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COMMUTING NONSELFADJOINT OPERATORS AND THEIR CHARACTERISTIC OPERATOR-FUNCTIONS*

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Abstract. In this paper we present some generalizations of results of M. S. Livšic [4, 6], concerning regular colligations $(A_1, A_2, H, \Phi, E, \sigma_1, \sigma_2, \gamma, \tilde{\gamma})$ $(\sigma_1 > 0)$ of a pair of commuting nonselfadjoint operators A_1, A_2 with finite dimensional imaginary parts, their complete characteristic functions and a class $\Omega(\sigma_1, \sigma_2)$ of operator-functions $W(x_1, x_2, z) : E \to E$ in the case of an inner function $W(1, 0, z)$ of the class $\Omega(\sigma_1)$. We consider regular colligations $(A_1, \dots, A_n, H, \Phi, E, \sigma_1, \dots, \sigma_n, \{\gamma_{k1}\}_2^n, \{\tilde{\gamma}_{k1}\}_2^n)$ $(\sigma_1 > 0)$ of n-tuples $(n > 2)$ of commuting nonselfadjoint operators $A_1, A_2, \dots, A_n$ with finite dimensional imaginary parts, their complete characteristic functions and a description of a class $\Omega_\Gamma(\sigma_1, \dots, \sigma_n)$ of operator-functions $W(x_1, \dots, x_n, z) : E \to E$ in the case when $W(1, 0, \dots, 0, z)$ is not inner function of the class $\Omega(\sigma_1)$ $(\sigma_1 > 0,\ n \geq 2)$. We essentially use the conditions for the operators $\{\sigma_k\}_1^n$, $\{\gamma_{k1}\}_2^n$, $\{\tilde{\gamma}_{k1}\}_2^n$ that V. A. Zolotarev has considered in [9].

Introduction. Let us remind some basic definitions and theorems from the theory of the operator colligations.

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Let H and E be Hilbert spaces.

Definition 1 [3]. *Let $A_1, A_2, \dots, A_n$ be bounded linear nonselfadjoint operators in H, $\sigma_1, \sigma_2, \dots, \sigma_n$ be bounded linear selfadjoint operators in E, Φ be a bounded linear mapping of H into E. A set*

$$X : (A_1, A_2, \dots, A_n, H, \Phi, E, \sigma_1, \sigma_2, \dots, \sigma_n)$$

is said to be a colligation if

$$(A_k - A_k^*)/i = \Phi^* \sigma_k \Phi, \quad k = 1, 2, \dots, n. \tag{1}$$

In the following statements we assume that $\dim E < +\infty$, $\bigcap_k \ker \sigma_k = \{0\}$ and the operator σ_1 is an invertible operator.

Definition 2 [4]. *The operator-function in E*

$$S(x_1, \dots, x_n, z) = I - i\Phi(x_1 A_1 + \dots + x_n A_n - zI)^{-1} \Phi^* (x_1 \sigma_1 + \dots + x_n \sigma_n) \tag{2}$$

where $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $z \in \mathbb{C}$, is said to be the complete characteristic function of the colligation X.

Definition 3 [1]. *An operator-function $W(l) : E \to E$ is said to be a function of the class $\Omega(\sigma_1)$ if it has the following properties:*

1) *$W(l)$ is a meromorphic function in the open upper half plane* $\operatorname{Im} l > 0$*;*
2) *$W(l)$ is holomorphic in a neighbourhood $|l| > a$ of $l = \infty$ and $W(\infty) = I$;*
3) *$W^*(l) \sigma_1 W(l) > \sigma_1$* $(\operatorname{Im} l > 0)$*;*
4) *$W^*(l) \sigma_1 W(l) = \sigma_1$* $(\operatorname{Im} l = 0)$*.*

Theorem 1 [1]. *Let σ_1 be a given selfadjoint invertible operator in E. A given operator-function $W(l)$ belongs to the class $\Omega(\sigma_1)$ iff $W(l)$ is the characteristic function of some single operator colligation $(A, H, \Phi, E, \sigma_1)$.*

A colligation X is said to be commutative if $A_k A_s = A_s A_k$, $k, s = 1, 2, \dots, n$.

Definition 4 [2]. *A commutative colligation*

$$X = (A_1, \dots, A_n, H, \Phi, E, \sigma_1, \dots, \sigma_n)$$

is said to be regular if there exists a set of selfadjoint operators $\{\gamma_{ks}\}$, $k, s = 1, 2, \ldots, n$, *which satisfy the conditions*

$$\sigma_k \Phi A_s^* - \sigma_s \Phi A_k^* = \gamma_{ks} \Phi, \quad k, s = 1, 2, \ldots, n \tag{3}$$

and $\gamma_{sk} = -\gamma_{ks}$.

The operators $\tilde{\gamma}_{ks}$, $k, s = 1, 2, \ldots, n$, defined by the relations

$$\tilde{\gamma}_{ks} = \gamma_{ks} + i(\sigma_k \Phi \Phi^* \sigma_s - \sigma_s \Phi \Phi^* \sigma_k),$$

satisfy the relations

$$\sigma_k \Phi A_s - \sigma_s \Phi A_k = \tilde{\gamma}_{ks} \Phi. \tag{4}$$

Let σ_1 is a positive operator. We use the results of V. Zolotarev [9] concerning the solutions of the corresponding open system of a regular colligation X

$$\begin{aligned} i\frac{\partial f(x)}{\partial x_k} + A_k f(x) &= \Phi^* \sigma_k u(x), \quad k = 1, 2, \ldots, n, \\ f(x)_{|\Gamma_+} &= f_0(x), \\ v(x) &= u(x) - i\Phi f(x), \end{aligned}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, $\Gamma_+ = \partial \mathbb{R}_+^n$, $f(x)$ is a vector-function in E. We can assume that for the commutative regular colligation X in the case of $n > 2$ are given only the selfadjoint operators in E

$$\gamma_{21}, \gamma_{31}, \ldots, \gamma_{n1}$$

that satisfy the conditions

$$\sigma_1^{-1}\sigma_k \sigma_1^{-1}\gamma_{s1} + \sigma_1^{-1}\gamma_{k1}\sigma_1^{-1}\sigma_s = \sigma_1^{-1}\gamma_{s1}\sigma_1^{-1}\sigma_k + \sigma_1^{-1}\sigma_s\sigma_1^{-1}\gamma_{k1}, \tag{5}$$

$$\sigma_1^{-1}\gamma_{k1}\sigma_1^{-1}s_{s1} = \sigma_1^{-1}\gamma_{s1}\sigma_1^{-1}\gamma_{k1}, \tag{6}$$

$$\sigma_k \Phi A_1^* - \sigma_1 \Phi A_k^* = \gamma_{k1}\Phi \tag{7}$$

$(k, s = 2, 3, \ldots, n)$ and the operators $\sigma_1, \sigma_2, \ldots, \sigma_n$ satisfy the conditions

$$\sigma_1^{-1}\sigma_k\sigma_1^{-1}\sigma_s = \sigma_1^{-1}\sigma_s\sigma_1^{-1}\sigma_k \quad (k, s = 2, 3, \ldots, n). \tag{8}$$

