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ISOMORPHISM OF COMMUTATIVE MODULAR GROUP ALGEBRAS*

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Abstract. Let K be a field of characteristic $p > 0$ and let G be a direct sum of cyclic groups, such that its torsion part is a p-group. If there exists a K-isomorphism $KH \cong KG$ for some group H, then it is shown that $H \cong G$.

Let G be a direct sum of cyclic groups, a divisible group or a simply presented torsion abelian group. Then $KH \cong KG$ as K-algebras for all fields K and some group H if and only if $H \cong G$.

1. Introduction. Let G be an abelian group, tG be its torsion subgroup and G_p be a p-primary component of G. Throughout this article R and K will denote commutative rings with identities and $U(R)$ will be the multiplicative group of a ring R.

Let us denote by $U(RG)$ and $U_p(RG)$ the unit group and its p-primary component (i. e. its Sylow p-subgroup), respectively and by $V(RG)$ and $V_p(RG) =$

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$S(RG)$ the group of normalized units (i. e. the units of augmentation 1) and its p-primary component (i. e. its normed Sylow p-subgroup) in a group algebra RG, respectively.

In this paper the groups $V(RG;H)$ (see 2.2) and $S(KG)$, and their decompositions into a restricted (bounded) direct product (i.e. a direct sum) of cyclic p-groups are being examined. Some criteria are obtained for $V(RG;H)$ and $S(KG)$ when they are direct sums of cyclic p-groups, and G is an arbitrary abelian group, H is a pure p-subgroup of G and R is an arbitrary ring, K is a ring without nilpotent elements, and $\mathrm{char}R = \mathrm{char}K = p$-prime number. The proofs are based on Kulikov's theorem (see [11, p. 144 and p. 550] or [7, p. 106, Theorem 17.1]).

Besides, the isomorphism problem for commutative modular group algebras is being discussed. Namely, we prove that the group algebra KG over a field K determines G up to isomorphism for the cases when:

$(*)$ G is a direct sum of cyclic groups, the torsion subgroup of which is a p-group, and $\mathrm{char}K = p > 0$.

$(**)$ G is a direct sum of cyclic groups, or a divisible group or a simply presented torsion group, and K is every field (every field of prime characteristic).

Thus, we conclude that KG determines the isomorphism class of the group G in cases $(*)$ and $(**)$, i.e. a full system of invariants of the K-algebra KG is the group G.

2. Unit groups in commutative modular group algebras.

2.1. Preliminary lemmas.

Lemma 1. *Let R be a commutative ring with identity and prime characteristic p.*

(1) *Let $r \in R$. Then $r \in U(R)$ if and only if $r^p \in U(R^p)$.*

(2) $$U^p(R) = U(R^p).$$

P r o o f. (1) Let $r \in U(R)$, i.e. does exist $\alpha \in R$ with $r\alpha = 1$. Hence $r^p.\alpha^p = 1$, i.e. $r^p \in U(R^p)$. Now let $r^p \in U(R^p)$, i.e. does exist $\beta \in R^p$ with $r^p.\beta = 1$, i.e. $r.r^{p-1}.\beta = 1$. Finally $r \in U(R)$.

(2) Let $x \in U^p(R)$, i.e. $x = \gamma^p$, $\gamma \in U(R)$. From (1), $\gamma^p \in U(R^p)$, i.e. $x \in U(R^p)$ and $U^p(R) \subseteq U(R^p)$. Now let $y \in U(R^p)$. Therefore does exist

$\delta \in R$ and $y = \delta^p$. But $\delta^p \in U(R^p)$ and by (1), $\delta \in U(R)$, i.e. $y \in U^p(R)$. Finally $U(R^p) \subseteq U^p(R)$ and the lemma is true. $\square$

Lemma 2. *Let R be a commutative ring with identity of prime characteristic p and let G be an abelian group. For every ordinal number σ we have:*

$$(RG)^{p^\sigma} = R^{p^\sigma} G^{p^\sigma}. \tag{3}$$

$$U^{p^\sigma}(RG) = U(R^{p^\sigma} G^{p^\sigma}). \tag{4}$$

$$V^{p^\sigma}(RG) = V(R^{p^\sigma} G^{p^\sigma}). \tag{5}$$

$$U_p^{p^\sigma}(RG) = U_p(R^{p^\sigma} G^{p^\sigma}). \tag{6}$$

$$S^{p^\sigma}(RG) = S(R^{p^\sigma} G^{p^\sigma}). \tag{7}$$

Proof. Let $\sigma = 1$. Further the proof goes on a standard way by means of a transfinite induction.

