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## ISOMORPHISM OF COMMUTATIVE MODULAR GROUP ALGEBRAS\*

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ABSTRACT. Let  $K$  be a field of characteristic  $p > 0$  and let  $G$  be a direct sum of cyclic groups, such that its torsion part is a  $p$ -group. If there exists a  $K$ -isomorphism  $KH \cong KG$  for some group  $H$ , then it is shown that  $H \cong G$ .

Let  $G$  be a direct sum of cyclic groups, a divisible group or a simply presented torsion abelian group. Then  $KH \cong KG$  as  $K$ -algebras for all fields  $K$  and some group  $H$  if and only if  $H \cong G$ .

**1. Introduction.** Let  $G$  be an abelian group,  $tG$  be its torsion subgroup and  $G_p$  be a  $p$ -primary component of  $G$ . Throughout this article  $R$  and  $K$  will denote commutative rings with identities and  $U(R)$  will be the multiplicative group of a ring  $R$ .

Let us denote by  $U(RG)$  and  $U_p(RG)$  the unit group and its  $p$ -primary component (i. e. its Sylow  $p$ -subgroup), respectively and by  $V(RG)$  and  $V_p(RG) =$

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$S(RG)$  the group of normalized units (i. e. the units of augmentation 1) and its  $p$ -primary component (i. e. its normed Sylow  $p$ -subgroup) in a group algebra  $RG$ , respectively.

In this paper the groups  $V(RG; H)$  (see 2.2) and  $S(KG)$ , and their decompositions into a restricted (bounded) direct product (i.e. a direct sum) of cyclic  $p$ -groups are being examined. Some criteria are obtained for  $V(RG; H)$  and  $S(KG)$  when they are direct sums of cyclic  $p$ -groups, and  $G$  is an arbitrary abelian group,  $H$  is a pure  $p$ -subgroup of  $G$  and  $R$  is an arbitrary ring,  $K$  is a ring without nilpotent elements, and  $\text{char}R = \text{char}K = p$ -prime number. The proofs are based on Kulikov's theorem (see [11, p. 144 and p. 550] or [7, p. 106, Theorem 17.1]).

Besides, the isomorphism problem for commutative modular group algebras is being discussed. Namely, we prove that the group algebra  $KG$  over a field  $K$  determines  $G$  up to isomorphism for the cases when:

(\*)  $G$  is a direct sum of cyclic groups, the torsion subgroup of which is a  $p$ -group, and  $\text{char}K = p > 0$ .

(\*\*)  $G$  is a direct sum of cyclic groups, or a divisible group or a simply presented torsion group, and  $K$  is every field (every field of prime characteristic).

Thus, we conclude that  $KG$  determines the isomorphism class of the group  $G$  in cases (\*) and (\*\*), i.e. a full system of invariants of the  $K$ -algebra  $KG$  is the group  $G$ .

## 2. Unit groups in commutative modular group algebras.

### 2.1. Preliminary lemmas.

**Lemma 1.** *Let  $R$  be a commutative ring with identity and prime characteristic  $p$ .*

(1) *Let  $r \in R$ . Then  $r \in U(R)$  if and only if  $r^p \in U(R^p)$ .*

(2) 
$$U^p(R) = U(R^p).$$

*Proof.* (1) Let  $r \in U(R)$ , i.e. does exist  $\alpha \in R$  with  $r\alpha = 1$ . Hence  $r^p \cdot \alpha^p = 1$ , i.e.  $r^p \in U(R^p)$ . Now let  $r^p \in U(R^p)$ , i.e. does exist  $\beta \in R^p$  with  $r^p \cdot \beta = 1$ , i.e.  $r \cdot r^{p-1} \cdot \beta = 1$ . Finally  $r \in U(R)$ .

(2) Let  $x \in U^p(R)$ , i.e.  $x = \gamma^p$ ,  $\gamma \in U(R)$ . From (1),  $\gamma^p \in U(R^p)$ , i.e.  $x \in U(R^p)$  and  $U^p(R) \subseteq U(R^p)$ . Now let  $y \in U(R^p)$ . Therefore does exist

$\delta \in R$  and  $y = \delta^p$ . But  $\delta^p \in U(R^p)$  and by (1),  $\delta \in U(R)$ , i.e.  $y \in U^p(R)$ . Finally  $U(R^p) \subseteq U^p(R)$  and the lemma is true.  $\square$

**Lemma 2.** *Let  $R$  be a commutative ring with identity of prime characteristic  $p$  and let  $G$  be an abelian group. For every ordinal number  $\sigma$  we have:*

- (3)  $(RG)^{p^\sigma} = R^{p^\sigma} G^{p^\sigma}.$
- (4)  $U^{p^\sigma}(RG) = U(R^{p^\sigma} G^{p^\sigma}).$
- (5)  $V^{p^\sigma}(RG) = V(R^{p^\sigma} G^{p^\sigma}).$
- (6)  $U_p^{p^\sigma}(RG) = U_p(R^{p^\sigma} G^{p^\sigma}).$
- (7)  $S^{p^\sigma}(RG) = S(R^{p^\sigma} G^{p^\sigma}).$

