{18}$$

and the invariant density is of the form $d\pi = dB \wedge dE \wedge dF$. If $\alpha \neq 1, 2, 4$ and $(1 - \alpha)B - \beta \neq 0$, then we obtain the measurable subset

$$(Bx+y)^2 + 2\left\{h\left[(1-\alpha)B - \beta\right]^{\frac{2-\alpha}{1-\alpha}} + BE\right\}x + 2Ey + F = 0, \quad (\alpha \neq 1, 2, 4,\ (1-\alpha)B - \beta \neq 0,\ h \neq 0;\ h = \text{const}) \tag{19}$$

and the invariant density is $d\pi = |(1-\alpha)B - \beta|^{\frac{\alpha-4}{1-\alpha}} dB \wedge dE \wedge dF$.

Consider the group $\bar{H}_3^8 = (Y_3, Y_4, \alpha Y_1 + Y_5$. The measurable subset is determined by

$$(Bx+y)^2 + 2Dx + 2Ey + (D - BE)\left[h - \alpha \ln|D - BE|\right] + E^2 = 0, \quad (h = \text{const}) \tag{20}$$

and the invariant density is of the form $d\pi = (D - BE)^{-2} dB \wedge dD \wedge dE$.

Consider the group $\bar{H}_3^9 = (Y_3, Y_4, \alpha Y_2 + Y_5) | \alpha \neq 0)$. If $\alpha = 2$, then we find the measurable subset of parabolas

$$(21) \qquad \begin{gathered} (Bx + y)^2 + 2(BE + h)x + 2Ey + F = 0, \\ (F - E^2 \neq 0,\ h \neq 0;\ h = \text{const}) \end{gathered}$$

and the invariant density is $d\pi = |F - E^2|^{-1} dB \wedge dE \wedge dF$. For $\alpha \neq 2$ we obtain the measurable subset

$$(22) \qquad \begin{gathered} (Bx + y)^2 + 2Dx + 2Ey + h(D - BE)^{\frac{2}{2-\alpha}} + E^2 = 0, \\ (\alpha \neq 2;\ h = \text{const}) \end{gathered}$$

and the corresponding density is $d\pi = (D - BE)^{-2} dB \wedge dD \wedge dE$.

Next let us consider the group $\bar{H}_3^{10} = (\alpha Y_1 + Y_4, Y_2 + 2Y_5, Y_3 | \alpha \neq 0)$. The measurable subset of parabolas is

$$(23) \qquad \begin{gathered} (Bx + y)^2 + 2Dx + 2Ey + (D - BE)\left(2\alpha B - h\sqrt[3]{D - BE}\right) + E^2 = 0, \\ (h = \text{const}) \end{gathered}$$

and the invariant density is of the form $d\pi = (D - BE)^{-2} dB \wedge dD \wedge dE$.

Thus we proved the following

Theorem 3. *The measurable subsets of parabolas* (2) *under the three-parametric subgroups of* H_5 *are* (10) – (23).

6. Measurable subsets of parabolas under the two-parametric subgroups of H_5. Now we shall examine the two-parametric subgroups of H_5. The group $\bar{H}_2^1 = (Y_1, Y_2)$ is intransitive on the set (2) and it determines the invariant variety

$$D - BE = h_1 B, \quad B^2 F - D^2 = h_2 B^2,$$

where $h_1 B \neq 0$; $h_1, h_2 = \text{const}$. Then the subset of parabolas

$$(24) \qquad \begin{gathered} (Bx + y)^2 + 2B(E + h_1)x + 2Ey + (E + h_1)^2 + h_2 = 0, \\ (h_1 B \neq 0;\ h_1, h_2 = \text{const}) \end{gathered}$$

is measurable under $\bar{H}_2^1$ and the invariant density is $d\pi = |B|^{-1} dB \wedge dE$.

Consider the group $\bar{H}_2^2 = (Y_2, Y_3)$. We find the measurable subset

$$(25) \qquad \begin{gathered} (Bx + y)^2 + 2B(E + h_1)x + 2Ey + E^2 + h_2 = 0, \\ (h_1 B \neq 0;\ h_1, h_2 = \text{const}) \end{gathered}$$

and the invariant density is of the form $d\pi = |B|^{-1} dB \wedge dE$.