If we define the operators γ_{ks} and $\tilde{\gamma}_{ks}$ $(k, s = 1, 2, 3, \ldots, n)$ with the formulae

$$\begin{aligned} \gamma_{ks} &= \sigma_s \sigma_1^{-1} \gamma_{k1} - \sigma_k \sigma_1^{-1} \gamma_{s1}, \\ \tilde{\gamma}_{ks} &= \gamma_{ks} + i(\sigma_k \Phi\Phi^* \sigma_s - \sigma_s \Phi\Phi^* \sigma_k), \end{aligned} \tag{9}$$

then these operators satisfy the relations (3) and

$$\begin{aligned} \tilde{\gamma}_{ks} &= \tilde{\gamma}_{ks}^* = -\tilde{\gamma}_{sk}, \\ \sigma_k \Phi A_s - \sigma_s \Phi A_k &= \tilde{\gamma}_{ks} \Phi, \\ \sigma_1^{-1} \sigma_k \sigma_1^{-1} \tilde{\gamma}_{s1} + \sigma_1^{-1} \tilde{\gamma}_{k1} \sigma_1^{-1} \sigma_s &= \sigma_1^{-1} \tilde{\gamma}_{s1} \sigma_1^{-1} \sigma_k + \sigma_1^{-1} \sigma_s \sigma_1^{-1} \tilde{\gamma}_{k1}, \\ \sigma_1^{-1} \tilde{\gamma}_{k1} \sigma_1^{-1} \tilde{\gamma}_{s1} &= \sigma_1^{-1} \tilde{\gamma}_{s1} \sigma_1^{-1} \tilde{\gamma}_{k1}, \\ \tilde{\gamma}_{ks} &= \sigma_s \sigma_1^{-1} \tilde{\gamma}_{k1} - \sigma_k \sigma_1^{-1} \tilde{\gamma}_{s1} \end{aligned}$$

for all $k, s = 1, 2, \ldots, n$. We will include the operators $\{\gamma_{k1}\}_2^n$, $\{\tilde{\gamma}_{k1}\}_2^n$ in the notation of the regular colligation, i.e.

$$X = (A_1, A_2, \ldots, A_n, H, \Phi, E, \sigma_1, \sigma_2, \ldots, \sigma_n, \{\gamma_{k1}\}_2^n, \{\tilde{\gamma}_{k1}\}_2^n).$$

1. The output realisation of colligations. Let H and E are like above stated, let

$$X = (A_1, A_2, \ldots, A_n, H, \Phi, E, \sigma_1, \sigma_2, \ldots, \sigma_n, \{\gamma_{k1}\}_2^n, \{\tilde{\gamma}_{k1}\}_2^n)$$

be a commutative regular colligation, where σ_1, is an invertible operator, $\sigma_1 > 0$, the operators $\{\sigma_k\}_1^n$, $\{\gamma_{k1}\}_2^n$, $\{\tilde{\gamma}_{k1}\}_2^n$ satisfy the conditions (8), (5), (6), (7) and

$$\tilde{\gamma}_{k1} = \gamma_{k1} + i(\sigma_k \Phi\Phi^* \sigma_1 - \sigma_1 \Phi\Phi^* \sigma_k), \quad k = 2, 3, \ldots, n. \tag{10}$$

We denote

$$v_h(x_1, \ldots, x_n) = \Phi e^{i(x_1 A_1 + \cdots + x_n A_n)} h, \quad h \in H, \tag{11}$$

and the principal subspace of the colligation X

$$\hat{H} = \operatorname*{span}_{m_1, \ldots, m_n \in \mathbb{N}_0} \{A_1^{m_1} \ldots A_n^{m_n} \Phi^*(E)\}.$$

Let $\tilde{H}$ be a set of solutions (11) of the equations

$$\sigma_1 \frac{\partial v}{\partial x_k} - \sigma_k \frac{\partial v}{\partial x_1} + i\tilde{\gamma}_{k1} v = 0, \quad k = 2, 3, \dots, n, \tag{12}$$

and let an operator $U : \hat{H} \to \tilde{H}$ is defined by the equality

$$Uh = \Phi e^{i(x_1 A_1 + \dots + x_n A_n)} h = v_h(x_1, \dots, x_n) \quad (h \in \hat{H}). \tag{13}$$

From the existence of the limit

$$\lim_{t \to +\infty} (e^{itA_1} h, e^{itA_1} h), h \in \hat{H} \tag{14}$$

for the dissipative operator A_1 it follows that

$$(h, h) = \lim_{t \to +\infty} (e^{itA_1} h, e^{itA_1} h) + \int_0^{+\infty} (\sigma_1 \Phi e^{itA_1} h, \Phi e^{itA_1} h) dt. \tag{15}$$

Then using the equality (15) it follows that the formula

$$\begin{aligned} \langle v_{h_1}(x_1, \dots, x_n), v_{h_2}(x_1, \dots, x_n) \rangle &= \lim_{x_1 \to +\infty} (e^{itA_1} h_1, e^{itA_2} h_2) \\ &+ \int_0^{+\infty} (\sigma_1 v_{h_1}(x_1, 0, \dots, 0), v_{h_2}(x_1, 0, \dots, 0)) dx_1 \end{aligned} \tag{16}$$

defines a scalar product in $\tilde{H}$ and the operator U is an isometric one.

In [5, 6] M. S. Livšic has considered a commutative regular colligation $(A_1, A_2, H, \Phi, E, \sigma_1, \sigma_2)$ in the case of a dissipative operator A_1 satisfying the condition

$$\lim_{t \to +\infty} (e^{itA_1} h, e^{itA_1} h) = 0$$

for every h, $h \in H$.