**Proof.** Let  $\sigma = 1$ . Further the proof goes on a standard way by means of a transfinite induction.

(3) is evidently. (4) Since  $(RG)^p = R^p G^p$  by (3), then  $U(R^p G^p) = U((RG)^p) = U^p(RG)$  from Lemma 1, because  $RG$  is a commutative ring with identity and  $\text{char}RG = p$ . (5) Certainly from (4),  $V(R^p G^p) = U(R^p G^p) \cap V(RG) = U^p(RG) \cap V(RG) = V^p(RG)$ , since  $V(RG)$  is pure in  $U(RG)$  as its direct factor. (6) follows immediately from (4). (7) follows immediately from (5). The lemma is proved.  $\square$

**Lemma 3.** *Let  $G$  be an abelian group and  $K$  be a commutative ring with identity of prime characteristic  $p$  without nilpotent elements. Then*

(8)  $S(KG) = 1$  if and only if  $G_p = 1.$

**Proof.** If  $S(KG) = 1$ , then  $G_p=1$ , since  $G_p \subseteq S(KG)$ . Let  $G_p = 1$ ,  $c = \sum_{1 \leq i \leq n} \mu_i g_i \in S(KG)$  ( $\mu_i \in K, g_i \in G$ ),  $\sum_{1 \leq i \leq n} \mu_i = 1$  and  $c^{p^m} = 1$  for any  $m \in \mathbb{N}$ . Therefore  $\sum_{1 \leq i \leq n} \mu_i^{p^m} g_i^{p^m} = 1$ . But  $g_{j-1}^{p^m} \neq g_j^{p^m}$  ( $j=2, \dots, n+1, g_{n+1}=g_1$ ). Indeed, let  $g_{j-1}^{p^m} = g_j^{p^m}$ , i.e.  $(g_{j-1} \cdot g_j^{-1})^{p^m} = 1$ , i.e.  $g_{j-1} \cdot g_j^{-1} \in G_p = 1$  and  $g_{j-1} = g_j$  — a contradiction. Hence  $g_1^{p^m} = 1$ , i.e.  $g_1 \in G_p = 1$  and  $g_1 = 1$ ;  $\mu_1^{p^m} = 1$ , i.e.  $(\mu_1 - 1)^{p^m} = 0$  and  $\mu_1 = 1$ ;  $\mu_2^{p^m} = \dots = \mu_n^{p^m} = 0$ , i.e.  $\mu_2 = \dots = \mu_n = 0$ . Finally  $c = 1$ , i.e.  $S(KG) = 1$ . So, the lemma is proved.  $\square$

**2.2. Direct sums of cyclic groups of the Sylow  $p$ -subgroups of modular group algebra.** Let  $H$  be a subgroup of an abelian group  $G$ , i.e.  $H \leq$

$G$ . Following May [14, 15], we define the subgroup  $\mathcal{K}(H) \stackrel{def}{=} \text{kernel}(V(RG) \rightarrow V(R(G/H)))$  where the homomorphism  $V(RG) \rightarrow V(R(G/H))$  is induced by the natural map (epimorphism)  $G \rightarrow G/H$ . Thus, evidently  $\mathcal{K}(H) = V(RG) \cap (1 + RG.I(H))$ , where  $I(H)$  denotes the augmentation ideal of  $RH$ , and  $RG.I(H) \stackrel{def}{=} I(RG; H)$  [17] denotes the relative augmentation ideal of  $RG$ , i.e.  $I(RG; H) = \langle h - 1 \mid h \in H \rangle \triangleleft RG$ . If  $x \in I(RG; H)$ , then  $x = \sum_{h \in H} x_{ah} \cdot (h - 1)$ ,  $x_{ah} \in RG$ ,  $a \in G$ , i.e.  $x_{ah} = \sum_{a \in G} \alpha_{ah} a$ ,  $\alpha_{ah} \in R$  and  $x = \sum_{h \in H} \sum_{a \in G} \alpha_{ah} a (h - 1) = \sum_{h \in H} \sum_{a \in G} \alpha_{ah} ah - \sum_{h \in H} \sum_{a \in G} \alpha_{ah} a = \sum_{g \in G} \alpha_g g$  and  $\sum_{g \in aH} \alpha_g = 0$ ,  $a \in G$ , i.e.  $x = \sum_{g \in G} \alpha_g g$ ,  $\alpha_g \in R$  and  $\sum_{g \in aH} \alpha_g = 0$  for every  $a \in G$  [17]. If  $H = G$ , then  $I(RG; G) = I(RG) = I(G)$  is the augmentation ideal of  $RG$ . If  $H = 1$ , then  $I(RG; H) = 0$ . Besides obviously  $V(RH) \leq \mathcal{K}(H)$ .

Let  $\bar{x} \in V(RG; H) \stackrel{def}{=} 1 + I(RG; H)$ , i.e.  $\bar{x} = 1 + x$ , where  $x = \sum_{g \in G} r_g g \in I(RG; H)$ ,  $r_g \in R$ ,  $\sum_{g \in aH} r_g = 0$  for each  $a \in G$ , i.e.  $\bar{x} = 1 + \sum_{g \in G} r_g g$ ,  $r_g \in R$ ,  $\sum_{g \in aH} r_g = 0$  for each  $a \in G$ , i.e.

$$(***) \quad \bar{x} = \sum_{g \in G} \bar{r}_g g, \bar{r}_g \in R \text{ and } \sum_{g \in aH} \bar{r}_g = \begin{cases} 1, & a \in H \\ 0, & a \notin H \end{cases} \text{ for each } a \in G.$$

Let  $H$  be an abelian  $p$ -group and  $\text{char}R = p$  be a prime number. Thus  $\mathcal{K}(H) = 1 + I(RG; H) \stackrel{def}{=} V(RG; H) \leq V(RG)$  is a  $p$ -group and consequently  $V(RG; H) = S(RG; H) \leq S(RG)$ . Besides if  $G = H$  then  $V(RG) = V(RG; G) = 1 + I(G) = \mathcal{K}(G)$  is a  $p$ -group (see also [14]).