(3) is evidently. (4) Since $(RG)^p = R^p G^p$ by (3), then $U(R^p G^p) = U((RG)^p) = U^p(RG)$ from Lemma 1, because RG is a commutative ring with identity and char$RG = p$. (5) Certainly from (4), $V(R^p G^p) = U(R^p G^p) \cap V(RG) = U^p(RG) \cap V(RG) = V^p(RG)$, since $V(RG)$ is pure in $U(RG)$ as its direct factor. (6) follows immediately from (4). (7) follows immediately from (5). The lemma is proved. $\square$

Lemma 3. *Let G be an abelian group and K be a commutative ring with identity of prime characteristic p without nilpotent elements. Then*

$$S(KG) = 1 \text{ if and only if } G_p = 1\,. \tag{8}$$

Proof. If $S(KG) = 1$, then G_p=1, since $G_p \subseteq S(KG)$. Let $G_p = 1$, $c = \sum_{1 \le i \le n} \mu_i g_i \in S(KG)$ ($\mu_i \in K$, $g_i \in G$), $\sum_{1 \le i \le n} \mu_i = 1$ and $c^{p^m} = 1$ for any $m \in \mathbb{N}$. Therefore $\sum_{1 \le i \le n} \mu_i^{p^m} g_i^{p^m} = 1$. But $g_{j-1}^{p^m} \neq g_j^{p^m}$ (j=2, ..., n+1, g_{n+1}=g_1). Indeed, let $g_{j-1}^{p^m} = g_j^{p^m}$, i.e. $(g_{j-1}.g_j^{-1})^{p^m} = 1$, i.e. $g_{j-1}.g_j^{-1} \in G_p = 1$ and $g_{j-1} = g_j$ — a contradiction. Hence $g_1^{p^m} = 1$, i.e. $g_1 \in G_p = 1$ and $g_1 = 1$; $\mu_1^{p^m} = 1$, i.e. $(\mu_1 - 1)^{p^m} = 0$ and $\mu_1 = 1$; $\mu_2^{p^m} = \cdots = \mu_n^{p^m} = 0$, i.e. $\mu_2 = \cdots = \mu_n = 0$. Finally $c = 1$, i.e. $S(KG) = 1$. So, the lemma is proved. $\square$

2.2. Direct sums of cyclic groups of the Sylow p-subgroups of modular group algebra. Let H be a subgroup of an abelian group G, i.e. $H \leq$

G. Following May [14, 15], we define the subgroup $\mathcal{K}(H) \stackrel{def}{=} \mathrm{kernel}(V(RG) \to V(R(G/H)))$ where the homomorphism $V(RG) \to V(R(G/H))$ is induced by the natural map (epimorphism) $G \to G/H$. Thus, evidently $\mathcal{K}(H) = V(RG) \cap (1+RG.I(H))$, where $I(H)$ denotes the augmentation ideal of RH, and $RG.I(H) \stackrel{def}{=} I(RG; H)$ [17] denotes the relative augmentation ideal of RG, i.e. $I(RG;H) = \langle h-1 \,|\, h \in H\rangle \lhd RG$. If $x \in I(RG;H)$, then $x = \sum_{h\in H} x_{ah}.(h-1)$, $x_{ah} \in RG$, $a \in G$, i.e. $x_{ah} = \sum_{a\in G} \alpha_{ah}a$, $\alpha_{ah} \in R$ and $x = \sum_{h\in H}\sum_{a\in G} \alpha_{ah}a(h-1) = \sum_{h\in H}\sum_{a\in G} \alpha_{ah}ah - \sum_{h\in H}\sum_{a\in G} \alpha_{ah}a = \sum_{g\in G} \alpha_g g$ and $\sum_{g\in aH} \alpha_g = 0$, $a \in G$, i.e. $x = \sum_{g\in G} \alpha_g g$, $\alpha_g \in R$ and $\sum_{g\in aH} \alpha_g = 0$ for every $a \in G$ [17]. If $H = G$, then $I(RG;G) = I(RG) = I(G)$ is the augmentation ideal of RG. If $H = 1$, then $I(RG;H) = 0$. Besides obviously $V(RH) \le \mathcal{K}(H)$.