Studying the group $\bar{H}_2^3 = (Y_2, Y_4)$ we obtain the measurable subset

$$\begin{gathered}(Bx+y)^2 + 2Dx + 2h_1 y + h_2 = 0,\\ (D - h_1 B \neq 0;\ h_1, h_2 = \text{const})\end{gathered} \tag{26}$$

and the invariant density is $d\pi = (D - h_1 B)^2 dB \wedge dD$.

Studying the group $\bar{H}_2^4 = (Y_2, Y_5)$ we get the measurable subset of parabolas

$$\begin{gathered}(Bx+y)^2 + 2Dx + 2h_1 B^{-1} Dy + h_2 B^{-2} D^2 = 0,\\ (BD \neq 0,\ h_1 \neq 1;\ h_1, h_2 = \text{const})\end{gathered} \tag{27}$$

and the invariant density $d\pi = |BD|^{-1} dB \wedge dD$.

We now consider the group $\bar{H}_2^5 = (Y_1, \alpha Y_2 + Y_3)$. If $\alpha = 0$, then we find the measurable subset

$$\begin{gathered}(h_1 x + y)^2 + 2(h_1 E + h_2)x + 2Ey + F = 0,\\ (h_2 \neq 0;\ h_1, h_2 = \text{const})\end{gathered} \tag{28}$$

and the invariant density is $d\pi = dE \wedge dF$. If $\alpha \neq 0$, then the measurable subset is determined by

$$\begin{gathered}(Bx+y)^2 + 2B(E + h_1)x + 2Ey + h_1^2 h_2 - 2\alpha^{-1} h_1 \ln|h_1 B| + (E + h_1)^2 = 0,\\ (h_1 B \neq 0;\ h_1, h_2 = \text{const})\end{gathered} \tag{29}$$

and has the invariant density $d\pi = |B|^{-1} dB \wedge dE$.

Consider the group $\bar{H}_2^6 = (Y_1, \alpha Y_2 + Y_5)$. If $\alpha = 1$, then we get the measurable subset

$$\begin{gathered}(h_1 x + y)^2 + 2Dx + 2Ey + h_1^{-2}\left[h_2(D - h_1 E)^2 + D^2\right] = 0,\\ (D - h_1 E \neq 0,\ h_1 \neq 0;\ h_1, h_2 = \text{const})\end{gathered} \tag{30}$$

and the invariant density is $d\pi = (D - h_1 E)^{-2} dD \wedge dE$. For $\alpha = 2$ we obtain the measurable subset

$$\begin{gathered}(Bx+y)^2 + 2(BE + h_1)x + 2Ey + B^{-2}\left[(BE + h_1)^2 + h_2\right] = 0,\\ (h_1 B \neq 0;\ h_1, h_2 = \text{const})\end{gathered} \tag{31}$$

and the invariant density is $d\pi = dB \wedge dE$. If $\alpha \neq 1, 2$, then we find the measurable subset

$$\begin{gathered}(Bx+y)^2 + 2B\left(E + h_1 B^{\frac{1}{1-\alpha}}\right)x + 2Ey + h_2 B^{\frac{2}{1-\alpha}} + \left(E + h_1 B^{\frac{1}{1-\alpha}}\right)^2 = 0,\\ (h_1 B \neq 0;\ h_1, h_2 = \text{const})\end{gathered} \tag{32}$$

and the invariant density $d\pi = |B|^{\frac{\alpha-2}{1-\alpha}} dB \wedge dE$.

Considering the group $\bar{H}_2^7 = (Y_3, \alpha Y_1 + Y_4 | \alpha \neq 0)$ we get the measurable subset

$$(33) \qquad \begin{gathered}(Bx+y)^2 + 2(BE+h_1)x + 2Ey + E^2 + 2\alpha h_1 B + h_2 = 0,\\ (h_1 \neq 0;\ h_1, h_2 = \text{const})\end{gathered}$$

and the invariant density $d\pi = dB \wedge dE$.

Now we examine the group $\bar{H}_2^8 = (Y_3, \alpha Y_1 + Y_5)$. The measurable subset is determined by

$$(34) \qquad \begin{gathered}(Bx+y)^2 + 2B(E+h_1B)x + 2Ey + E^2 + h_1B^2(h_2 - \alpha \ln |h_1 B^2|) = 0,\\ (h_1 B \neq 0;\ h_1, h_2 = \text{const})\end{gathered}$$

and the invariant density is $d\pi = (B)^{-2} dB \wedge dE$.

