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SPECTRAL GEOMETRY ON CERTAIN ALMOST HERMITIAN MANIFOLDS

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ABSTRACT. On compact Riemannian and Kähler manifolds the spectra of the real and complex Laplacians determine the geometry of the manifolds to a considerable extent, though not completely, as isospectral manifolds need not to be isometric. The literature on the geometric consequences of isospectrallity is extensive (e.g., [1]–[3], [5]–[11]). In [3] we considered such consequences for the classes of almost Hermitian Einstein manifolds satisfying $\rho = \rho^*$, as well as inequality relations between ρ and ρ^*, where ρ is the scalar curvature and ρ^* is the * - scalar curvature; these included the almost and nearly Kähler Einstein manifolds. In this paper we consider the implications of isospectrality for the class of almost Hermitian manifolds satisfying $R_c = R_{c^*}$, where $R_c = (R_{ij})$ is the Ricci curvature tensor and $R_{c^*} = (R_{ij^*})$ is the Ricci * – tensor, and prove that complex projective space (CP^n, g_0, J_0), where g_0 is the Fubini – Study metric, is characterized by the spectrum in this class.

1. Preliminaries. Let (M,g) be a Riemannian manifold of real dimension $m = 2n \geq 2$ with metric $g = (g_{ij})$. If $R = (R_{hijk})$ is the Riemann curvature tensor, $R_c = (R_{hk}) = g^{ij}R_{hijk}$ the Ricci curvature tensor and $\rho = g^{hk}R_{hk}$ the scalar curvature, then the Einstein tensor $E = (E_{ij})$ is given by

$$E_{ij} \equiv R_{ij} - \frac{\rho}{m} g_{ij}. \tag{1.1}$$

where (M,g) is Einstein if $E = 0$.

If (M,g) is a compact connected C^∞ manifold and $\Delta = -(d\delta + \delta d)$ is the Laplace operator on p–forms, $0 \leq p \leq 2n$, (0–forms corresponding to differentiable functions on M) with respect to the metric g, then the spectrum of the Laplacian are the eigenvalues of Δ,

$$\operatorname{Spec}^p(M,g) = \{\lambda_{i,p} | 0 \geq \lambda_{1,p} \geq \lambda_{2,p} \geq \cdots \geq \lambda_{k,p} \geq \cdots \downarrow -\infty\} \tag{1.2}$$

where each eigenvalue is repeated as often as its multiplicity. Further $\operatorname{Spec}^{2n-p}(M,g) = \operatorname{Spec}^p(M,g)$ when M is orientable.

Relevant to the study of the spectrum is the Minakshisundaram–Pleijel–Gaffney asymptotic formula

$$\sum_{k=0}^{\infty} \exp(\lambda_{k,p} t) \; t \overset{\sim}{\to} 0 \; \frac{1}{(4\pi t)^n} \sum_{i=0}^{\infty} a_{i,p} t^i, \tag{1.3}$$

where the first three coefficients are given by [9].

$$a_{0,p} = \binom{2n}{p} \int_M dM = \binom{2n}{p} \mathrm{vol}(M), \tag{1.4}$$

$$a_{1,p} = [\frac{1}{6}\binom{2n}{p} - \binom{2n-2}{p-1}] \int_M \rho dM, \tag{1.5}$$

$$a_{2,p} = \int_M [c_1(2n,p)\rho^2 + c_2(2n,p)|R_c|^2 + c_3(2n,p)|R|^2] dM; \tag{1.6}$$

where

$$c_1(2n,p) = \frac{1}{72}\binom{2n}{p} - \frac{1}{6}\binom{2n-2}{p-1} + \frac{1}{2}\binom{2n-4}{p-2}, \tag{1.7}$$

$$c_2(2n,p) = -\frac{1}{180}\binom{2n}{p} + \frac{1}{2}\binom{2n-2}{p-1} - 2\binom{2n-4}{p-2}, \tag{1.8}$$

$$c_3(2n,p) = \frac{1}{180}\binom{2n}{p} - \frac{1}{12}\binom{2n-2}{p-1} + \frac{1}{2}\binom{2n-4}{p-2}. \tag{1.9}$$

2. Almost Hermitian manifolds and the Bochner curvature tensor. Let (M, g, J) be an almost Hermitian of real dimension $m = 2n \geq 2$ with almost complex structure $J = (F_i^j)$ and almost Hermitian metric $g = (g_{ij})$; that is, $g(JX, JY) = g(X, Y)$ for all X, Y in the tangent space $T_p(M)$. In dimension $2n = 2$, (M, g) is Kähler Einstein and has holomorphic sectional curvature $\varkappa = \frac{\rho}{2}$.