Let $\overline{x} \in V(RG;H) \stackrel{def}{=} 1 + I(RG;H)$, i.e. $\overline{x} = 1 + x$, where $x = \sum_{g\in G} r_g g \in I(RG;H)$, $r_g \in R$, $\sum_{g\in aH} r_g = 0$ for each $a \in G$, i.e. $\overline{x} = 1 + \sum_{g\in G} r_g g$, $r_g \in R$, $\sum_{g\in aH} r_g = 0$ for each $a \in G$, i.e.

$$(***) \qquad \overline{x} = \sum_{g\in G} \overline{r_g} g,\ \overline{r_g} \in R \ \text{ and } \sum_{g\in aH} \overline{r_g} = \begin{cases} 1, & a \in H \\ 0, & a \notin H \end{cases} \qquad \text{for each } a \in G.$$

Let H be an abelian p-group and $\mathrm{char}R = p$ be a prime number. Thus $\mathcal{K}(H) = 1 + I(RG;H) \stackrel{def}{=} V(RG;H) \le V(RG)$ is a p-group and consequently $V(RG;H) = S(RG;H) \le S(RG)$. Besides if $G = H$ then $V(RG) = V(RG;G) = 1 + I(G) = \mathcal{K}(G)$ is a p-group (see also [14]).

The group $V(RG;H)$ is being examined in the researches [14, 15], [18] and [17], but in the last two articles G is an abelian p-group, $G \neq H$.

The next lemma is proved in [17], for the case when G is an abelian p-group.

Lemma 4. *Let L be a subring of a commutative ring R with identity, let $\mathrm{char}R = p$ be prime, and let A and B be subgroups of an abelian group G such that $A \cap B$ is p-torsion. Then*

$$(9) \qquad V(RG;A) \cap V(LB) = V(LB; B\cap A)\,.$$

P r o o f. Elementary we have that $V(LB;B\cap A) \subseteq V(LB)$, $V(LB;B\cap A) \subseteq V(RG;A)$ and hence $V(LB;B\cap A) \subseteq V(RG;A) \cap V(LB)$.

Let now $x \in V(RG;A) \cap V(LB)$, i.e.

$$x = \sum_{b\in B} x_b.b, \ x_b \in L \ \text{ and } \ \sum_{b\in B} x_b = 1,$$

and $\sum\limits_{b\in \overline{b}A} x_b = \begin{cases} 1, & \overline{b} \in A \\ 0, & \overline{b} \notin A \end{cases}$ for each $\overline{b} \in B$. Besides, $\overline{b}A \cap B = \overline{b}(A \cap B)$, since $\overline{b} \in B$. Hence $\sum\limits_{b\in \overline{b}(A\cap B)} x_b = \begin{cases} 1, & \overline{b} \in A \cap B \\ 0, & \overline{b} \notin A \cap B \end{cases}$ for each $\overline{b} \in B$, i.e. $x \in V(LB; B\cap A)$ and $V(RG;A) \cap V(LB) \subseteq V(LB; B \cap A)$. So, the lemma is true. □

If $A \leq G$ and $B \leq G$ and $L \leq R$, then $V(RG;A) \cap V(LB) \subseteq V(LB; B \cap A)$.

Let R be a commutative ring with identity and prime characteristic p. Nako Nachev in [17] shows that if B is a basic subgroup of the p-group G, then $V(RG;B)$ is a direct sum of cyclic groups.

Theorem 1. *Let R be a commutative ring with identity of prime characteristic p and let H be a pure p-subgroup of the abelian group G. The group $V(RG;H)$ is a direct sum of cyclic p-groups if and only if the group H is a direct sum of cyclic p-groups.*

Proof. If $V(RG;H)$ is a direct sum of cyclic p-groups, then the same is H, because $H \subseteq V(RG;H)$. Now let H be a direct sum of cyclic groups. Thus from the criterion of Kulikov (cf. [11] and [7]), $H = \bigcup\limits_{n=1}^{\infty} M_n$, $M_1 \subseteq \cdots \subseteq M_n \subseteq \cdots$ and $M_n \cap H^{p^n} = 1$. But therefore $V(RG;H) = V(RG; \bigcup\limits_{n=1}^{\infty} M_n) = \bigcup\limits_{n=1}^{\infty} V(RG;M_n)$. Indeed, $V(RG;M_n) \subseteq V(RG; \bigcup\limits_{n=1}^{\infty} M_n)$ for each $n \in \mathbb{N}$ and consequently, $\bigcup\limits_{n=1}^{\infty} V(RG;M_n) \subseteq V(RG; \bigcup\limits_{n=1}^{\infty} M_n)$. Besides, let $\overline{x} = \sum\limits_{g\in G} \overline{r_g} g \in V(RG; \bigcup\limits_{n=1}^{\infty} M_n)$. Hence $\sum\limits_{g\in a(\bigcup_{n=1}^{\infty} M_n)} \overline{r_g} = \begin{cases} 1, & a \in \bigcup\limits_{n=1}^{\infty} M_n \\ 0, & a \notin \bigcup\limits_{n=1}^{\infty} M_n \end{cases}$, i.e. $\sum\limits_{g\in \bigcup_{n=1}^{\infty}(aM_n)} \overline{r_g} = \begin{cases} 1, & a \in \bigcup\limits_{n=1}^{\infty} M_n \\ 0, & a \notin \bigcup\limits_{n=1}^{\infty} M_n \end{cases}$, since $a(\bigcup\limits_{n=1}^{\infty} M_n) = \bigcup\limits_{n=1}^{\infty}(aM_n)$, for every $a \in G$. Finally, $\sum\limits_{g\in aM_k} \overline{r_g} = \begin{cases} 1, & a \in M_k \\ 0, & a \notin M_k \end{cases}$ for any $k \in \mathbb{N}$, because $a \notin \bigcup\limits_{n=1}^{\infty} M_n$ if and only if $a \notin M_n$ for every $n \in \mathbb{N}$. Therefore $\overline{x} \in V(RG;M_k)$ for this $k \in \mathbb{N}$, i.e. $\overline{x} \in$