We study the group $\bar{H}_2^9 = (Y_3, \alpha Y_2 + \beta Y_4 + Y_5 | \alpha \neq 0)$. If $\alpha = 1, \beta = 0$, then we find the measurable subset

$$(35) \qquad \begin{gathered}(h_1 x + y)^2 + 2Dx + 2Ey + E^2 + h_2(D - h_1 E) = 0,\\ (D - h_1 E \neq 0;\ h_1, h_2 = \text{const})\end{gathered}$$

and the invariant density is of the form $d\pi = (D - h_1 E)^{-2} dD \wedge dE$. For $\alpha = 1, \beta \neq 0$ we obtain the measurable subset

$$(36) \qquad \begin{gathered}(Bx+y)^2 + 2(BE + h_1 e^{-\frac{B}{\beta}})x + 2Ey + E^2 + h_2 e^{-\frac{2B}{\beta}} = 0,\\ (h_1 \beta \neq 0;\ h_1, h_2 = \text{const})\end{gathered}$$

and the invariant density is $d\pi = e^{\frac{B}{\beta}} dB \wedge dE$. If $\alpha = 2$, then the measurable subset is given by

$$(37) \qquad \begin{gathered}(Bx+y)^2 + 2(BE+h_1)x + 2Ey + E^2 + h_2(B+\beta)^{-2} = 0,\\ (B + \beta \neq 0,\ h_1 \neq 0;\ h_1, h_2 = \text{const})\end{gathered}$$

and the invariant density is $d\pi = dB \wedge dE$. If $\alpha \neq 1, 2$, then we get the measurable subset

$$(38) \qquad \begin{gathered}(Bx+y)^2 + 2\left\{BE + h_1\left[(1-\alpha)B - \beta\right]^{\frac{2-\alpha}{1-\alpha}}\right\}x + 2Ey + E^2\\ + h_2\left[(1-\alpha)B - \beta\right]^{\frac{2}{1-\alpha}} = 0,\\ ((1-\alpha)B - \beta \neq 0,\ h_1 \neq 0;\ h_1, h_2 = \text{const})\end{gathered}$$

and the invariant density $d\pi = |(1-\alpha)B - \beta|^{\frac{\alpha-2}{1-\alpha}} dB \wedge dE$.

Consider the group $\bar{H}_2^{10} = (Y_4, \alpha Y_2 + Y_3)$. If $\alpha = 0$, then we obtain the measurable subset

$$(39) \qquad \begin{gathered}(Bx+y)^2 + 2(BE+h_1)x + 2Ey + E^2 + h_2 = 0,\\ (h_1 \neq 0;\ h_1, h_2 = \text{const})\end{gathered}$$

and the invariant density is $d\pi = dB \wedge dE$. For $\alpha \neq 0$, we get the measurable subset of parabolas

$$(Bx + y)^2 + 2(BE + h_1 e^{\alpha E})x + 2Ey + E^2 + h_2 = 0, \quad (h_1 \neq 0;\ h_1, h_2 = \text{const}) \tag{40}$$

and the invariant density $d\pi = e^{-\alpha E} dB \wedge dE$.

Examine the group $\bar{H}_2^{11} = (Y_4, \alpha Y_2 + Y_5)$. If $\alpha = 2$, then we find the measurable subset

$$(Bx + y)^2 + 2(BE + h_1)x + 2Ey + h_2 E^2 = 0, \quad (h_1 \neq 0;\ h_1, h_2 = \text{const}) \tag{41}$$

and the invariant density is $d\pi = dB \wedge dE$. For $\alpha \neq 2$ the measurable subset is characterized by

$$(Bx + y)^2 + 2E(B + h_1 E^{1-\alpha})x + 2Ey + h_2 E^2 = 0, \quad (h_1 E \neq 0;\ h_1, h_2 = \text{const}) \tag{42}$$

and the invariant density is $d\pi = |E|^{\alpha - 2} dB \wedge dE$.

Now we study the group $\bar{H}_2^{12} = (Y_2 + 2Y_5, \alpha Y_1 + Y_4 | \alpha \neq 0)$. We get the measurable subset

$$\begin{gathered}(Bx + y)^2 + 2\left[BE - h_1(\alpha B^2 - 2E)^{\frac{3}{2}}\right] x + 2Ey + E^2 \\ -(\alpha B^2 - 2E)^{\frac{3}{2}} \left[2\alpha h_1 B + h_2(\alpha B^2 - 2E)^{\frac{1}{2}}\right] = 0, \\ (\alpha B^2 - 2E > 0,\ h_1 \neq 0;\ h_2 = \text{const})\end{gathered} \tag{43}$$

and the invariant density $d\pi = (\alpha B^2 - 2E)^{-\frac{3}{2}} dB \wedge dE$.