The group  $V(RG; H)$  is being examined in the researches [14, 15], [18] and [17], but in the last two articles  $G$  is an abelian  $p$ -group,  $G \neq H$ .

The next lemma is proved in [17], for the case when  $G$  is an abelian  $p$ -group.

**Lemma 4.** *Let  $L$  be a subring of a commutative ring  $R$  with identity, let  $\text{char}R = p$  be prime, and let  $A$  and  $B$  be subgroups of an abelian group  $G$  such that  $A \cap B$  is  $p$ -torsion. Then*

$$(9) \quad V(RG; A) \cap V(LB) = V(LB; B \cap A).$$

**Proof.** Elementary we have that  $V(LB; B \cap A) \subseteq V(LB)$ ,  $V(LB; B \cap A) \subseteq V(RG; A)$  and hence  $V(LB; B \cap A) \subseteq V(RG; A) \cap V(LB)$ .

Let now  $x \in V(RG; A) \cap V(LB)$ , i.e.

$$x = \sum_{b \in B} x_b \cdot b, \quad x_b \in L \quad \text{and} \quad \sum_{b \in B} x_b = 1,$$

and  $\sum_{b \in \bar{b}A} x_b = \begin{cases} 1, & \bar{b} \in A \\ 0, & \bar{b} \notin A \end{cases}$  for each  $\bar{b} \in B$ . Besides,  $\bar{b}A \cap B = \bar{b}(A \cap B)$ ,

since  $\bar{b} \in B$ . Hence  $\sum_{b \in \bar{b}(A \cap B)} x_b = \begin{cases} 1, & \bar{b} \in A \cap B \\ 0, & \bar{b} \notin A \cap B \end{cases}$  for each  $\bar{b} \in B$ , i.e.  $x \in V(LB; B \cap A)$  and  $V(RG; A) \cap V(LB) \subseteq V(LB; B \cap A)$ . So, the lemma is true.  $\square$

If  $A \leq G$  and  $B \leq G$  and  $L \leq R$ , then  $V(RG; A) \cap V(LB) \subseteq V(LB; B \cap A)$ .

Let  $R$  be a commutative ring with identity and prime characteristic  $p$ . Nako Nachev in [17] shows that if  $B$  is a basic subgroup of the  $p$ -group  $G$ , then  $V(RG; B)$  is a direct sum of cyclic groups.

**Theorem 1.** *Let  $R$  be a commutative ring with identity of prime characteristic  $p$  and let  $H$  be a pure  $p$ -subgroup of the abelian group  $G$ . The group  $V(RG; H)$  is a direct sum of cyclic  $p$ -groups if and only if the group  $H$  is a direct sum of cyclic  $p$ -groups.*

**Proof.** If  $V(RG; H)$  is a direct sum of cyclic  $p$ -groups, then the same is  $H$ , because  $H \subseteq V(RG; H)$ . Now let  $H$  be a direct sum of cyclic groups. Thus from the criterion of Kulikov (cf. [11] and [7]),  $H = \bigcup_{n=1}^{\infty} M_n$ ,  $M_1 \subseteq \dots \subseteq M_n \subseteq \dots$  and  $M_n \cap H^{p^n} = 1$ . But therefore  $V(RG; H) = V(RG; \bigcup_{n=1}^{\infty} M_n) = \bigcup_{n=1}^{\infty} V(RG; M_n)$ . Indeed,  $V(RG; M_n) \subseteq V(RG; \bigcup_{n=1}^{\infty} M_n)$  for each  $n \in \mathbb{N}$  and consequently,  $\bigcup_{n=1}^{\infty} V(RG; M_n) \subseteq V(RG; \bigcup_{n=1}^{\infty} M_n)$ . Besides, let  $\bar{x} = \sum_{g \in G} \bar{r}_g g \in$

$$V(RG; \bigcup_{n=1}^{\infty} M_n). \quad \text{Hence} \quad \sum_{g \in a(\bigcup_{n=1}^{\infty} M_n)} \bar{r}_g = \begin{cases} 1, & a \in \bigcup_{n=1}^{\infty} M_n \\ 0, & a \notin \bigcup_{n=1}^{\infty} M_n \end{cases}, \quad \text{i.e.} \quad \sum_{g \in \bigcup_{n=1}^{\infty} (aM_n)} \bar{r}_g =$$

$$\begin{cases} 1, & a \in \bigcup_{n=1}^{\infty} M_n \\ 0, & a \notin \bigcup_{n=1}^{\infty} M_n \end{cases}, \quad \text{since} \quad a(\bigcup_{n=1}^{\infty} M_n) = \bigcup_{n=1}^{\infty} (aM_n), \quad \text{for every } a \in G. \quad \text{Finally,}$$