The following proposition is a generalization of Theorem 2 in [5] in the case of a commutative regular colligation

$$X = (A_1, A_2, \dots, A_n, H, \Phi, E, \sigma_1, \sigma_2, \dots, \sigma_n, \{\gamma_{k1}\}_2^n, \{\tilde{\gamma}_{k1}\}_2^n)$$

for $n > 2$ and with an arbitrary dissipative operator A_1 (i.e. in the case of an arbitrary limit (14)).

Theorem 2. *Let*

$$X = (A_1, A_2, \dots, A_n, H, \Phi, E, \sigma_1, \sigma_2, \dots, \sigma_n, \{\gamma_{k1}\}_2^n, \{\tilde{\gamma}_{k1}\}_2^n)$$

be a commutative regular colligation with a positive and invertible operator σ_1 *and let* $\hat{H}$ *be the principal subspace of* X*. Then the colligation*

$$\hat{X} = (A_1, \dots, A_n, \hat{H}, \Phi, E, \sigma_1, \sigma_2, \dots, \sigma_n, \{\gamma_{k1}\}_2^n, \{\tilde{\gamma}_{k1}\}_2^n)$$

is unitary equivalent to the colligation

$$\tilde{X} = (\tilde{A}_1, \dots, \tilde{A}_n, \tilde{H}, \tilde{\Phi}, E, \sigma_1, \sigma_2, \dots, \sigma_n, \{\gamma_{k1}\}_2^n, \{\tilde{\gamma}_{k1}\}_2^n),$$

where $\tilde{A}_k = -i\partial/\partial x_k$*,* $k = 1, 2, \dots, n$*,* $\tilde{H}$ *is a set of solutions* (11) *of the equations* (12) *such that*

1) $\tilde{H}$ *is Hilbert space with respect to the scalar product* (16)*;*

2) *if* $v(x_1, \dots, x_n)$ *belongs to* $\tilde{H}$ *then* $\tilde{A}_k v(x_1, \dots, x_n)$ *belongs to* $\tilde{H}$ *for every* $k = 1, 2, \dots, n$*;*

3) *the operators* $\tilde{A}_1, \dots, \tilde{A}_n$ *are bounded in* $\tilde{H}$*.*

The next equality holds

$$\begin{aligned} &\lim_{\xi \to +\infty} \langle e^{i\xi \tilde{A}_1} v_h(x_1, \dots, x_n), e^{i\xi \tilde{A}_1} v_h(x_1, \dots, x_n) \rangle \\ &= \lim_{\xi \to +\infty} (e^{i\xi A_1} h, e^{i\xi A_1} h) \quad (v_h \in \tilde{H}). \end{aligned} \tag{17}$$

P r o o f. The scalar product (16) in $\tilde{H}$ and the equations (12) imply that the set $\tilde{H}$ and the operators $\tilde{A}_1, \dots, \tilde{A}_n$ satisfy the conditions 1), 2), 3). The operator $U : \hat{H} \to \tilde{H}$ defined by the equality (13), the form of the operators $\tilde{A}_1, \dots, \tilde{A}_n$ and the solutions $v_h(x_1, \dots, x_n)$ of the equations (12) show the relation between $\tilde{A}_k$ and A_k

$$\tilde{A}_k = U A_k U^* \quad (k = 1, 2, \dots, n)$$

and $\tilde{\Phi} = \Phi U^*$. Now using the conditions (1) and (7) for the operators $A_1, \dots, A_n$ of the commutative regular colligation $\hat{X}$ it follows that

$$\begin{aligned} (\tilde{A}_k - \tilde{A}_k^*)/i &= \tilde{\Phi}^* \sigma_k \tilde{\Phi}, \\ \sigma_k \tilde{\Phi} \tilde{A}_1^* - \sigma_1 \tilde{\Phi} \tilde{A}_k^* &= \gamma_{k1} \tilde{\Phi}, \\ \sigma_k \tilde{\Phi} \tilde{A}_1 - \sigma_1 \tilde{\Phi} \tilde{A}_k &= \tilde{\gamma}_{k1} \tilde{\Phi} \end{aligned}$$

for all $k = 1, 2, \dots, n$. Hence the set

$$\tilde{X} = (\tilde{A}_1, \dots, \tilde{A}_n, \tilde{H}, \tilde{\Phi}, E, \sigma_1, \dots, \sigma_n, \{\gamma_{k1}\}_2^n, \{\tilde{\gamma}_{k1}\}_2^n)$$

is a commutative regular colligation that is unitary equivalent to the colligation $\hat{X}$.

It is easy to see that the equality (17) follows from the relation

$$\frac{\partial}{\partial\xi}e^{i\xi\tilde{A}_1}v_h(x_1,\dots,x_n)=\frac{\partial}{\partial x_1}e^{i\xi\tilde{A}_1}v_h(x_1,\dots,x_n).$$

The functions $v_h(x_1,\dots,x_n)$ and $v_h(x_1,0,\dots,0)$ are said to be output representation and the mode of an element h correspondingly. If a mode $v_0(x_1)$ is given then the corresponding output representation $v(x_1,\dots,x_n)$ is defined uniquely by equations (12) and the condition $v(x_1,\dots,x_n)=v_0(x_1)$ in the region of an existence and an uniqueness of the solutions (see [8]).

Remarks. Analogously we can prove Theorem 2 for an arbitrary commutative regular colligation

$$(A_1,\dots,A_n,H,\Phi,E,\sigma_1,\dots,\sigma_n,\{\gamma_{ks}\}_1^n,\{\tilde{\gamma}_{ks}\}_1^n)\quad(\sigma_1>0)$$

using the conditions (3) and (4).