Hence, we can deduce:

Theorem 4. *The measurable subsets of parabolas* (2) *under the two-parametric subgroups of* H_5 *are* (24) – (43).

7. Measurable subsets of parabolas under the one-parametric subgroups of H_5. Finally, we shall consider the one-parametric subgroups of H_5. The associated group $\bar{H}_1 = (Y_1)$, corresponding to the group $H_1 = (X_1)$, is intransitive and it determines the invariant variety

$$B = h_1,\ D - BE = h_2,\ B^2 F - D^2 = h_3,$$

where $h_1 \neq 0$, $h_2 \neq 0$, $h_1, h_2, h_3 = \text{const}$. Then the subset of parabolas

$$(h_1 x + y)^2 + 2(h_1 E + h_2)x + 2Ey + h_1^{-2}\left[(h_1 E + h_2)^2 + h_3\right] = 0, \quad (h_1 \neq 0,\ h_2 \neq 0;\ h_1, h_2, h_3 = \text{const}) \tag{44}$$

is measurable and its invariant density is $d\pi = dE$.

Examine the group $\bar{H}_1^2 = (Y_2)$. We get the measurable subset

$$(45)\qquad \begin{gathered} (Bx+y)^2 + 2h_1 Bx + 2h_2 y + h_3 = 0, \\ ((h_1 - h_2)B \neq 0;\ h_1, h_2, h_3 = \text{const}) \end{gathered}$$

and the invariant density is of the form $d\pi = |B|^{-1} dB$.

We consider the group $\bar{H}_1^3 = (Y_3)$. The measurable subset of parabolas is determined by the equation

$$(46)\qquad \begin{gathered} (h_1 x + y)^2 + 2(h_1 E + h_2)x + 2Ey + h_3 + E^2 = 0, \\ (h_2 \neq 0;\ h_1, h_2, h_3 = \text{const}) \end{gathered}$$

and the invariant density is $d\pi = dE$.

Considering the group $\bar{H}_1^4 = (Y_4)$ we find the measurable subset

$$(47)\qquad \begin{gathered} (Bx+y)^2 + 2(h_1 + h_2 B)x + 2h_2 y + h_3 = 0, \\ (h_1 \neq 0;\ h_1, h_2, h_3 = \text{const}) \end{gathered}$$

and the invariant density $d\pi = dB$.

Now we examine the group $\bar{H}_1^5 = (Y_5)$. The measurable subset is

$$(48)\qquad \begin{gathered} (Bx+y)^2 + 2h_1 B^2 x + 2h_2 By + h_3 B^2 = 0, \\ ((h_1 - h_2)B \neq 0;\ h_1, h_2, h_3 = \text{const}) \end{gathered}$$

and the invariant density is of the form $d\pi = |B|^{-1} dB$.

Consider the group $\bar{H}_1^6 = (\alpha Y_1 + Y_4 | \alpha \neq 0)$. We get the measurable subset

$$(49)\qquad \begin{gathered} (Bx+y)^2 + (\alpha B^3 - h_1 B - 2h_2)x + (\alpha B^2 - h_1)y + \tfrac{1}{4}\alpha^2 B^4 \\ -\tfrac{1}{2}\alpha h_1 B^2 - 2\alpha h_2 B - h_3 = 0, \\ (h_2 \neq 0,\ h_1, h_2, h_3 = \text{const}) \end{gathered}$$

and the invariant density $d\pi = dB$.

Examining the group $\bar{H}_1^7 = (\alpha Y_1 + Y_5 | \alpha \neq 0)$ we obtain the measurable subset

$$(50)\qquad \begin{gathered} (Bx+y)^2 + 2B^2(h_1 - \alpha \ln|B|)x + 2B(h_2 - \alpha \ln|B|)y \\ +B^2(h_3 - 2\alpha h_1 \ln|B| + \alpha^2 \ln^2|B|) = 0, \\ ((h_1 - h_2)B \neq 0;\ h_1, h_2, h_3 = \text{const}) \end{gathered}$$

and the invariant density is $d\pi = |B|^{-1} dB$.