$\sum_{g \in aM_k} \bar{r}_g = \begin{cases} 1, & a \in M_k \\ 0, & a \notin M_k \end{cases}$  for any  $k \in \mathbb{N}$ , because  $a \notin \bigcup_{n=1}^{\infty} M_n$  if and only if  $a \notin M_n$  for every  $n \in \mathbb{N}$ . Therefore  $\bar{x} \in V(RG; M_k)$  for this  $k \in \mathbb{N}$ , i.e.  $\bar{x} \in$

$\bigcup_{n=1}^{\infty} V(RG; M_n)$  and  $V(RG; \bigcup_{n=1}^{\infty} M_n) \subseteq \bigcup_{n=1}^{\infty} V(RG; M_n)$ . Finally  $V(RG; \bigcup_{n=1}^{\infty} M_n) = \bigcup_{n=1}^{\infty} V(RG; M_n)$ . Moreover the heights in the group  $V(RG; H)$  of the elements of  $V(RG; M_n)$  are completely bounded for every  $n \in \mathbb{N}$ . This statement is valid, since Lemma 2 and Lemma 4 imply that,  $V(RG; M_n) \cap V^{p^n}(RG; H) \subseteq V(RG; M_n) \cap V^{p^n}(RG) = V(RG; M_n) \cap V(R^{p^n}G^{p^n}) = V(R^{p^n}G^{p^n}; M_n \cap G^{p^n}) = 1$ , because  $M_n \cap G^{p^n} = (M_n \cap H) \cap G^{p^n} = M_n \cap (H \cap G^{p^n}) = M_n \cap H^{p^n} = 1$ . Finally  $V(RG; M_n) \cap V^{p^n}(RG; H) = 1$  for each  $n \in \mathbb{N}$  and therefore from Kulikov's criterion,  $V(RG; H)$  is a direct sum of cyclic  $p$ -groups. This proves the theorem.  $\square$

**Remark.** The author has showed [4] more generally that  $V(RG; H)/H$  is a direct sum of cyclics, provided  $H$  is. Thus  $H$  is a direct factor of  $V(RG; H)$  with a direct sum of cyclics complement. The same assertion was suggested by the referee. The author wish to express his indebtedness to him for the helpful comments and conclusions.

**Corollary 1** (Molloy [16]). *Let  $R$  be a commutative ring with identity and with prime characteristic  $p$  and let  $G$  be an abelian  $p$ -group. The group  $V(RG)$  is a direct sum of cyclic groups if and only if the group  $G$  is a direct sum of cyclic groups.*

*Proof.* We can easily see that, the statement holds from Theorem 1 by  $H = G$ . So, the corollary is true.  $\square$

**Problem 1.** *Let  $H$  be  $p$ -torsion and  $H \leq G$ , where  $G$  is an abelian group, and let  $R$  be a commutative ring with identity of prime characteristic  $p$ . Then whether  $V(RG; H)$  is a direct sum of cyclic  $p$ -groups if and only if  $H$  is a direct sum of cyclic  $p$ -groups? However this is probably not true (when  $H$  is not pure in  $G$ ) in general.*

**Corollary 2.** *Let  $G$  be an abelian group and  $K$  be a commutative ring with identity and prime characteristic  $p$  without nilpotent elements. The group  $S(KG)$  is a direct sum of cyclic groups if and only if the group  $G_p$  is a direct sum of cyclic groups.*

*Proof.* It is well-known that,  $S(KG) = 1 + I(KG; G_p) = S(KG; G_p)$ . Indeed  $S(KG; G_p) \subseteq S(KG)$ . If now  $\bar{x} \in S(KG)$ , then  $\bar{x} = \sum_{g \in G} f_g g$ ,  $f_g \in K$ ,  $\sum_{g \in G} f_g = 1$ . Let  $\bar{x}^{p^i} = 1$  for any  $i \in \mathbb{N}$ , i.e.  $1 = \sum_{g \in G} f_g^{p^i} g^{p^i} = \sum_{g \in G_p} f_g^{p^i} g^{p^i} + \sum_{g \in G \setminus G_p} f_g^{p^i} g^{p^i} = \sum_{\substack{g \in G_p \\ g^{p^i} = 1}} f_g^{p^i} g^{p^i} + \sum_{\substack{g \in G_p \\ g^{p^i} \neq 1}} f_g^{p^i} g^{p^i} + \sum_{g \in G \setminus G_p} f_g^{p^i} g^{p^i}$ . Consequently

$$\sum_{g \in G \setminus G_p} f_g^{p^i} = \left( \sum_{g \in G \setminus G_p} f_g \right)^{p^i} = 0, \text{ i.e. } \sum_{g \in G \setminus G_p} f_g = 0, \text{ i.e. } \sum_{g \in gG_p} f_g = 0 \text{ and}$$