2. The consonance between operator-functions and linear manifolds. Let E be a finite dimensional Hilbert space. Let $\sigma_1,\sigma_2,\dots,\sigma_n$ be bounded linear selfadjoint operators in E, σ_1 be a positive and invertible operator, $\bigcap\limits_k \ker\sigma_k=\{0\}$, let $\{\gamma_{k1}\}_2^n$ be bounded linear selfadjoint operators in E, satisfying the conditions (8), (5) and (6), Let us denote by $M(l,\sigma,\gamma)$ ($l=(l_1,\dots,l_n)\in\mathbb{C}^n$, $\sigma=(\sigma_1,\dots,\sigma_n)$, $\gamma=(\gamma_{21},\dots,\gamma_{n1})$) a linear manifold of solutions u of the equations

$$(l_k\sigma_1-l_1\sigma_k+\gamma_{k1})u=0,k=2,3,\dots,n.$$

If $u\in M(l,\sigma,\gamma)$ then u satisfies the equations

$$(l_k\sigma_s-l_s\sigma_k+\gamma_{ks})u=0,\quad k,s=1,2,\dots,n, \tag{18}$$

where the selfadjoint operators $\{\gamma_{ks}\}$ are defined by (9).

Let us consider the algebraic curve

$$\Gamma(\sigma,\gamma)=\{(l_1,\dots,l_n)\in\mathbb{C}^n:\Gamma_{k1}^{(\sigma,\gamma)}(l_1,l_k)=0,\ k=2,3,\dots,n\},$$

where $\Gamma_{k1}^{(\sigma,\gamma)}(l_1,l_k)=\det(l_k\sigma_1-l_1\sigma_k+\gamma_{k1})$, $k=2,3,\dots,n$.

Let $W(x_1, \ldots, x_n, z) : E \to E$ be an operator-function, holomorphic in a region

$$K_a = \{(x_1, \ldots, x_n, z) \in \mathbb{C}^{n+1} : |z| > a(|x_1|^2 + \cdots + |x_n|^2)^{1/2}\}$$

$(a > 0)$ and homogeneous with respect to the variables $x_1, \ldots, x_n, z$:

$$W(tx_1, \ldots, tx_n, tz) = W(x_1, \ldots, x_n, z).$$

Definition 5. *The operator-function $W(x_1, \ldots, x_n, z)$ and the linear manifold $M(l, \sigma, \gamma)$ are said to be consonant with respect to the algebraic curve $\Gamma(\sigma, \gamma)$ if the restriction*

$$W(x_1, \ldots, x_n, l_1 x_1 + \ldots + l_n x_n)_{|M(l,\sigma,\gamma)}$$

to the linear manifold $M(l, \sigma, \gamma)$ does not depend on $(x_1, \ldots, x_n)$ when $(l_1, \ldots, l_n) \in \Gamma(\sigma, \gamma)$ $((x_1, \ldots, x_n, l_1 x_1 + \cdots + l_n x_n) \in K_a)$.

The operator-function

$$\hat{W}(l_1, \ldots, l_n) = W(x_1, \ldots, x_n, l_1 x_1 + \ldots + l_n x_n)_{|M(l,\sigma,\gamma)}$$

will be called the trunk of $W(x_1, \ldots, x_n, z)$ with respect to $M(l, \sigma, \gamma)$.

The next theorem gives an important property of the characteristic operator-function of a commutative regular colligation X.

Theorem 3. *Let $(A_1, \ldots, A_n, H, \Phi, E, \sigma_1, \ldots, \sigma_n, \{\gamma_{k1}\}_2^n, \{\tilde{\gamma}_{k1}\}_2^n)$ be a commutative regular colligation, where the operators $\{\sigma_k\}_1^n$, $\{\gamma_{k1}\}_2^n$ and $\{\tilde{\gamma}_{k1}\}_2^n$ satisfy the conditions* (8), (5), (6), (7). *Then the complete characteristic function* (2) *of the colligation and the linear manifold $M(l, \sigma, \gamma)$ are consonant with respect to the algebraic curve $\Gamma(\sigma, \gamma)$.*

P r o o f. Let $l = (l_1, \ldots, l_n) \in \Gamma(\sigma, \gamma)$. Then $\dim M(l, \sigma, \gamma) \neq 0$ (see [7]), i.e. there exists $u_0 \in M(l, \sigma, \gamma)$ such that

$$(l_k \sigma_1 - l_1 \sigma_k + \gamma_{k1}) u_0 = 0, \quad k = 2, 3, \ldots, n. \tag{19}$$

Using the conditions for the regular colligations (7) and (19) we obtain

$$(A_1 - l_1 I)\Phi^* \sigma_k u_0 = (A_k - l_k I)\Phi^* \sigma_1 u_0, \quad k = 2, 3, \ldots, n.$$

Then $(A_k - l_kI)^{-1}\Phi^*\sigma_k u_0 = (A_1 - l_1I)\Phi^*\sigma_1 u_0$ and it is easy to see that

$$(x_1A_1 + \cdots + x_nA_n - (l_1x_1 + \cdots + l_nx_n))^{-1}\Phi^*(x_1\sigma_1 + \cdots + x_n\sigma_n)$$

does not depend on $x_1, \ldots, x_n$ onto $M(l, \sigma, \gamma)$. Hence

$$\begin{aligned} &S(x_1, \ldots, x_n, l_1x_1 + \cdots + l_nx_n) \\ &= I - i\Phi(x_1A_1 + \cdots + x_nA_n - (l_1x_1 + \cdots + l_nx_n)I)^{-1}\Phi^*(x_1\sigma_1 + \cdots + x_n\sigma_n) \end{aligned}$$

onto $M(l, \sigma, \gamma)$ does not depend on $x_1, \ldots, x_n$ and the proof is completed. □

We shall consider some properties of an operator-function which is consonant to $M(l, \sigma, \gamma)$ with respect to $\Gamma(\sigma, \gamma)$ if the operator σ_1, is positive.