$\bigcup_{n=1}^{\infty} V(RG; M_n)$ and $V(RG; \bigcup_{n=1}^{\infty} M_n) \subseteq \bigcup_{n=1}^{\infty} V(RG; M_n)$. Finally $V(RG; \bigcup_{n=1}^{\infty} M_n) = \bigcup_{n=1}^{\infty} V(RG; M_n)$. Moreover the heights in the group $V(RG; H)$ of the elements of $V(RG; M_n)$ are completely bounded for every $n \in \mathbb{N}$. This statement is valid, since Lemma 2 and Lemma 4 imply that, $V(RG; M_n) \cap V^{p^n}(RG; H) \subseteq V(RG; M_n) \cap V^{p^n}(RG) = V(RG; M_n) \cap V(R^{p^n} G^{p^n}) = V(R^{p^n} G^{p^n}; M_n \cap G^{p^n}) = 1$, because $M_n \cap G^{p^n} = (M_n \cap H) \cap G^{p^n} = M_n \cap (H \cap G^{p^n}) = M_n \cap H^{p^n} = 1$. Finally $V(RG; M_n) \cap V^{p^n}(RG; H) = 1$ for each $n \in \mathbb{N}$ and therefore from Kulikov's criterion, $V(RG; H)$ is a direct sum of cyclic p-groups. This proves the theorem. □

Remark. The author has showed [4] more generally that $V(RG; H)/H$ is a direct sum of cyclics, provided H is. Thus H is a direct factor of $V(RG; H)$ with a direct sum of cyclics complement. The same assertion was suggested by the referee. The author wish to express his indebtedness to him for the helpful comments and conclusions.

Corollary 1 (Mollov [16])**.** *Let R be a commutative ring with identity and with prime characteristic p and let G be an abelian p-group. The group $V(RG)$ is a direct sum of cyclic groups if and only if the group G is a direct sum of cyclic groups.*

P r o o f. We can easily see that, the statement holds from Theorem 1 by $H = G$. So, the corollary is true. □

Problem 1. *Let H be p-torsion and $H \leq G$, where G is an abelian group, and let R be a commutative ring with identity of prime characteristic p. Then whether $V(RG; H)$ is a direct sum of cyclic p-groups if and only if H is a direct sum of cyclic p-groups? However this is probably not true (when H is not pure in G) in general.*

Corollary 2. *Let G be an abelian group and K be a commutative ring with identity and prime characteristic p without nilpotent elements. The group $S(KG)$ is a direct sum of cyclic groups if and only if the group G_p is a direct sum of cyclic groups.*

P r o o f. It is well-known that, $S(KG) = 1 + I(KG; G_p) = S(KG; G_p)$. Indeed $S(KG; G_p) \subseteq S(KG)$. If now $\overline{x} \in S(KG)$, then $\overline{x} = \sum_{g \in G} f_g g$, $f_g \in K$, $\sum_{g \in G} f_g = 1$. Let $\overline{x}^{p^i} = 1$ for any $i \in \mathbb{N}$, i.e. $1 = \sum_{g \in G} f_g^{p^i} g^{p^i} = \sum_{g \in G_p} f_g^{p^i} g^{p^i} + \sum_{g \in G \setminus G_p} f_g^{p^i} g^{p^i} = \sum_{\substack{g \in G_p \\ g^{p^i} = 1}} f_g^{p^i} g^{p^i} + \sum_{\substack{g \in G_p \\ g^{p^i} \neq 1}} f_g^{p^i} g^{p^i} + \sum_{g \in G \setminus G_p} f_g^{p^i} g^{p^i}$. Consequently