$\sum_{g \in G_p} f_g^{p^i} = \left( \sum_{g \in G_p} f_g \right)^{p^i} = 1, \text{ i.e. } \sum_{g \in G_p} f_g = 1.$  Finally  $\sum_{g \in aG_p} f_g = \begin{cases} 1, & a \in G_p \\ 0, & a \notin G_p \end{cases}$  for every  $a \in G$ . Thus  $\bar{x} \in S(KG; G_p)$  and it follows that  $S(KG) \subseteq S(KG; G_p)$ , i.e.  $S(KG) = S(KG; G_p)$ . Then the statement holds immediately from Theorem 1, where  $H = G_p$  since  $G_p$  is pure in  $G$ . Thus the proof of the corollary is completed.  $\square$

It can be seen trivial that if  $H$  is a  $p$ -group,  $H \leq G$ ,  $G$  is an abelian group and  $R$  is a commutative ring with identity and with prime characteristic  $p$ , then  $V(RG; H)$  is a bounded group if and only if  $H$  is a bounded group. Besides it is well to note that [5, 6] (cf. also [17]) if  $B$  is basic in  $p$ -torsion  $G$ , then  $V(RG; B)$  is basic in  $V(RG)$  provided  $R$  is perfect. This follows directly by virtue of Theorem 1 and other elementary conclusions.

### 2.3. Simply presentedness of the Sylow $p$ -subgroup of modular group algebra.

**Theorem 2.** *Let  $G$  be a torsion abelian group and  $K$  be a perfect commutative ring with identity of prime characteristic  $p$  without nilpotent elements (perfect field of characteristic  $p$ ). Then the group  $S(KG)$  is simply presented if and only if  $G_p$  is simply presented.*

**Proof.** It is well-known that,  $G = \prod_p G_p = G_p \times \prod_{q \neq p} G_q = G_p \times M$ , where  $q$  is a prime number and  $M = \prod_{q \neq p} G_q$  is a  $p$ -divisible group, i.e.  $M^p = M$ , because  $G_q^p = G_q$  for every prime  $q \neq p$ .

By [5, Proposition 8]  $S(KG) \cong S(KG_p) \times S((KG_p)M)$  and if  $S(KG)$  is simply presented, then  $S(KG_p)$  is simply presented as its direct factor. Hence from [14], we conclude that  $G_p$  is a simply presented group.

Now let  $G_p$  be simply presented. Again by [5],  $S(KG) \cong S(KM) \times S((KM)G_p)$ . But  $M_p = 1$  and Lemma 3 implies that,  $S(KM) = 1$ . Therefore  $S(KG) \cong S((KM)G_p)$ , where  $KM$  is a perfect commutative ring with 1, without nilpotent elements and  $\text{char} KM = p$ . By virtue of the same technique (in a slight modified variant) described in [14],  $S((KM)G_p)$  is simply presented, i.e.  $S(KG)$  is simply presented. So, the theorem is proved.  $\square$

**3. Isomorphism of commutative (modular) group algebras.** Now we shall present some assertions for the isomorphism problem of commutative modular group algebras of abelian  $p$ -groups and  $p$ -mixed abelian groups:

(10) (Berman, 1967 [1]). Let  $K$  be a field,  $\text{char}K = p > 0$  and  $G$  be a countable abelian  $p$ -group. If  $H$  is a group such that  $KH \cong KG$  as  $K$ -algebras, then  $H \cong G$ .

(11) (Berman–Mollov, 1969 [2]). Let  $K$  be a field,  $\text{char}K = p > 0$  and  $G$  be a direct sum of cyclic  $p$ -groups. If  $H$  is a group, then  $KH \cong KG$  as  $K$ -algebras if and only if  $H \cong G$ .

*Proof.* The isomorphism  $KG \cong KH$  implies  $V(KG) \cong V(KH)$  and by Corollary 1,  $H$  is a direct sum of cyclic  $p$ -groups. But  $KG \cong KH$  and therefore the Ulm–Kaplansky invariants of  $G$  and  $H$  are equal (see [2]). These invariants serve to classify the direct sums of cyclic  $p$ -groups and hence,  $G \cong H$ . The proof is finished.  $\square$

(12) (May, 1988 [14]). Let  $K$  be a field,  $\text{char}K = p > 0$  and  $G$  be a  $p$ -local Warfield abelian group. If  $H$  is a group such that  $KH \cong KG$  as  $K$ -algebras, then  $H \cong G$ .

(13) (May, 1988 [14]). Let  $K$  be a field,  $\text{char}K = p > 0$  and  $G$  be a simply presented abelian  $p$ -group. If  $H$  is a group, then  $KH \cong KG$  as  $K$ -algebras if and only if  $H \cong G$ .

**Definition 1** (Ullery, 1989 [19]). *The abelian  $p$ -group  $G$  is called  $\lambda$ -elementary if  $\lambda$  is a limit ordinal number and there exists a totally projective abelian  $p$ -group  $A$  such that  $G$  is  $\sigma$ -balanced (isotype and  $\sigma$ -nice) in  $A$  for all  $\sigma < \lambda$  and the factor-group  $A/G$  has a totally projective reduced part.*

(14) (Ullery, 1989 [19]). Let  $K$  be a field,  $\text{char}K = p > 0$  and  $G$  be an  $\lambda$ -elementary abelian  $p$ -group. If  $H$  is a group, then the  $K$ -isomorphism  $KH \cong KG$  implies  $H \cong G$ .