Define the Bochner curvature tensor $B = (B_{hijk})$ by

$$\begin{aligned} B_{hijk} \equiv{}& R_{hijk} - \frac{1}{2n+4}(R_{ij}g_{hk} - R_{ik}g_{hj} + R_{hk}g_{ij} - R_{hj}g_{ik} + F_{ij}F_h^r R_{rk} \\ & - F_{ik}F_h^r R_{rj} + F_{hk}F_i^r R_{rj} - F_{hj}F_i^r R_{rk} - 2F_{jk}F_h^r R_{ri} - 2F_{hi}F_j^r R_{rk}) \\ & + \frac{\rho}{(2n+2)(2n+4)}(g_{ij}g_{hk} - g_{ik}g_{hj} + F_{ij}F_{hk} - F_{ik}F_{hj} - 2F_{hi}F_{jk}). \end{aligned} \tag{2.1}$$

Lemma 2.1. (see e.g.,[4]) *If (M,g) is a Kahler manifold of nonzero constant holomorphic sectional curvature $\varkappa$, then the Riemann curvature tensor, Ricci curvature tensor and scalar curvature are given, respectively by*

$$R_{hijk} = \frac{\varkappa}{4}(g_{hk}g_{ij} - g_{hj}g_{ik} + F_{hk}F_{ij} - F_{hj}F_{ik} - 2F_{hi}F_{jk}), \tag{2.2}$$

$$R_{ij} = \frac{n+1}{2}\varkappa g_{ij}, \tag{2.3}$$

$$\rho = n(n+1)\varkappa. \tag{2.4}$$

Hence, (M,g) is Einstein and $B = 0$.

By Schur's theorem, a *Kähler manifold of dimension $2n \geq 4$ with constant holomorphic sectional curvature $\varkappa$ is of constant holomorphic curvature; that is, $\varkappa$ is a global constant on the manifold.*

Lemma 2.2. [4] *An almost Hermitian manifold (M,g) of dimension $2n \geq 4$ with curvature tensor given by (2.2) is Kähler and has constant holomorphic sectional curvature $\varkappa = \dfrac{\rho}{n(n+1)}$. Consequently we have*

Corollary 2.3. *If (M,g) is an almost Hermitian Einstein manifold of dimension $2n \geq 4$ with $B = 0$ and $\rho \neq 0$, then the conclusion of Lemma 2.2 follows.*

Lemma 2.4. [4] *If (M,g) is a Kähler manifold then*

$$R_{ij} = R_{ij^*} \equiv F^{kq}F_i^r R_{krqj} = -\frac{1}{2}F^{kq}F_i^r R_{kqjr}, \tag{2.5}$$

$$\rho = \rho^* \equiv F^{kq}F^{jr} R_{krqj} = -\frac{1}{2}F^{kq}F^{jr} R_{kqjr}, \tag{2.6}$$

$$R_{ij} = F_i^h F_j^k R_{hk}. \tag{2.7}$$

The last equality in (2.5), as in (2.6), is by way of the first Bianchi identity.

Lemma 2.5. *If (M,g) is a Hermitian manifold satisfying (2.5), then (2.6) and (2.7) follow.*

Proof. Equation (2.6) follows on contracting (2.5) with g^{ij}. To prove (2.7), multiply (2.5) by $F_p^i F_s^j$, then $R_{ij}F_p^i F_s^j = F_p^i F_s^j F^{kq} F_i^r R_{krqj} = -F_s^j F^{kq} R_{kpqj} = F_s^j F^{kq} R_{qpkj} = F_s^j F^{kq} R_{kjqp} = R_{ps}$.

Lemma 2.6. *If (M,g) is an almost Hermitian manifold with (2.5), then the square length of the Bochner tensor is given by*

$$|B|^2 = |R|^2 - \frac{8}{n+2}|R_c|^2 + \frac{2}{(n+1)(n+2)}\rho^2. \quad \bullet \tag{2.8}$$

Proof. The equation follows from a rather lenghty calculation of $|B_{hijk}|^2$ using (2.1), (2.5), (2.6), (2.7).

3. Spectral geometry on almost Hermitian manifolds satisfying $R_c = R_{c^*}$.

We shall assume in 3 that (M,g,J) and (M',g',J') are compact almost Hermitian manifolds satisfying $R_{ij} = R_{ij^*}$ in the respective metrics g and g'.

The main results in this section are the following:

Theorem 3.1. *a) If* $\mathrm{Spec}^p(M,g) = \mathrm{Spec}^p(M',g')$ *for $p = 0, 1$ or 2, then in dimension $2n = 2$, (M,g) is of constant holomorphic curvature $\varkappa$ if and only if (M',g') is.*

b) In dimension $2n \geq 4$, if $\mathrm{Spec}^p(M,g,J) = \mathrm{Spec}^p(M',g')$, then (M,g) is a Kähler manifold of constant holomorphic sectional curvature $\varkappa$ if and only if (M',g') is, in the following cases: $p = 0$ and $4 \leq 2n \leq 10$; $p = 1$ and $16 \leq 2n \leq 102$; $p = 2$ and $2n = 6, 8, 14$ or $18 \leq 2n \leq 188$; $p = 0$ and 1 and $2n \geq 4$; $p = 0$ and 2 and $2n \neq 12$.

