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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR PARTIAL DIFFERENTIAL-FUNCTIONAL EQUATIONS OF THE FIRST ORDER WITH DEVIATING ARGUMENT OF THE DERIVATIVE OF UNKNOWN FUNCTION

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Abstract. We consider the existence and uniqueness problem for partial differential-functional equations of the first order with the initial condition for which the right-hand side depends on the derivative of unknown function with deviating argument.

1. Introduction. We consider the following Cauchy problem

$$(1) \qquad \begin{cases} D_x z(x,y) = f(x,y,z(\cdot), D_y z(\alpha(x,y),\beta(x))), & (x,y) \in E \\ z(0,y) = v(y), & y \in [0,b], \end{cases}$$

where $E = [0,a]\times[0,b]$, $a, b > 0$, $f \in C(E\times C(E;\mathcal{R}^n)\times\mathcal{R}^n;\mathcal{R}^n)$, $v \in C([0,b];\mathcal{R}^n)$, $\alpha \in C(E;[0,a])$, $\beta \in C([0,a];[0,b])$ and $D_p g$ means the partial derivative of g with respect to p.

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The problem $D_x z(x,y) = f(x,y,z(\cdot),D_y z(x,y))$, $z(0,y) = v(y)$ has been investigated intensively in the literature (see [5], [6], [7] for references). But there are only few results concerning equations with $D_y z$ depending on a deviating argument (see [1]–[3], [8]–[10]). In the present paper we consider such an equation. But notice that the deviating argument we consider is of peculiar type and it is not a generalization of not deviating argument (x,y).

The proof of our result is based on Bielecki method of changing norm (see [4]).

2. Notations and assumptions. We denote $E_x = \{(s,y) \in E : 0 \le s \le x\}$, $C = C(E;\mathcal{R}^n)$,

$$\|z\|_{0,x} = \sup\{|z(s,y)| : (s,y) \in E_x\},$$

$$\|z\|_K = \sup\{|z(x,y)|e^{-Kx} : (x,y) \in E\}$$

for $z \in C$, $x \in [0,a]$, $K \in \mathcal{R}$, where $|\cdot|$ means a fixed norm in $\mathcal{R}^n$. Notice that $\|\cdot\|_K$ is a norm equivalent to the supremum norm and C is a Banach space with this norm.

$D_q f$ means the partial derivative of f with respect to the fourth argument

We need the following assumptions.

(H_1) The function f satisfies the following Volterra condition: for all $(x,y) \in E$, $u,v \in C$, $q \in \mathcal{R}^n$ if $u(s,y) = v(s,y)$ for $(s,y) \in E_x$, then $f(x,y,u(\cdot),q) = f(x,y,v(\cdot),q)$.

(H_2) The functions f, $D_y f$, $D_q f$, α, $D_y\alpha$, β, v, $D_y v$ are continuous and there exists $L > 0$ such that

$$|f(x,y,p,q) - f(x,y,P,Q)| \le L(\|p-P\|_{0,x} + |q-Q|),$$

$$|D_y f(x,y,p,q) - D_y f(x,y,P,Q)| \le L(\|p-P\|_{0,x} + |q-Q|),$$

$$|D_q f(x,y,p,q) - D_q f(x,y,P,Q)| \le L(\|p-P\|_{0,x} + |q-Q|),$$

for $(x,y) \in E$, $p,P \in C$, $q,Q \in \mathcal{R}^n$.

(H_3) $\alpha(x,y) \le x$ for $(x,y) \in E$ and

$$l = \sup\{|D_y\alpha(x,y) D_q f(x,y,p,q)| : (x,y) \in E,\ p \in C,\ q \in \mathcal{R}^n\} < 1$$

Notice that (H_3) implies that the growth of $f(x,y,p,\cdot)$ is at most linear except for $D_y\alpha(x,y)=0$.

3. The result.

Now we state

Theorem. *Suppose that assumptions (H_1), (H_2), (H_3) are satisfied, then there exists exactly one solution $\hat{z}$ of* (1) *such that $D_xD_y\hat{z}$ and $D_yD_x\hat{z}$ exist and they are continuous functions.*