**Definition 2** (Ullery, 1990 [20]). *Let  $\mathcal{K}_1$  be a special class of abelian groups consisting all  $\mu$ -elementary abelian groups of Hill, where  $\mu$  is a limit ordinal and, all totally projective abelian groups.*

(15) (Ullery, 1990 [20]). Let  $K$  be a field,  $\text{char}K = p > 0$  and  $G$  be an abelian  $p$ -group of the class  $\mathcal{K}_1$ . If  $H$  is a group, then the  $K$ -isomorphism  $KH \cong KG$  implies  $H \cong G$ .

(16) (Karpilovsky, 1982 [9]). Let  $K$  be a field,  $\text{char}K = p > 0$  and  $G$  be a mixed abelian group such that  $tG$  is an algebraically compact  $p$ -group. Then the  $K$ -isomorphism  $KH \cong KG$  for some group  $H$  implies that  $H \cong G$ .

(17) (Ullery, 1992 [21]). Let  $K$  be a field,  $\text{char}K = p > 0$  and  $G$  be a mixed abelian group where  $tG$  is a countable  $p$ -group and the torsion free rank of  $G$  is 1. Then the  $K$ -isomorphism  $KH \cong KG$  for some group  $H$  implies that  $H \cong G$ .

Now we formulate the main results.

**3.1. Isomorphism of commutative (modular) group algebras of direct sums of cyclic groups.**

**Theorem 3** (ISOMORPHISM). *Let  $K$  be a field,  $\text{char}K = p > 0$ ,  $G$  be a splitting abelian group and  $tG$  be a direct sum of cyclic  $p$ -groups. Then  $KH \cong KG$  as  $K$ -algebras for some group  $H$  if and only if  $H \cong G$ .*

*Proof.* First, we obtain that  $tH$  is a  $p$ -group. By [14],  $V(KG)/\mathcal{K}(tG) \cong G/tG$  is a torsion-free group (see 2.2), hence  $tV(KG) \subseteq \mathcal{K}(tG)$ . But  $\mathcal{K}(tG)$  is a  $p$ -group (cf. again 2.2) and therefore  $tV(KG) = \mathcal{K}(tG)$  is a  $p$ -group. We may assume that  $KG = KH$  (or  $KG = KG'$ ,  $H \cong G' \leq V(KG)$ ). Consequently  $V(KG) = V(KH)$  and  $tV(KG) = tV(KH)$ . Then  $tV(KH)$  is a  $p$ -group and thus  $tH$  is a  $p$ -group, since  $tH \subseteq tV(KH)$ . Finally  $tG = G_p$  and  $tH = H_p$ . Besides  $KH = KG$  implies that, the Ulm-Kaplansky invariants of  $G_p$  and  $H_p$  are equal (see [12] or [9], [10]). But,  $U(KH) = U(KG)$  and  $S(KH) = U_p(KH) = U_p(KG) = S(KG)$  ( $S(KH) = tV(KH)$  and  $S(KG) = tV(KG)$ ). By Corollary 2,  $S(KG) = S(KH)$  is a direct sum of cyclic groups, i.e.  $H_p$  is one also. Hence  $tH \cong tG$ , since the invariants of Ulm-Kaplansky serve to classify the direct sums of cyclic groups. Moreover,  $tH$  is a direct sum of cyclic groups and by [15],  $H$  is a splitting group, i.e.  $H$  splits, because  $KH = KG$  splits, since  $G$  splits. Finally  $G \cong tG \times G/tG$  and  $H \cong tH \times H/tH$ . The  $K$ -isomorphism  $KH \cong KG$  implies  $H/tH \cong G/tG$  (see [12]). Therefore the isomorphism  $tH \cong tG$  is equivalent to  $H \cong G$ . This completes the proof of the theorem.  $\square$

The next theorem follows immediately from Theorem 3, since if  $G$  is a direct sum of cyclic groups, then  $G$  is a splitting group (cf. [11, p. 171]). But now we will obtain a new proof.

**Theorem 4** (ISOMORPHISM). *Let  $K$  be a field,  $\text{char}K = p > 0$ ,  $G$  be a direct sum of cyclic groups and  $tG$  be a  $p$ -group. Then  $KH \cong KG$  as  $K$ -algebras for some group  $H$  if and only if  $H \cong G$ .*

*Proof.* First, analogically to Theorem 3,  $tH = H_p$  is  $p$ -torsion. Secondly,  $tG$  is a direct sum of cyclic groups, since  $tG \subseteq G$  and then  $tH \cong tG$  by the fact that [15],  $V(KG) = G \times T$ , where  $T$  is a direct sum of cyclic  $p$ -groups, and hence  $V(KG) = V(KH)$  is a direct sum of cyclic groups, i.e.  $H$  is a direct sum of cyclic groups. Consequently,  $H \cong tH \times H/tH$  and  $G \cong tG \times G/tG$ . It was shown in [12] that, from  $KH \cong KG$  follows that  $H/tH \cong G/tG$ . Hence,  $G \cong H$ . This completes the proof of the theorem.  $\square$

We can see trivially that Theorem 4 implies (11). If  $tG$  is not a  $p$ -group, then probably  $H \not\cong G$ . It is interesting to know, what the full system of invariants

in this case are?