$\sum\limits_{g\in G\setminus G_p} f_g^{p^i} = (\sum\limits_{g\in G\setminus G_p} f_g)^{p^i} = 0$, i.e. $\sum\limits_{g\in G\setminus G_p} f_g = 0$, i.e. $\sum\limits_{g\in gG_p} f_g = 0$ and $\sum\limits_{g\in G_p} f_g^{p^i} = (\sum\limits_{g\in G_p} f_g)^{p^i} = 1$, i.e. $\sum\limits_{g\in G_p} f_g = 1$. Finally $\sum\limits_{g\in aG_p} f_g = \begin{cases} 1, & a\in G_p \\ 0, & a\notin G_p \end{cases}$ for every $a\in G$. Thus $\overline{x}\in S(KG;G_p)$ and it follows that $S(KG)\subseteq S(KG;G_p)$, i.e. $S(KG)=S(KG;G_p)$. Then the statement holds immediately from Theorem 1, where $H=G_p$ since G_p is pure in G. Thus the proof of the corollary is completed. □

It can be seen trivial that if H is a p-group, $H\leq G$, G is an abelian group and R is a commutative ring with identity and with prime characteristic p, then $V(RG;H)$ is a bounded group if and only if H is a bounded group. Besides it is well to note that [5, 6] (cf. also [17]) if B is basic in p-torsion G, then $V(RG;B)$ is basic in $V(RG)$ provided R is perfect. This follows directly by virtue of Theorem 1 and other elementary conclusions.

2.3. Simply presentedness of the Sylow p-subgroup of modular group algebra.

Theorem 2. *Let G be a torsion abelian group and K be a perfect commutative ring with identity of prime characteristic p without nilpotent elements (perfect field of characteristic p). Then the group $S(KG)$ is simply presented if and only if G_p is simply presented.*

Proof. It is well-known that, $G=\prod\limits_p G_p = G_p\times\prod\limits_{q\neq p} G_q = G_p\times M$, where q is a prime number and $M=\prod\limits_{q\neq p} G_q$ is a p-divisible group, i.e. $M^p=M$, because $G_q^p=G_q$ for every prime $q\neq p$.

By [5, Proposition 8] $S(KG)\cong S(KG_p)\times S((KG_p)M)$ and if $S(KG)$ is simply presented, then $S(KG_p)$ is simply presented as its direct factor. Hence from [14], we conclude that G_p is a simply presented group.

Now let G_p be simply presented. Again by [5], $S(KG)\cong S(KM)\times S((KM)G_p)$. But $M_p=1$ and Lemma 3 implies that, $S(KM)=1$. Therefore $S(KG)\cong S((KM)G_p)$, where KM is a perfect commutative ring with 1, without nilpotent elements and $\mathrm{char}KM=p$. By virtue of the same technique (in a slight modified variant) described in [14], $S((KM)G_p)$ is simply presented, i.e. $S(KG)$ is simply presented. So, the theorem is proved. □

3. Isomorphism of commutative (modular) group algebras. Now we shall present some assertions for the isomorphism problem of commutative modular group algebras of abelian p-groups and p-mixed abelian groups:

(10) (Berman, 1967 [1]). Let K be a field, char$K = p > 0$ and G be a countable abelian p-group. If H is a group such that $KH \cong KG$ as K-algebras, then $H \cong G$.
(11) (Berman–Mollov, 1969 [2]). Let K be a field, char$K = p > 0$ and G be a direct sum of cyclic p-groups. If H is a group, then $KH \cong KG$ as K-algebras if and only if $H \cong G$.

Proof. The isomorphism $KG \cong KH$ implies $V(KG) \cong V(KH)$ and by Corollary 1, H is a direct sum of cyclic p-groups. But $KG \cong KH$ and therefore the Ulm–Kaplansky invariants of G and H are equal (see [2]). These invariants serve to classify the direct sums of cyclic p-groups and hence, $G \cong H$. The proof is finished. □
(12) (May, 1988 [14]). Let K be a field, char$K = p > 0$ and G be a p-local Warfield abelian group. If H is a group such that $KH \cong KG$ as K-algebras, then $H \cong G$.
(13) (May, 1988 [14]). Let K be a field, char$K = p > 0$ and G be a simply presented abelian p-group. If H is a group, then $KH \cong KG$ as K-algebras if and only if $H \cong G$.