Theorem 4. *If an operator-function $W(x_1, \ldots, x_n, z) : E \to E$ and the linear manifold $M(l, \sigma, \gamma)$ are consonant with respect to $\Gamma(\sigma, \gamma)$ and σ_1 is a positive and invertible operator then $W(x_1, \ldots, x_n, z)$ is defined uniquely by its trunk $\hat{W}(l_1, \ldots, l_n)$ with respect to $M(l, \sigma, \gamma)$.*

Proof. Let $t_2, \ldots, t_n$ be real numbers such that $\sigma_1 + t_2\sigma_2 + \cdots + t_n\sigma_n > 0$. It is easy to see that

$$u = \bigcap_{k=2}^{n} \ker(l_k\sigma_1 - (z - l_2t_2 - \cdots - l_nt_n)\sigma_k + \gamma_{k1})$$

if and only if $u \in \bigcap_k \ker(l_k(\sigma_1 + t_2\sigma_2 + \cdots + t_n\sigma_n) - \sigma_kA + B\gamma_{k1})$, where $A = zI + t_2\sigma_1^{-1}\gamma_{21} + \cdots + t_n\sigma_1^{-1}\gamma_{n1}$, $B = (\sigma_1 + t_2\sigma_2 + \ldots + t_n\sigma_n)\sigma_1^{-1}$. Then the roots $l_k^{(1)}, l_k^{(2)}, \ldots, l_k^{(m_k)}$ of the equations

$$\det(l_k(\sigma_1 + t_2\sigma_2 + \cdots + t_n\sigma_n) - \sigma_kA + B\gamma_{k1}) = 0, \quad k = 1, 2, \ldots, n,$$

are real and the corresponding subspaces

$$\ker(l_k^{(p_k)}(\sigma_1 + t_2\sigma_2 + \cdots + t_n\sigma_n) - \sigma_kA + B\gamma_{k1}), \quad p_k = 1, 2, \ldots, m_k,$$

are orthogonal with respect to the positive operator $\sigma_1 + t_2\sigma_2 + \cdots + t_n\sigma_n$ for all $k = 2, 3, \ldots, n$ and the space E has the representation

$$\begin{aligned} E &= \sum_{p_2, p_3, \ldots, p_n} \oplus \bigcap_k \ker(l_k^{(p_k)}(\sigma_1 + t_2\sigma_2 + \cdots + t_n\sigma_n) - \sigma_kA + B\gamma_{k1}) \\ &= \sum_{p_2, p_3, \ldots, p_n} \oplus \bigcap_k \ker(l_k^{(p_k)}\sigma_1 - (z - l_2^{(p_2)}t_2 - \cdots - l_n^{(p_n)}t_n)\sigma_k + \gamma_{k1}). \end{aligned}$$

Then

$$\begin{aligned}
&W(1,t_2,\dots,t_n,z)\\
&\quad=\sum_{p_2,p_3,\dots,p_n=1}^{m_2,m_3,\dots,m_n} W(1,t_2,\dots,t_n,z)P^{(t_2,\dots,t_n)}(l_2^{(p_2)},\dots,l_n^{(p_n)},z)\\
&\quad=\sum_{p_2,p_3,\dots,p_n} \hat{W}(l_1^{(p_2,\dots,p_n)},l_2^{(p_2)},\dots,l_n^{(p_n)})P^{(t_2,\dots,t_n)}(l_2^{(p_2)},\dots,l_n^{(p_n)},z),
\end{aligned}$$

where $l^{(p2,\dots,p_n)}=z-l_2^{(p_2)}t_2-l_3^{(p_3)}t_3-\dots-l_n^{(p_n)}t_n$, $P^{(t_2,\dots,t_n)}(l_2^{(p_2)},\dots,l_n^{(p_n)},z)$ are the orthoprojectors with respect to the operator $\sigma_1+t_2\sigma_2+\dots+t_n\sigma_n>0$ onto $\bigcap\limits_k \ker(l_k^{(p_k)}\sigma_1-(z-l_2^{(p_2)}t_2-\dots-l_n^{(p_n)}t_n)\sigma_k+\gamma_{k1})$. Consequently for the operator-function $W(1,t_2,\dots,t_n)$ we obtain the following restoration formula

$$\begin{aligned}
&W(1,t_2,\dots,t_n)\\
&\quad=\sum_{p_2,p_3,\dots,p_n} \hat{W}(l_1^{(p_2,\dots,p_n)},l_2^{(p_2)},\dots,l_n^{(p_n)})P^{(t_2,\dots,t_n)}(l_2^{(p_2)},\dots,l_n^{(p_n)},z),
\end{aligned} \tag{20}$$

where $l^{(p2,\dots,p_n)}=z-l_2^{(p_2)}t_2-\dots-l_n^{(p_n)}t_n$. This completes the proof. □

Theorem 4 shows that the next corollaries concerning the relations between an operator-function $W(x_1,\dots,x_n,z)$ and its trunk $\hat{W}(l_1,\dots,l_n)$, the complete characteristic function of a regular colligation and its trunk (i.e. the joint characteristic function of a colligation), the complete characteristic function and its values at an arbitrary point, are true.

Corollary 1. *If the trunk $\hat{W}(l_1,\dots,l_n)$ of the operator-function $W(x_1,\dots,x_n,z)$ is isometric with respect to $\sigma_1,\dots,\sigma_n$ for real $l_1,\dots,l_n$ and $\sigma_1>0$ then $W(x_1,\dots,x_n,z)$ is isometric with respect to $x_1\sigma_1+\dots+x_n\sigma_n$ for real $x_1,\dots,x_n,z$ in the region K_a.*

Corollary 2. *The complete characteristic function $S(x_1,\dots,x_n,z)$ of a commutative regular colligation*

$$X=(A_1,A_2,\dots,A_n,H,\Phi,E,\sigma_1,\sigma_2,\dots,\sigma_n,\{\gamma_{k1}\}_2^n,\{\tilde{\gamma}_{k1}\}_2^n)$$

with $\sigma_1>0$ is uniquely determined by the joint characteristic function of this colligation

$$S(x_1,\dots,x_n,l_1x_1+\dots+l_nx_n)_{|M(l,\sigma,\gamma)}.$$

Corollary 3. *A complete characteristic function is determined uniquely by* $\sigma_1, \ldots, \sigma_n$, $\sigma_1 > 0$, $\gamma_{21}, \ldots, \gamma_{n1}$ *and its values* $S(x_1^0, \ldots, x_n^0, z)$ *at an arbitrary fixed point* $(x_1^0, \ldots, x_n^0)$.