**Theorem 5** (ISOMORPHISM). *Let  $G$  be a direct sum of cyclic groups and  $H$  is a group. Then  $KH \cong KG$  as  $K$ -algebras over all fields  $K$  if and only if  $H \cong G$ .*

*Proof.* Clearly,  $G \cong tG \times G/tG$ . Also, it is known that (see [12, p. 148]) an isomorphism of  $KH$  and  $KG$  implies that  $G$  and  $H$  are isomorphic modulo their torsion subgroups, i.e.  $G/tG \cong H/tH$ . Since  $G/tG$  is a direct sum of cyclic groups (a free group), then the same is  $H/tH$  and from [7, p. 91, Theorem 14.4 or p. 143, Theorem 28.2]; [11],  $H \cong tH \times H/tH$ . Suppose that  $K_p$  is a field with  $\text{char}K_p = p \neq 0$ . Because  $G_p$  is a direct sum of cyclic groups and  $V_p(K_pG) = V_p(K_pH)$ , therefore  $H_p \subseteq V_p(K_pH)$  is a direct sum of cyclic groups by Corollary 2, for every prime  $p$ . Moreover,  $G_p$  and  $H_p$  have the same Ulm-Kaplansky invariants for each prime  $p$ . Thus,  $tG = \prod_p G_p \cong \prod_p H_p = tH$ , i.e.  $tG \cong tH$ , since  $G_p \cong H_p$  for all primes  $p$ . Finally,  $G \cong H$ . So, everything is proved.  $\square$

### 3.2. Isomorphism of commutative (modular) group algebras of simply presented torsion groups.

**Definition 3.** *The torsion abelian group  $G$  is said to be simply presented if all its  $p$ -primary components are simply presented (for all prime integers  $p$ ) — (see [8]).*

**Theorem 6** (ISOMORPHISM). *Let  $G$  be a simply presented torsion abelian group and  $H$  is a group. Then  $KH \cong KG$  as  $K$ -algebras over all fields  $K$  if and only if  $H \cong G$ .*

*Proof.* Let  $p$  be an arbitrary prime and let  $K_p$  be a field with  $\text{char}K_p = p > 0$ . Hence  $S(K_pH) \cong S(K_pG)$  and since  $G_p$  is simply presented, by Theorem 2  $H_p$  is simply presented because we may precisely assume that  $K_p$  is perfect. Therefore  $H_p \cong G_p$  for this  $p$ , because  $G_p$  and  $H_p$  have isomorphic divisible parts ([12] or [9, 10]) and the reduced simply presented  $p$ -groups are invariants of the functions of Ulm-Kaplansky (see [8]), and they are invariants of a commutative modular group algebra. Besides  $G/tG \cong H/tH$  (cf. [12]) and  $H$  is a torsion abelian group, i.e.  $H = tH$  since  $G$  is torsion, as  $G = tG$  and  $1 \cong H/tH$ . Furthermore,  $G = \prod_p G_p \cong \prod_p H_p = H$ , i.e. finally,  $G \cong H$ . This completes the proof of the theorem.  $\square$

**Proposition 1.** *Let  $K$  be a field,  $\text{char}K = p > 0$ , let  $G$  be a torsion abelian group and let  $G_p$  be simply presented. Then  $KH \cong KG$  as  $K$ -algebras*

for some group  $H$  implies  $H_p \cong G_p$ .

The proof is analogous to this of Theorem 6.

**3.3. Isomorphism of commutative (modular) group algebras of divisible groups.**

**Theorem 7 (ISOMORPHISM).** *Let  $G$  be a divisible abelian group and  $H$  is a group. Then  $KH \cong KG$  as  $K$ -algebras over all fields  $K$  if and only if  $H \cong G$ .*

*Proof.* Certainly,  $tG$  is divisible since  $tG$  is pure in  $G$  and hence  $G \cong tG \times G/tG$ . Similarly for  $G_p$ , i.e.  $G_p$  is a divisible group for each primes  $p$ . Suppose that again,  $K_p$  is a field and  $\text{char}K_p = p \neq 0$  assuming that  $K_p$  is perfect. Hence by Lemma 2,  $V^p(K_pG) = V(K_p^pG^p) = V(K_pG)$ , i.e.  $V(K_pG) = V(K_pH)$  is  $p$ -divisible, for every prime number  $p$ . Thus  $H$  is  $p$ -divisible as  $p$ -pure in  $V(K_pH)$ , for every  $p$ . Furthermore  $H$  and  $tH$  are divisible (see [7]). Similarly for  $H_p$ . Consequently  $H \cong tH \times H/tH$ . Suppose that,  $(K_pG)(p) \stackrel{\text{def}}{=} \{x \in K_pG \mid x^p = 0\}$  and  $(K_pH)(p) \stackrel{\text{def}}{=} \{y \in K_pH \mid y^p = 0\}$ . Evidently  $(K_pG)(p) \cong (K_pH)(p)$ . We well-know that,  $(K_pG)(p) = I(K_pG; G[p])$  and  $(K_pH)(p) = I(K_pH; H[p])$  (see [9] or [10]). Hence  $|I(K_pG; G[p])| = |I(K_pH; H[p])|$  and  $|G[p]| = |H[p]|$  (cf. [9] and [10]). But  $G[p]$  and  $H[p]$  are bounded and thus  $G[p] \cong H[p]$ . We see that,  $G_p[p] = G[p]$  and  $H_p[p] = H[p]$ . Furthermore,  $G_p \cong H_p$  (see [7, p. 126, Exercise 1]). Thus  $tG = \prod_p G_p \cong \prod_p H_p = tH$ . But  $G/tG \cong H/tH$  [12], and hence,  $G \cong H$ . So, the theorem is proved.  $\square$

**3.4. The isomorphism problem for commutative (modular) group algebras.** From (16) it follows that:

(18) Let  $K$  be a field,  $\text{char}K = p > 0$  and let  $G$  be a group with  $tG$  a divisible  $p$ -group. Then  $KH \cong KG$  as  $K$ -algebras for some group  $H$  if and only if  $H \cong G$ .