Definition 1 (Ullery, 1989 [19]). *The abelian p-group G is called λ-elementary if λ is a limit ordinal number and there exists a totally projective abelian p-group A such that G is σ-balanced (isotype and σ-nice) in A for all $\sigma < \lambda$ and the factor-group A/G has a totally projective reduced part.*

(14) (Ullery, 1989 [19]). Let K be a field, char$K = p > 0$ and G be an λ-elementary abelian p-group. If H is a group, then the K-isomorphism $KH \cong KG$ implies $H \cong G$.

Definition 2 (Ullery, 1990 [20]). *Let $\mathcal{K}_1$ be a special class of abelian groups consisting all μ-elementary abelian groups of Hill, where μ is a limit ordinal and, all totally projective abelian groups.*

(15) (Ullery, 1990 [20]). Let K be a field, char$K = p > 0$ and G be an abelian p-group of the class $\mathcal{K}_1$. If H is a group, then the K-isomorphism $KH \cong KG$ implies $H \cong G$.
(16) (Karpilovsky, 1982 [9]). Let K be a field, char$K = p > 0$ and G be a mixed abelian group such that tG is an algebraically compact p-group. Then the K-isomorphism $KH \cong KG$ for some group H implies that $H \cong G$.
(17) (Ullery, 1992 [21]). Let K be a field, char$K = p > 0$ and G be a mixed abelian group where tG is a countable p-group and the torsion free rank of G is 1. Then the K-isomorphism $KH \cong KG$ for some group H implies that $H \cong G$.

Now we formulate the main results.

3.1. Isomorphism of commutative (modular) group algebras of direct sums of cyclic groups.

Theorem 3 (ISOMORPHISM). *Let K be a field, char$K = p > 0$, G be a splitting abelian group and tG be a direct sum of cyclic p-groups. Then $KH \cong KG$ as K-algebras for some group H if and only if $H \cong G$.*

Proof. First, we obtain that tH is a p-group. By [14], $V(KG)/\mathcal{K}(tG) \cong G/tG$ is a torsion–free group (see 2.2), hence $tV(KG) \subseteq \mathcal{K}(tG)$. But $\mathcal{K}(tG)$ is a p-group (cf. again 2.2) and therefore $tV(KG) = \mathcal{K}(tG)$ is a p-group. We may assume that $KG = KH$ (or $KG = KG'$, $H \cong G' \leq V(KG)$). Consequently $V(KG) = V(KH)$ and $tV(KG) = tV(KH)$. Then $tV(KH)$ is a p-group and thus tH is a p-group, since $tH \subseteq tV(KH)$. Finally $tG = G_p$ and $tH = H_p$. Besides $KH = KG$ implies that, the Ulm-Kaplansky invariants of G_p and H_p are equal (see [12] or [9], [10]). But, $U(KH) = U(KG)$ and $S(KH) = U_p(KH) = U_p(KG) = S(KG)$ $(S(KH) = tV(KH)$ and $S(KG) = tV(KG))$. By Corollary 2, $S(KG) = S(KH)$ is a direct sum of cyclic groups, i.e. H_p is one also. Hence $tH \cong tG$, since the invariants of Ulm-Kaplansky serve to classify the direct sums of cyclic groups. Moreover, tH is a direct sum of cyclic groups and by [15], H is a splitting group, i.e. H splits, because $KH = KG$ splits, since G splits. Finally $G \cong tG \times G/tG$ and $H \cong tH \times H/tH$. The K-isomorphism $KH \cong KG$ implies $H/tH \cong G/tG$ (see [12]). Therefore the isomorphism $tH \cong tG$ is equivalent to $H \cong G$. This completes the proof of the theorem. □

The next theorem follows immediately from Theorem 3, since if G is a direct sum of cyclic groups, then G is a splitting group (cf. [11, p. 171]). But now we will obtain a new proof.

Theorem 4 (ISOMORPHISM). *Let K be a field, char$K = p > 0$, G be a direct sum of cyclic groups and tG be a p-group. Then $KH \cong KG$ as K-algebras for some group H if and only if $H \cong G$.*

Proof. First, analogically to Theorem 3, $tH = H_p$ is p-torsion. Secondly, tG is a direct sum of cyclic groups, since $tG \subseteq G$ and then $tH \cong tG$ by the fact that [15], $V(KG) = G \times T$, where T is a direct sum of cyclic p-groups, and hence $V(KG) = V(KH)$ is a direct sum of cyclic groups, i.e. H is a direct sum of cyclic groups. Consequently, $H \cong tH \times H/tH$ and $G \cong tG \times G/tG$. It was shown in [12] that, from $KH \cong KG$ follows that $H/tH \cong G/tG$. Hence, $G \cong H$. This completes the proof of the theorem. □

We can see trivially that Theorem 4 implies (11). If tG is not a p-group, then probably $H \ncong G$. It is interesting to know, what the full system of invariants in this case are?