Corollary 3.2. *If* $\mathrm{Spec}^p(M,g,J) = \mathrm{Spec}^p(CP^n, g_0, J_0)$, *then (M,g,J) is Kähler and holomorphically isotermic to (CP^n, g_0, J_0) in the following cases: $p = 0$ and $2 \leq 2n \leq 10$; $p = 1$ and $2n = 2$ or $16 \leq 2n \leq 102$; $p = 2$ and $2n = 2, 6, 8, 14$ or $18 \leq 2n \leq 188$; $p = 0$ and 1 and $2n \geq 2$, so that $(CP^n, g_0, J_)$) is characterized by the spectrum in every dimension in the class of almost Hermitian manifolds satisfying $R_c = R_{c^*}$; $p = 0$ and 2 and $2n \neq 12$.*

Proof of Theorem 3.1. Letting $p = 0$ in (1.4)–(1.9) gives:

$$a_{0,0} = \int_M dM = \mathrm{vol}(M), \tag{3.1}$$

$$a_{1,0} = \frac{1}{6}\int_M \rho dM, \tag{3.2}$$

$$a_{2,0} = \frac{1}{360}\int_M [5\rho^2 - 2|R_c|^2 + 2|R|^2]dM. \tag{3.3}$$

a) In dimension $2n = 2$, $|R|^2 = \rho^2$ and since g is an Einstein metric then by (1.1), $|R_c|^2 = \frac{\rho^2}{2}$ and $a_{2,0} = \frac{1}{60}\int_M \rho^2 dM$. If, say, (M,g,J) has constant holomorphic curvature $\varkappa$, then since $a_{0,0} = a'_{0,0}$, $a_{1,0} = a'_{1,0}$ and $a_{2,0} = a'_{2,0}$, it follows that $\mathrm{vol}(M) = \mathrm{vol}(M')$ and therefore, $\int_{M'} \rho' dM' = 2\varkappa \mathrm{vol}(M')$ and $\int_{M'} \rho'^2 dM' = 4\varkappa^2 \mathrm{vol}(M')$. We then have equality in the Schwarz inequality $\left(\int_{M'} \rho' dM'\right)^2 \leq \left(\int_{M'} \rho'^2 dM'\right)\left(\int_{M'} dM'\right)$, so that ρ' is constant and $\rho' = \rho$.

The proofs in the remaining cases are similar upon taking $p = 1$ and 2, respectively, in (1.4)–(1.9). For $p = 2$ the results follows, as well, as a consequence of the case $p = 0$ since $\mathrm{Spec}^0(M,g) = \mathrm{Spec}^0(M',g')$ by duality.

b) In dimension $4 \leq 2n \leq 10$, substituting (2.8) and (1.1) in (3.3) gives

$$\begin{aligned}&\int_M [\frac{5n^2+4n+3}{n(n+1)}\rho^2 + \frac{-2n+12}{n+2}|E|^2 + 2|B|^2] dM \\ &= \int_{M'} [\frac{5n^2+4n+3}{n(n+1)}\rho'^2 + \frac{-2n+12}{n+2}|E'|^2 + 2|B'|^2] dM',\end{aligned} \tag{3.4}$$

since $a_{2,0} = a'_{2,0}$, with the coefficient of $|E|^2$ positive.

If, say (M,g,J) is Kähler with constant holomorphic sectional curvature $\varkappa$, then by Lemma 2.1 (M,g) is Einstein and $B = 0$. Since ρ is constant, then $a_{0,0} = a'_{0,0}$, $a_{1,0} = a'_{1,0}$ and the Schwarz inequality imply $\left(\int_{M'} \rho'^2 dM'\right)\left(\int_{M'} dM'\right) \geq \left(\int_{M'} \rho' dM'\right)^2 = \left(\int_M \rho dM\right)^2 = \rho^2[vol(M)]^2 = \rho^2 \mathrm{vol}(M')\mathrm{vol}(M) = \mathrm{vol}(M') \int_M \rho^2 dM$. Then by (3.4) $|B'|^2 = 0$, and $|E'|^2 = 0$, so $\int_{M'} \rho'^2 dM' = \int_M \rho^2 dM$, implying equality in the Schwarz inequality. Thus, ρ' is constant and $\rho' = \rho$. Hence, by Corollary 2.3, (M',g',J') is Kähler of constant holomorphic sectional curvature $\varkappa' = \varkappa$.