Theorem 5. *If the operator-function* $W(x_1, \ldots, x_n, z) : E \to E$ *and the linear manifold* $M(l, \sigma, \gamma)$ *are consonant with respect to* $\Gamma(\sigma, \gamma)$ *and* $\sigma_1 > 0$ *then* $W(x_1, \ldots, x_n, z)$ *satisfies the following partial differential equations:*

$$\frac{\partial W}{\partial x_k} + \frac{\partial W}{\partial z}(x_1\sigma_1 + \cdots + x_n\sigma_n)^{-1}(z\sigma_k - x_1\gamma_{k1} - \cdots - x_n\gamma_{kn}) = 0, \quad k = 1, 2, \ldots, n,$$

when the operator $x_1\sigma_1 + \cdots + x_n\sigma_n$ *is an invertible one. (The operators* $\{\gamma_{ks}\}$ *are defined by the formulae* (9).*)*

Proof. Let $l \in \Gamma(\sigma, \gamma)$. Then

$$\left(\frac{\partial W}{\partial x_k} + l_k \frac{\partial W}{\partial z}\right)_{\Big| M(l,\sigma,\gamma)} = 0, \quad k = 1, 2, \ldots, n, \tag{21}$$

because the trunk $\hat{W}(l_1, \ldots, l_n) = W(x_1, \ldots, x_n, l_1x_1 + \ldots + l_nx_n)_{|M(l,\sigma,\gamma)}$ does not depend on $x_1, \ldots, x_n$ onto $M(l, \sigma, \gamma)$. Using the equations (18) we obtain

$$(l_k(x_1\sigma_1 + \cdots + x_n\sigma_n) - (l_1x_1 + \ldots + l_nx_n)\sigma_k + x_1\gamma_{k1} + \cdots + x_n\gamma_{kn})_{|M(l,\sigma,\gamma)} = 0$$

and

$$l_kI_{|M(l,\sigma,\gamma)} = (x_1\sigma_1 + \cdots + x_n\sigma_n)^{-1}(z\sigma_k - x_1\gamma_{k1} - \cdots - x_n\gamma_{kn})_{|M(l,\sigma,\gamma)} \tag{22}$$

for all $k = 1, 2, \ldots, n$, when the operator $(x_1\sigma_1 + \cdots + x_n\sigma_n)^{-1}$ exists. From (21) and (22) it follows that

$$\left(\frac{\partial \hat{W}}{\partial x_k} + \frac{\partial \hat{W}}{\partial z}(x_1\sigma_1 + \cdots + x_n\sigma_n)^{-1}(z\sigma_k - x_1\gamma_{k1} - \cdots - x_n\gamma_{kn})\right)_{\Big| M(l,\sigma,\gamma)} = 0,$$

where $z = l_1x_1 + \ldots + l_nx_n$. And now the restoration formula (20) and Theorem 4 show that

$$\frac{\partial W}{\partial x_k} + \frac{\partial W}{\partial z}(x_1\sigma_1 + \cdots + x_n\sigma_n)^{-1}(z\sigma_k - x_1\gamma_{k1} - \cdots - x_n\gamma_{kn}) = 0$$

onto the space E. The proof is completed. $\square$

3. Description of a class of characteristic functions. Let E, $\sigma_1, \sigma_2, \ldots, \sigma_n$, $\{\gamma_{k1}\}_2^n$, $(n \geq 2)$, $M(l, \sigma, \gamma)$, $\Gamma(\sigma, \gamma)$ are like in Section 2. By $\Delta_\Gamma(\sigma_1, \ldots, \sigma_n)$ we will denote the set of all $(n-1)$-tuples $\tilde{\eta} = (\tilde{\eta}_{21}, \ldots, \tilde{\eta}_{n1})$ of selfadjoint operators in E such that

$$\Gamma_{k1}^{(\sigma,\tilde{\eta})}(l_1, l_k) = 0, \quad k = 2, 3, \ldots, n$$

for any $l \in \Gamma(\sigma, \gamma)$ (i.e. $\Gamma(\sigma, \gamma) = \Gamma(\sigma, \tilde{\eta})$),

$$\sigma_1^{-1}\sigma_k\sigma_1^{-1}\tilde{\eta}_{s1} + \sigma_1^{-1}\tilde{\eta}_{k1}\sigma_1^{-1}\sigma_s = \sigma_1^{-1}\tilde{\eta}_{s1}\sigma_1^{-1}\sigma_k + \sigma_1^{-1}\sigma_s\sigma_1^{-1}\tilde{\eta}_{k1},$$

$$\sigma_1^{-1}\tilde{\eta}_{k1}\sigma_1^{-1}\tilde{\eta}_{s1} = \sigma_1^{-1}\tilde{\eta}_{s1}\sigma_1^{-1}\tilde{\eta}_{k1}, \quad k, s = 2, 3, \ldots, n.$$

We define the operators $\tilde{\eta}_{ks}$ $(k, s = 1, 2, \ldots, n)$ by the equalities

$$\tilde{\eta} = \sigma_s\sigma_1^{-1}\tilde{\eta}_{k1} - \sigma_k\sigma_1^{-1}\tilde{\eta}_{s1}. \tag{23}$$

Definition 6. *An operator-function $W(x_1, \ldots, x_n, z) : E \to E$ is said to be a function of the class $\Omega_\Gamma(\sigma_1, \ldots, \sigma_n)$ if $W(x_1, \ldots, x_n, z)$ satisfies the following conditions:*

1) *$W(x_1, \ldots, x_n, z)$ has the form*

$$W(x_1, \ldots, x_n, z) = I - iR(x_1, \ldots, x_n, z)(x_1\sigma_1 + \cdots + x_n\sigma_n), \tag{24}$$

where the function $R(x_1, \ldots, x_n, z)$ is holomorphic in a region

$$K_a = \{(x_1, \ldots, x_n, z) \in \mathbb{C}^{n+1} \; : \; |z| > (|x_1|^2 + \cdots + |x_n|^2)^{1/2}\}$$

$(a > 0)$ and $R(tx_1, \ldots tx_n, tz) = \dfrac{1}{t}R(x_1, \ldots, x_n, z)$;

2) *$W(x_1, \ldots, x_n, z)$ and the linear manifold $M(l, \sigma, \gamma)$ are consonant with respect to the algebraic curve $\Gamma(\sigma, \gamma)$ and the trunk*

$$\hat{W}(l_1, \ldots, l_n) = W(x_1, \ldots, x_n, l_1x_1 + \ldots + l_nx_n)_{|M(l,\sigma,\gamma)}$$

is an isometric mapping of $M(l, \sigma, \gamma)$ onto $M(l, \sigma, \tilde{\gamma})$ with respect to $\sigma_1, \ldots, \sigma_n$, where $\tilde{\gamma} \in \Delta_\Gamma(\sigma_1, \ldots, \sigma_n)$, $(x_1, \ldots, x_n, l_1x_1 + \ldots + l_nx_n) \in K_a$, $M(l, \sigma, \tilde{\gamma})$ is linear manifold of solutions u of equations $(l_k\sigma_1 - l_1\sigma_k + \tilde{\gamma}_{k1})u = 0$, $k = 2, 3, \ldots, n$;

3) *$W(1, 0, \dots, 0, z)$ belongs to the class $\Omega(\sigma_1)$.*

We need the next two propositions for the proof of the main result of this section.