Theorem 5 (ISOMORPHISM). *Let G be a direct sum of cyclic groups and H is a group. Then $KH \cong KG$ as K-algebras over all fields K if and only if $H \cong G$.*

P r o o f. Clearly, $G \cong tG \times G/tG$. Also, it is known that (see [12, p. 148]) an isomorphism of KH and KG implies that G and H are isomorphic modulo their torsion subgroups, i.e. $G/tG \cong H/tH$. Since G/tG is a direct sum of cyclic groups (a free group), then the same is H/tH and from [7, p. 91, Theorem 14.4 or p. 143, Theorem 28.2]; [11], $H \cong tH \times H/tH$. Suppose that K_p is a field with $\mathrm{char} K_p = p \neq 0$. Because G_p is a direct sum of cyclic groups and $V_p(K_pG) = V_p(K_pH)$, therefore $H_p \subseteq V_p(K_pH)$ is a direct sum of cyclic groups by Corollary 2, for every prime p. Moreover, G_p and H_p have the same Ulm-Kaplansky invariants for each prime p. Thus, $tG = \prod_p G_p \cong \prod_p H_p = tH$, i.e. $tG \cong tH$, since $G_p \cong H_p$ for all primes p. Finally, $G \cong H$. So, everything is proved. □

3.2. Isomorphism of commutative (modular) group algebras of simply presented torsion groups.

Definition 3. *The torsion abelian group G is said to be simply presented if all its p-primary components are simply presented (for all prime integers p) — (see [8]).*

Theorem 6 (ISOMORPHISM). *Let G be a simply presented torsion abelian group and H is a group. Then $KH \cong KG$ as K-algebras over all fields K if and only if $H \cong G$.*

P r o o f. Let p be an arbitrary prime and let K_p be a field with $\mathrm{char} K_p = p > 0$. Hence $S(K_pH) \cong S(K_pG)$ and since G_p is simply presented, by Theorem 2 H_p is simply presented because we may precisely assume that K_p is perfect. Therefore $H_p \cong G_p$ for this p, because G_p and H_p have isomorphic divisible parts ([12] or [9, 10]) and the reduced simply presented p-groups are invariants of the functions of Ulm-Kaplansky (see [8]), and they are invariants of a commutative modular group algebra. Besides $G/tG \cong H/tH$ (cf. [12]) and H is a torsion abelian group, i.e. $H = tH$ since G is torsion, as $G = tG$ and $1 \cong H/tH$. Furthermore, $G = \prod_p G_p \cong \prod_p H_p = H$, i.e. finally, $G \cong H$. This completes the proof of the theorem. □

Proposition 1. *Let K be a field, $\mathrm{char} K = p > 0$, let G be a torsion abelian group and let G_p be simply presented. Then $KH \cong KG$ as K-algebras* *for some group H implies $H_p \cong G_p$.*

The proof is analogous to this of Theorem 6.

3.3. Isomorphism of commutative (modular) group algebras of divisible groups.

Theorem 7 (ISOMORPHISM). *Let G be a divisible abelian group and H is a group. Then $KH \cong KG$ as K-algebras over all fields K if and only if $H \cong G$.*

Proof. Certainly, tG is divisible since tG is pure in G and hence $G \cong tG \times G/tG$. Similarly for G_p, i.e. G_p is a divisible group for each primes p. Suppose that again, K_p is a field and $\mathrm{char}K_p = p \neq 0$ assuming that K_p is perfect. Hence by Lemma 2, $V^p(K_pG) = V(K_p^pG^p) = V(K_pG)$, i.e. $V(K_pG) = V(K_pH)$ is p-divisible, for every prime number p. Thus H is p-divisible as p-pure in $V(K_pH)$, for every p. Furthermore H and tH are divisible (see [7]). Similarly for H_p. Consequently $H \cong tH \times H/tH$. Suppose that, $(K_pG)(p) \overset{def}{=} \{x \in K_pG \,|\, x^p = 0\}$ and $(K_pH)(p) \overset{def}{=} \{y \in K_pH \,|\, y^p = 0\}$. Evidently $(K_pG)(p) \cong (K_pH)(p)$. We well-know that, $(K_pG)(p) = I(K_pG; G[p])$ and $(K_pH)(p) = I(K_pH; H[p])$ (see [9] or [10]). Hence $|I(K_pG; G[p])| = |I(K_pH; H[p])|$ and $|G[p]| = |H[p]|$ (cf. [9] and [10]). But $G[p]$ and $H[p]$ are bounded and thus $G[p] \cong H[p]$. We see that, $G_p[p] = G[p]$ and $H_p[p] = H[p]$. Furthermore, $G_p \cong H_p$ (see [7, p. 126, Exercise 1]). Thus $tG = \prod_p G_p \cong \prod_p H_p = tH$. But $G/tG \cong H/tH$ [12], and hence, $G \cong H$. So, the theorem is proved. □