P r o o f. Let us define

$$F(U,u)(x,y)=\int_0^x\int_0^y u(s,t)dtds+v(y)+$$
$$+\int_0^x f\Big(s,0,U,v'(\beta(s))+\int_0^{\alpha(s,y)} u(\sigma,\beta(s))\,d\sigma\Big)ds$$

$$G(U,u)(x,y)=D_yf\Big(x,y,U,v'(\beta(x))+\int_0^{\alpha(x,y)} u(s,\beta(x))\,ds\Big)+$$
$$+D_y\alpha(x,y)D_qf\Big(x,y,U,v'(\beta(x))+\int_0^{\alpha(x,y)} u(s,\beta(x))\,ds\Big)u(\alpha(x,y),\beta(x))$$

for $U,u\in C$, $(x,y)\in E$. We prove that there exists exactly one pair $(\hat{U},\hat{u})$ of continuous functions, which are solutions of the equations

$$U=F(U,u)\quad\text{and}\quad u=G(U,u). \tag{2}$$

Let $K>L$ be fixed. For every U_1, U_2, $u\in C$ assumption (H_2) gives

$$|F(U_1,u)(x,y)-F(U_2,u)(x,y)|\le\int_0^x L\|U_1-U_2\|_{0,s}ds\le$$

$$\le L\int_0^x\|U_1-U_2\|_K\,e^{Ks}ds\le\frac{L}{K}e^{Ks}\|U_1-U_2\|_K,$$

so

$$\|F(U_1,u)-F(U_2,u)\|_K\le\frac{L}{K}\|U_1-U_2\|_K.$$

Therefore $F(\cdot,u)$ is a contraction and for each $u \in C$ there exists exactly one fixed point $U(u)$ of $F(\cdot,u)$. Moreover

$$
\begin{aligned}
&|U(u_1)(x,y) - U(u_2)(x,y)| = |F(U(u_1),u_1)(x,y) - F(U(u_2),u_2)(x,y)| \le \\
&\le |F(U(u_1),u_1)(x,y) - F(U(u_2),u_1)(x,y)| + \\
&\qquad\qquad + |F(U(u_2),u_1)(x,y) - F(U(u_2),u_2)(x,y)| \le \\
&\le \frac{L}{K}\|U(u_1) - U(u_2)\|_K \, e^{Kx} + \int_0^x \int_0^y |u_1(s,t) - u_2(s,t)| \, ds \, dt + \\
&\qquad\qquad + L \int_0^x \int_0^{\alpha(s,t)} |u_1(\sigma,\beta(s)) - u_2(\sigma,\beta(s))| \, d\sigma \, ds \le \\
&\le \frac{L}{K}\|U(u_1) - U(u_2)\|_K \, e^{Kx} + b \int_0^x \|u_1 - u_2\|_K \, e^{Ks} ds + \\
&\qquad\qquad + L \int_0^x \int_0^s \|u_1 - u_2\|_K \, e^{K\sigma} d\sigma ds \le \\
&\le (\frac{L}{K}\|U(u_1) - U(u_2)\|_K + (\frac{b}{K} + \frac{aL}{K})\|u_1 - u_2\|_K)e^{Kx},
\end{aligned}
$$

hence

$$
\|U(u_1) - U(u_2)\|_K \le l_K \|u_1 - u_2\|_K,
$$

where $l_K = (K-L)^{-1}(b + aL)$. For sufficiently large K let us define

$$
W_K = \{u \in C : \|u\|_K \le M\},
$$

where

$$
M = (1 - (Ll_K + \frac{L}{K} + l))^{-1} P, \quad P = 1 + \max_{(x,y)\in E} |D_y f(x,y,U(\Theta), v'(\beta(x)))|
$$

and $\Theta(x,y) = 0$ for $(x,y) \in E$. Denote also $G_U(u) = G(U(u),u)$. We prove that $G_U(W_K) \subset W_K$ for $K > 0$ such that $Ll_K + \frac{L}{K} + l < 1$.

If $u \in W_K$ then

$$|G(U(u),u)(x,y)| \le$$

$$\le \Big|D_y f\Big(x,y,U(u),v'(\beta(x))+\int_0^{\alpha(x,y)} u(s,\beta(x))\,ds\Big) - D_y f\Big(x,y,U(\Theta),v'(\beta(x))\Big)\Big| +$$
$$+|D_y f\Big(x,y,U(\Theta),v'(\beta(x))\Big)| + l|u(\alpha(x,y),\beta(x))| \le$$

$$\le L\|U(u)-U(\Theta)\|_{0,x} + L\int_0^{\alpha(x,y)} |u(s,\beta(x))|\,ds + P + l|u(\alpha(x,y),\beta(x))| \le$$
$$\le (L\|U(u)-U(\Theta)\|_K + \frac{L}{K}\|u\|_K + P + l\|u\|_K)e^{Kx} \le$$
$$\le (Ll_K + \frac{L}{K} + l)\|u\|_K\, e^{Kx} + Pe^{Kx},$$

therefore

$$\|G_U(u)\|_K \le (Ll_K + \frac{L}{K} + l)M + P = M.$$

We have just showed that $G_U(W_K) \subset W_K$ for sufficiently large K. We prove that G_U is a contraction on W_K with respect to the norm $\|\cdot\|_J$ for sufficiently large J and K.