Letting $p = 1$ in (1.4)–(1.9) gives

$$a_{0,1} = 2n \int_M dM = 2n\mathrm{vol}(M), \tag{3.5}$$

$$a_{1,1} = \frac{n-3}{3} \int_M \rho dM, \tag{3.6}$$

$$a_{2,1} = \frac{1}{180} \int_M [(5n-30)\rho^2 + (-2n+90)|R_c|^2 + (2n-15)|R|^2] dM. \tag{3.7}$$

In dimension $16 \leq 2n \leq 102$, substituting (2.8) and (1.1) in (3.7) gives

$$\begin{aligned}&\int_M [\frac{5n^3-26n^2+18n+15}{n(n+1)}\rho^2 + \frac{-2n^2+102n+60}{n+2}|E|^2 \\ &+(2n-15)|B|^2] dM = \int_{M'} [\frac{5n^3-26n^2+18n+15}{n(n+1)}\rho'^2 \\ &+\frac{-2n^2+102n+60}{n+2}|E'|^2 + (2n-15)|B'|^2] dM'\end{aligned} \tag{3.8}$$

since $a_{2,1} = a'_{2,1}$, with coefficients of ρ^2, $|E|^2$ and $|B|^2$ positive.

Letting $p = 2$ in (1.4)–(1.9) gives

$$a_{0,2} = (2n^2 - n)\int_M dM = (2n^2 - n)\mathrm{vol}(M), \tag{3.9}$$

$$a_{1,2} = \frac{2n^2 - 13n + 12}{6}\int_M \rho dM, \tag{3.10}$$

$$\begin{aligned} a_{2,2} &= \frac{1}{360}\int_M \big[(10n^2 - 125n + 300)\rho^2 + (-4n^2 + 362n - 1080)|R_c|^2 \\ &\quad +(4n^2 - 62n + 240)|R|^2\big]dM. \end{aligned} \tag{3.11}$$

In dimension $2n = 6, 8, 14$ or $18 \le 2n \le 188$, substituting (2.8) and (1.1) in (3.11) gives

$$\begin{aligned} &\int_M \Big[\frac{10n^4 - 117n^3 + 362n^2 - 183n - 60}{n(n+1)}\rho^2 + \frac{-4n^3 + 386n^2 - 852n - 240}{n+2}|E|^2 \\ &+(4n^2 - 62n + 240)|B|^2\Big]dM = \int_{M'} \Big[\frac{10n^4 - 117n^3 + 362n^2 - 183n - 60}{n(n+1)}\rho'^2 \\ &+\frac{-4n^3 + 386n^2 - 852n - 240}{n+2}|E'|^2 + (4n^2 - 62n + 240)|B'|^2\Big]dM'; \end{aligned} \tag{3.12}$$

since $a_{2,2} = a'_{2,2}$ with the coefficients of ρ^2, $|E|^2$ and $|B|^2$ positive.

If (M, g, J) is a Kähler with constant holomorphic sectional curvature $\varkappa$, then the implications of $B = 0$, $E = 0$, ρ constant, $a_{0,p} = a'_{0,p}$ and $a_{1,p} = a'_{1,p}$ for $p = 1$ and 2, and the Schwarz inequality are similar to the case $p = 0$.

For the case $p = 0$ and 1 in dimension $2n \ge 4$, multiplying (3.4) by $\frac{2n - 15}{2}$ and subtracting the resulting equation from (3.8) give

$$\int_M \Big[\frac{n+5}{n}\rho^2 + 10|E|^2\Big]dM = \int_{M'} \Big[\frac{n+5}{n}\rho'^2 + 10|E'|^2\Big]dM'. \tag{3.13}$$

If (M, g, J) is Kähler of constant holomorphic sectional curvature $\varkappa$, then $|B|^2 = |E|^2 = 0$, and since $\int_{M'} \rho'^2 dM \ge \int_M \rho^2 dM$, then $|E'|^2 = 0$, so that $\int_{M'} \rho'^2 dM = \int_M \rho^2 dM$ and by (3.4), $|B'|^2 = 0$. Then by Lemma 2.2, (M', g', J') is Kähler and of constant holomorphic sectional curvature $\varkappa' = \varkappa$.

For the case $p = 0$ and 2, we observe that from $p = 0$ and 2 above, the exceptional dimensions are $2n = 12, 16$ and $n \ge 190$. In dimension $2n \ne 12$, in a similar way as in the case for $p = 0$ and 1, we multiply (3.4), by $2n^2 - 31n + 120$ and subtract the resulting equation from (3.12) and the result follows.

Proof of Corollary 3.2. Since (CP^n, g_0, J_0) is the only Kähler manifold with a metric of positive constant holomorphic curvature $\varkappa$, then by Theorem 3.1, (M, g, J) is Kähler with constant holomorphic curvature $\varkappa$. Hence, (M, g, J) and (CP^n, g_0, J_0) are holomorphically isometric.

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