3.4. The isomorphism problem for commutative (modular) group algebras. From (16) it follows that:

(18) Let K be a field, $\mathrm{char}K = p > 0$ and let G be a group with tG a divisible p-group. Then $KH \cong KG$ as K-algebras for some group H if and only if $H \cong G$.

(19) Let K be a field, $\mathrm{char}K = p > 0$ and let G be a divisible group with tG a p-group. Then $KH \cong KG$ as K-algebras for some group H if and only if $H \cong G$.

Evidently (18) and (19) hold, since tG is divisible as pure in G.

If tG is not a p-group, then probably $H \ncong G$.

If G is algebraically compact (or cotorsion) and tG is p-torsion, then is $H \cong G$? If tG is not a p-group, then probably $H \ncong G$.

(20) Let K be a field, $\mathrm{char}K = p \neq 0$ and let G be a splitting abelian group with tG a countable p-group. Then $KH \cong KG$ as K-algebras for some group H if and only if $H \cong G$.

P r o o f. Assume that K is perfect. The algebra $KH \cong KG$ splits since $G \cong tG \times G/tG$ splits. From [21], H is a direct factor of $V(KH)$, because $tH \cong tG$ is a p-group (see again [21]). Hence (cf. [15]), $H \cong tH \times H/tH$. But $H/tH \cong G/tG$ and finally, $G \cong H$. The statement is proved. □

(21) Let K be a field, char$K = p \neq 0$ and let G be a splitting countable abelian group with tG a p-group. Then $KH \cong KG$ as K-algebras for some group H if and only if $H \cong G$.

The proof is trivial by following immediately (20).

Of some interest and importance is the following

Problem 2 (ISOMORPHISM PROBLEM). *Let K be a field of char$K = p > 0$ and let G be a splitting abelian group such that tG is a p-group. Then $KH \cong KG$ as K-algebras for some group H if and only if $H \cong G$.*

The proof of this problem splits to the following

Case 1) $KH \cong KG$ implies $tH \cong tG$.

Case 2) $V(KG) = G \times M$ for every abelian group G with tG a p-group, and hence by [15], G splits if and only if KG splits.

Case 3) We well-know that [12], $KH \cong KG$ implies $G/tG \cong H/tH$.

If 1), 2) and 3) are valid, then $KH \cong KG$ if and only if $H \cong G$. Indeed, $KH = KG$ splits since $G \cong tG \times G/tG$ splits. From Case 2), H is a direct factor of $V(KH)$, because $tH \cong tG$ is a p-group. Hence (see [15]), $H \cong tH \times H/tH$ and, therefore, finally by Case 1) and Case 3), $G \cong H$. So, everything is completely proved.

R. Brauer tags the following major problem (see [3, p. 112]): Whether the groups G_1 and G_2 are isomorphic ($G_1 \cong G_2$) if the group algebras KG_1 and KG_2 are K-isomorphic ($KG_1 \cong KG_2$) for all choices of the field K? Again the problem for abelian groups is reduced to the following procedure:

4) If G is a torsion-free abelian group, this is true by a result of Higman (see also May [12]).

5) If G is a mixed abelian group, this is however not true by a result of May (see May [13]).

There exist two nonisomorphic mixed countable abelian groups G and H of torsion–free rank one (G does not split, but H splits) such that for all choices of the field K, the group algebras KG and KH are isomorphic, i.e. $KG \cong KH$, but $G \ncong H$. As a corollary suppose that G is a countable splitting abelian group. Then when does $KH \cong KG$ as K-algebras for some group H implies $H \cong G$? Is this equivalent to the case when H is a splitting group?

If G is a countable group with torsion-free rank 1, when is H isomorphic to G? Now let G be algebraically compact (or cotorsion). Then is $H \cong G$?

6) If G is a torsion abelian group, this is probably true.

Certainly punkt 6) holds, if $K_pH \cong K_pG$ implies $H_p \cong G_p$ (for every prime p), when G and H are arbitrary groups, as G is abelian and for the field K_p, $\mathrm{char} K_p = p > 0$.

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