We consider the group $\bar{H}_1^8 = (\alpha Y_2 + Y_3 | \alpha \neq 0)$ and we find the measurable subset

$$(51)\qquad \begin{gathered} (Bx+y)^2 + 2B(h_1 + \alpha^{-1} \ln|B|)x + 2(h_2 + \alpha^{-1} \ln|B|)y + h_3 \\ +(h_2 + \alpha^{-1} \ln|B|)^2 = 0, \\ ((h_1 - h_2)B \neq 0;\ h_1, h_2, h_3 = \text{const}) \end{gathered}$$

and the invariant density is $d\pi = |B|^{-1}dB$.

Examine the group $\bar{H}_1^9 = (\alpha Y_2 + Y_5|\alpha \neq 0)$. If $\alpha = 1$, then we get the measurable subset of parabolas

$$(52) \qquad \begin{gathered}(h_1x+y)^2 + 2Dx + 2h_2Dy + h_3D^2 = 0,\\ ((1-h_1h_2)D \neq 0;\ h_1, h_2, h_3 = \text{const})\end{gathered}$$

and the density is of the form $d\pi = |D|^{-1}dD$. For $\alpha = 2$ the measurable subset is determined by the equation

$$(53) \qquad \begin{gathered}(Bx+y)^2 + 2h_1x + 2h_2B^{-1}y + h_3B^{-2} = 0,\\ ((h_1-h_2)B \neq 0;\ h_1, h_2, h_3 = \text{const})\end{gathered}$$

and the invariant density is $d\pi = |B|^{-1}dB$. If $\alpha \neq 1, 2$, then we obtain the measurable subset

$$(54) \qquad \begin{gathered}(Bx+y)^2 + 2h_1B^{\frac{2-\alpha}{1-\alpha}}y + 2h_2B^{\frac{1}{1-\alpha}}y + h_3B^{\frac{2}{1-\alpha}} = 0,\\ ((h_1-h_2)B \neq 0;\ h_1, h_2, h_3 = \text{const})\end{gathered}$$

and the invariant density is $d\pi = |B|^{-1}dB$.

Consider the group $\bar{H}_1^{10} = (\alpha Y_2 + \beta Y_4 + Y_5|\alpha\beta \neq 0)$. If $\alpha = 1$, then the measurable subset is

$$(55) \qquad \begin{gathered}[(h_1 - \beta\ln|E|)x + y]^2 + 2E(h_2 - \beta\ln|E|)x + 2Ey + h_3E^2 = 0,\\ ((h_1-h_2)E \neq 0,\ h_2 - \beta(\ln|E|-1) \neq 0;\ h_1, h_2, h_3 = \text{const})\end{gathered}$$

and the invariant density is of the form $d\pi = |E|^{-1}dE$. For $\alpha = 2$ we get the measurable subset

$$(56) \qquad \begin{gathered}(Bx+y)^2 + 2\left[h_1 - \beta h_2(B+\beta)^{-1}\right]x + 2h_2(B+\beta)^{-1} + h_3(B+\beta)^{-2} = 0,\\ (B+\beta \neq 0,\ h_1 - h_2 \neq 0;\ h_1, h_2, h_3 = \text{const})\end{gathered}$$

and the invariant density $d\pi = |B+\beta|^{-1}dB$. If $\alpha \neq 1, 2$, then the measurable subset is determined by the equation

$$(57) \qquad \begin{gathered}\left[(1-\alpha)^{-1}(h_2^{-1}E^{1-\alpha} + \beta)x + y\right]^2 + 2E^{2-\alpha}\\ \left[h_1 + (1-\alpha)^{-1}\beta E^{\alpha-1}\right]x + 2Ey + h_3E^2 = 0,\\ \Big\{h_2 \neq 0, \left[h_1 - h_2^{-1}(1-\alpha)^{-1}\right]E \neq 0,\\ (2-\alpha)h_1E^{1-\alpha} + (1-\alpha)^{-1}\beta \neq 0;\ h_1, h_2, h_3 = \text{const}\Big\}\end{gathered}$$

and the invariant density is $d\pi = |E|^{-1}dE$.

Let us summarize the results (44) – (57) by the following theorem:

Theorem 5. *The measurable subsets of parabolas* (2) *under the one-parametric subgroups of* H_5 *are* (44) – (57).

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