For $u, \bar{u} \in W_K,\ c = \max_{(x,y)\in E} |D_y\alpha(x,y)|$ we have

$$|G(U(u),u)(x,y) - G(U(\bar{u}),\bar{u})(x,y)| \le$$

$$\le L\|U(u)-U(\bar{u})\|_{0,x} + L\int_0^{\alpha(x,y)} |u(s,\beta(x)) - \bar{u}(s,\beta(x))|\,ds +$$

$$+|D_y\alpha(x,y)|\Big|D_q f\Big(x,y,U(u),v'(\beta(x)) + \int_0^{\alpha(x,y)} u(s,\beta(x))\,ds\Big) -$$

$$-D_q f\Big(x,y,U(\bar{u}),v'(\beta(x)) + \int_0^{\alpha(x,y)} \bar{u}(s,\beta(x))\,ds\Big)\Big||u(\alpha(x,y),\beta(x))|$$

$$+|D_y\alpha(x,y)||D_q f\Big(x,y,U(\bar{u}),v'(\beta(x)) + \int_0^{\alpha(x,y)} \bar{u}(s,\beta(x))\,ds\Big)| \times$$

$$\times|u(\alpha(x,y),\beta(x)) - \bar{u}(\alpha(x,y),\beta(x))| \le$$

$$\leq L\|U(u)-U(\bar u)\|_J\, e^{Jx}+\frac{L}{J}\|u-\bar u\|_J\, e^{Jx}+$$

$$+cL\Big(\|U(u)-U(\bar u)\|_{0,x}+\int\limits_0^{\alpha(x,y)}|u(s,\beta(x))-\bar u(s,\beta(x))|\,ds\Big)Me^{Kx}+$$

$$+l\|u-\bar u\|_J\, e^{Jx}$$

$$\leq (Ll_J+\frac{L}{J}+cL(l_J+\frac{1}{J})Me^{Kx}+l)e^{Jx}\|u-\bar u\|_J$$

so

$$\|G_U(u)-G_U(\bar u)\|_J\leq (Ll_J+\frac{L}{J}+cLM(l_J+\frac{1}{J})Me^{Ka}+l)\|u-\bar u\|_J.$$

Now it is clear that the operator G_U is contractive in W_K if K and J are sufficiently large and this operator has exactly one fixed point $\hat u$ in W_K. Since every fixed point $\bar u$ of G_U in C satisfies $|\bar u(0,y)|\leq P-1$ for $y\in[0,b]$ and $M>P$, then there exists K such that $\bar u\in W_K$. We get from the above that the function $\hat u$ is the unique fixed point of G_U in C. Of course the pair $(U(\hat u),\hat u)$ is the unique solution of (2). Now we demonstrate that $V=U(\hat u)$ is a solution of (1). From the definition of the operators F and G we get

$$V(0,y)\;=\;v(y),$$

$$D_yV(x,y)\;=\;\int\limits_0^x \hat u(s,y)\,ds+v'(y),$$

$$\hat u(x,y)\;=\;G(V,\hat u)(x,y)=\frac{d}{dy}f(x,y,V,v'(\beta(x))+\int\limits_0^{\alpha(x,y)}\hat u(s,\beta(x))\,ds)=$$

$$=\frac{d}{dy}f(x,y,V,D_yV(\alpha(x,y),\beta(x))),$$

$$D_xV(x,y)\;=\;\int\limits_0^y \hat u(x,t)\,dt+f(x,0,V,D_yV(\alpha(x,0),\beta(x)))=$$

$$=\;f(x,y,V,D_yV(\alpha(x,y),\beta(x))),$$

so V is a solution of (1) and it is obvious that $D_xD_yV = D_yD_xV = \hat{u}$, hence these derivatives are continuous. On the other hand, if z is a solution of (1) and $D_xD_yz = D_yD_xz$ then differentiating equation (1) with respect to y we obtain

$$\begin{aligned} D_yD_xz(x,y) &= D_yf(x,y,z,D_yz(\alpha(x,y),\beta(x))) + \\ &+ D_qf(x,y,z,D_yz(\alpha(x,y),\beta(x)))\, D_xD_yz(\alpha(x,y),\beta(x))\, D_y\alpha(x,y) \end{aligned}$$

and

$$D_yz(x,y) = \int_0^x D_xD_yz(s,y)\,ds + v'(y),$$

hence $D_xD_yz = G(z, D_xD_yz)$ and it is easy to verify that $z = F(z, D_xD_yz)$, so $z = U(D_xD_yz)$ and $D_xD_yz = \hat{u}$. We get that z is a solution of (1) if and only if (z, D_xD_yz) is a solution of (2). The proof is complete.

Remark. Under analogous assumptions we can prove a similar result for equation (1) with $y = (y_1, y_2, \dots, y_k) \in \mathcal{R}^k$.

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