(19) Let  $K$  be a field,  $\text{char}K = p > 0$  and let  $G$  be a divisible group with  $tG$  a  $p$ -group. Then  $KH \cong KG$  as  $K$ -algebras for some group  $H$  if and only if  $H \cong G$ .

Evidently (18) and (19) hold, since  $tG$  is divisible as pure in  $G$ .

If  $tG$  is not a  $p$ -group, then probably  $H \not\cong G$ .

If  $G$  is algebraically compact (or cotorsion) and  $tG$  is  $p$ -torsion, then is  $H \cong G$ ? If  $tG$  is not a  $p$ -group, then probably  $H \not\cong G$ .

(20) Let  $K$  be a field,  $\text{char}K = p \neq 0$  and let  $G$  be a splitting abelian group with  $tG$  a countable  $p$ -group. Then  $KH \cong KG$  as  $K$ -algebras for some group  $H$  if and only if  $H \cong G$ .

*Proof.* Assume that  $K$  is perfect. The algebra  $KH \cong KG$  splits since  $G \cong tG \times G/tG$  splits. From [21],  $H$  is a direct factor of  $V(KH)$ , because  $tH \cong tG$  is a  $p$ -group (see again [21]). Hence (cf. [15]),  $H \cong tH \times H/tH$ . But  $H/tH \cong G/tG$  and finally,  $G \cong H$ . The statement is proved.  $\square$

(21) Let  $K$  be a field,  $\text{char}K = p \neq 0$  and let  $G$  be a splitting countable abelian group with  $tG$  a  $p$ -group. Then  $KH \cong KG$  as  $K$ -algebras for some group  $H$  if and only if  $H \cong G$ .

The proof is trivial by following immediately (20).

Of some interest and importance is the following

**Problem 2** (ISOMORPHISM PROBLEM). *Let  $K$  be a field of  $\text{char}K = p > 0$  and let  $G$  be a splitting abelian group such that  $tG$  is a  $p$ -group. Then  $KH \cong KG$  as  $K$ -algebras for some group  $H$  if and only if  $H \cong G$ .*

The proof of this problem splits to the following

Case 1)  $KH \cong KG$  implies  $tH \cong tG$ .

Case 2)  $V(KG) = G \times M$  for every abelian group  $G$  with  $tG$  a  $p$ -group, and hence by [15],  $G$  splits if and only if  $KG$  splits.

Case 3) We well-know that [12],  $KH \cong KG$  implies  $G/tG \cong H/tH$ .

If 1), 2) and 3) are valid, then  $KH \cong KG$  if and only if  $H \cong G$ . Indeed,  $KH = KG$  splits since  $G \cong tG \times G/tG$  splits. From Case 2),  $H$  is a direct factor of  $V(KH)$ , because  $tH \cong tG$  is a  $p$ -group. Hence (see [15]),  $H \cong tH \times H/tH$  and, therefore, finally by Case 1) and Case 3),  $G \cong H$ . So, everything is completely proved.

R. Brauer tags the following major problem (see [3, p. 112]): Whether the groups  $G_1$  and  $G_2$  are isomorphic ( $G_1 \cong G_2$ ) if the group algebras  $KG_1$  and  $KG_2$  are  $K$ -isomorphic ( $KG_1 \cong KG_2$ ) for all choices of the field  $K$ ? Again the problem for abelian groups is reduced to the following procedure:

4) If  $G$  is a torsion-free abelian group, this is true by a result of Higman (see also May [12]).

5) If  $G$  is a mixed abelian group, this is however not true by a result of May (see May [13]).

There exist two nonisomorphic mixed countable abelian groups  $G$  and  $H$  of torsion-free rank one ( $G$  does not split, but  $H$  splits) such that for all choices of the field  $K$ , the group algebras  $KG$  and  $KH$  are isomorphic, i.e.  $KG \cong KH$ , but  $G \not\cong H$ . As a corollary suppose that  $G$  is a countable splitting abelian group. Then when does  $KH \cong KG$  as  $K$ -algebras for some group  $H$  implies  $H \cong G$ ? Is this equivalent to the case when  $H$  is a splitting group?

If  $G$  is a countable group with torsion-free rank 1, when is  $H$  isomorphic to  $G$ ? Now let  $G$  be algebraically compact (or cotorsion). Then is  $H \cong G$ ?

6) If  $G$  is a torsion abelian group, this is probably true.

Certainly punkt 6) holds, if  $K_p H \cong K_p G$  implies  $H_p \cong G_p$  (for every prime  $p$ ), when  $G$  and  $H$  are arbitrary groups, as  $G$  is abelian and for the field  $K_p$ ,  $\text{char} K_p = p > 0$ .

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