Lemma 1. *Let $W(x_1, \dots, x_n, z) : E \to E$ be an operator-function that satisfies the conditions 1) and 2) from Definition 6 and $R(x_1, \dots, x_n, z)$ be an operator-function, defined by the equality (24), Then R satisfies the equations*

$$\frac{\partial R}{\partial x_k}\sigma_1 - \frac{\partial R}{\partial x_1}\sigma_k - \frac{\partial R}{\partial z}\gamma_{k1} = 0, \tag{25}$$

$$\sigma_1\frac{\partial R}{\partial x_k} - \sigma_k\frac{\partial R}{\partial x_1} - \tilde{\gamma}_{k1}\frac{\partial R}{\partial z} = 0, \tag{26}$$

($k = 2, 3, \dots, n$), where $\tilde{\gamma} \in \Delta_\Gamma(\sigma_1, \dots, \sigma_n)$, $x_1\sigma_1 + \dots + x_n\sigma_n$ is an invertible operator and $x_1, \dots, x_n \in \mathbb{R}$.

P r o o f. From Theorem 5, the representation (24) of the operator-function $W(x_1, \dots, x_n, z)$ and the equality that $R(x_1, \dots, x_n, z)$ satisfies

$$x_1\frac{\partial R}{\partial x_1} + \dots + x_n\frac{\partial R}{\partial x_n} + z\frac{\partial R}{\partial z} + R = 0, \tag{27}$$

we obtain

$$\frac{\partial R}{\partial x_k}(x_1\sigma_1 + \dots + x_n\sigma_n) - \left(x_1\frac{\partial R}{\partial x_1} + \dots + x_n\frac{\partial R}{\partial x_n}\right)\sigma_k$$
$$-\frac{\partial R}{\partial z}(x_1\gamma_{k1} + x_2\gamma_{k2}\cdots + x_n\gamma_{kn}) = 0.$$

Using the conditions (5), (6) and (8) it is easy to see that

$$\left(\frac{\partial R}{\partial x_k} - \frac{\partial R}{\partial x_1}\sigma_k\sigma_1^{-1}\right)(x_1\sigma_1 + \dots + x_n\sigma_n) - \frac{\partial R}{\partial z}\gamma_{k1}\sigma_1^{-1}(x_1\sigma_1 + \dots + x_n\sigma_n) = 0$$

($k = 2, 3, \dots, n$). Hence R satisfies the equations (25) when the operator $(x_1\sigma_1 + \dots + x_n\sigma_n)$ is an invertible one.

From the condition 2) of Definition 6 it follows that $W^{-1}(x_1, \dots, x_n, z)$ and $M(l, \sigma, \tilde{\gamma})$ are consonant with respect to the algebraic curve $\Gamma(\sigma, \gamma)$ where $\tilde{\gamma} \in \Delta_\Gamma(\sigma_1, \dots, \sigma_n)$. Now Theorem 5 says that the operator-function $W^{-1}(x_1, \dots, x_n, z)$ satisfies the next equations:

$$\frac{\partial W^{-1}}{\partial x_k} + \frac{\partial W^{-1}}{\partial z}(x_1\sigma_1 + \dots + x_n\sigma_n)^{-1}(z\sigma_k - x_1\tilde{\gamma}_{k1} - \dots - x_n\tilde{\gamma}_{kn}) = 0 \tag{28}$$

($k = 1, 2, \ldots, n$) where the operators $\{\tilde{\gamma}_{ks}\}_1^n$ are defined by the equalities from the form (23)

$$\tilde{\gamma}_{ks} = \sigma_s \sigma_1^{-1} \tilde{\gamma}_{k1} - \sigma_k \sigma_1^{-1} \tilde{\gamma}_{s1}. \tag{29}$$

Suppose that $x_1, \ldots, x_n, z$ are real and $(x_1, \ldots, x_n, z) \in K_a$, from the Corollary 1 we have

$$W^{-1}(x_1, \ldots, x_n, z) = I + iR^*(x_1, \ldots, x_n, z)(x_1\sigma_1 + \cdots + x_n\sigma_n).$$

Hence

$$(x_1\sigma_1 + \cdots + x_n\sigma_n)\frac{\partial R}{\partial x_k} = (z\sigma_k - x_1\tilde{\gamma}_{k1} - \cdots - x_n\tilde{\gamma}_{kn})\frac{\partial R}{\partial z} + \sigma_k R = 0 \tag{30}$$

because the equalities (28) are true. Then using (28) and (30) we obtain

$$\begin{gathered}(x_1\sigma_1 + \cdots + x_n\sigma_n)\frac{\partial R}{\partial x_k} - (x_1\tilde{\gamma}_{k1} + \cdots + x_n\tilde{\gamma}_{kn})\frac{\partial R}{\partial z} \\ -\sigma_k\left(x_1\frac{\partial R}{\partial x_1} + \cdots + x_n\frac{\partial R}{\partial x_n}\right) = 0\end{gathered} \tag{31}$$

($k = 1, 2, \ldots, n$). Analogously from the relations (31) using (8), (29) we obtain the equations (26) and the lemma is proved. □

Let $W(x_1, \ldots, x_n, z) : E \to E$ be an operator-function that satisfies the conditions 1) and 2) of Definition 6 and let $R(x_1, \ldots, x_n, z)$ is defined by the equality (24). Now we introduce the operator-function

$$V(x_1, \ldots, x_n) = -\frac{1}{2\pi i}\int_{|z|=r} e^{iz} R(x_1, \ldots, x_n, z)dz, \tag{32}$$

where $r > a(|x_1|^2 + \cdots + |x_n|^2)^{1/2}$. $V(x_1, \ldots, x_n)$ is an entire function because $R(x_1, \ldots, x_n, z)$ is holomorphic in K_a. Let $\tilde{\gamma} = (\tilde{\gamma}_{21}, \ldots, \tilde{\gamma}_{n1}) \in \Delta_\Gamma(\sigma_1, \ldots, \sigma_n)$. Then it is easy to see that from Lemma 1 immediately follows the next proposition:

Lemma 2. *The operator-function $V(x_1, \ldots, x_n) : E \to E$, defined by (32), satisfies the next equations*

$$\sigma_1\frac{\partial V}{\partial x_k} - \sigma_k\frac{\partial V}{\partial x_1} + i\tilde{\gamma}_{k1}V = 0 \tag{33}$$

$$\frac{\partial V}{\partial x_k}\sigma_1 - \frac{\partial V}{\partial x_1}\sigma_k + iV\gamma_{k1} = 0, \quad k = 2, 3, \ldots, n. \tag{34}$$

In [4] M. S. Livšic has described a class $\Omega(\sigma_1, \sigma_2)$ $(\sigma_1 > 0)$ of operator-functions $W(x_1, x_2, z)$ in the case of an inner function $W(1, 0, z)$ of the class $\Omega(\sigma_1)$. The following theorem is a generalization of Theorem 10 in [4] in the case of a class $\Omega_\Gamma(\sigma_1, \dots, \sigma_n)$ $(\sigma_1 > 0)$ of operator-functions $W(x_1, \dots, x_n, z)$ when $n > 2$ and $W(1, 0, \dots, 0, z)$ is not inner function.

Theorem 6. *If a given operator-function* $W(x_1, \dots, x_n, z) : E \to E$ *belongs to the class* $\Omega_\Gamma(\sigma_1, \dots, \sigma_n)$ *with* $\sigma_1 > 0$ *then* $W(x_1, \dots, x_n, z)$ *is a complete characteristic function of a regular colligation*

$$X = (A_1, A_2, \dots, A_n, H, \Phi, E, \sigma_1, \sigma_2, \dots, \sigma_n, \{\gamma_{k1}\}_2^n, \{\tilde{\gamma}_{k1}\}_2^n).$$

Proof. Let $W(x_1, \dots, x_n, z) : E \to E$ is an arbitrary operator-function of the class $\Omega_\Gamma(\sigma_1, \dots, \sigma_n)$. Then $W(1, 0, \dots, 0, z)$ belongs to the class $\Omega(\sigma_1)$ $(\sigma_1 > 0)$ because the condition 3) holds. From Theorem 1 it follows that there exists a dissipative colligation $(A_1, H, \Phi, E, \sigma_1)$ with a characteristic operator-function $W(1, 0, \dots, 0, z)$, i.e. $W(1, 0, \dots, 0, z)$ has the representation

$$W(1, 0, \dots, 0, z) = I - i\Phi(A_1 - zI)^{-1}\Phi^*\sigma_1. \tag{35}$$

Then the operator-function $R(x_1, \dots, x_n, z)$ from the condition 1) of Definition 6 satisfies the condition

$$R(1, 0, \dots, 0, z) = \Phi(A_1 - zI)^{-1}\Phi^*.$$

Hence

$$V(x_1, 0, \dots, 0) = -\frac{1}{2\pi i}\int_{|z|=r} e^{iz} R(x_1, 0, \dots, 0, z)dz = \Phi e^{ix_1 A_1}\Phi^*,$$

where the operator-function $V(x_1, \dots, x_n)$ is defined by (32). Let $\tilde{\gamma} = (\tilde{\gamma}_{21}, \dots, \tilde{\gamma}_{n1}) \in \Delta_\Gamma(\sigma_1, \dots, \sigma_n)$ and the operators $\tilde{\gamma}_{ks}$, $k, s = 1, 2, \dots, n$, are defined by equalities from the form (23)

$$\tilde{\gamma}_{ks} = \sigma_s\sigma_1^{-1}\tilde{\gamma}_{k1} - \sigma_k\sigma_1^{-1}\tilde{\gamma}_{s1}.$$

Now we consider a set $\tilde{H}$ of solutions $y(x_1, \dots, x_n)$ of the equations

$$\sigma_1\frac{\partial y}{\partial x_k} - \sigma_k\frac{\partial y}{\partial x_1} + i\tilde{\gamma}_{k1}y = 0, \quad k = 2, 3, \dots, n \tag{36}$$

satisfying the conditions

$$y_h(x_1,\dots,x_n)_{|x=(x_1,0,\dots,0)} = \Phi e^{ix_1A_1}h, \tag{37}$$

($x_1 > 0$, $\operatorname{supp} y_h(x_1) \subseteq \mathbb{R}_+$), where $h \in \hat{H} = \operatorname*{span}_{n\in\mathbb{N}_0}\{A_1^n, \Phi^*(E)\}$ ($\hat{H}$ is the principal subspace of the colligation $(A_1, H, \Phi, E, \sigma_1)$), $y = y(x_1,\dots,x_n)$ is a vector-function in Hilbert space E. But the region in $\mathbb{R}^n$

$$K = \{x = (x_1,\dots,x_n) \in \mathbb{R}^n \ : \ \sum_{k=1}^{n} x_k\sigma_k > 0\},$$

containing the set $\{x_1 \geq 0, x_2 = \dots = x_n = 0\}$ (see [8]) is a region of an existence and an uniqueness of the solutions y_h of the problem (36), (37) and

$$V(x_1,\dots,x_n)g = y_{\Phi^*g}(x_1,\dots,x_n)$$

for any $g \in E$.

Using the existence of the limit $\lim_{\xi\to+\infty}(e^{i\xi A_1}h, e^{i\xi A_1}h)$ ($h \in \hat{H}$) for the dissipative colligation $(A_1, H, \Phi, E, \sigma_1)$ and the uniqueness of the solutions of the problem (36), (37) in the region K we introduce a scalar product in $\tilde{H}$ by the formula

$$\begin{aligned} \langle y_{h_1}(x_1,\dots,x_n), y_{h_2}(x_1,\dots,x_n)\rangle &= \lim_{\xi\to+\infty}(e^{i\xi A_1}h_1, e^{i\xi A_1}h_2) \\ &+ \int_0^{+\infty}(\sigma_1 y_{h_1}(x_1,0,\dots,0), y_{h_2}(x_1,0,\dots,0))dx_1 \end{aligned} \tag{38}$$

($y_{h_1}, y_{h_2} \in \tilde{H}$). It is clear that

$$\int_0^{+\infty}(\sigma_1 y_h(x_1,0,\dots,0), y_h(x_1,0,\dots,0))dx_1 = (h,h) - \lim_{\xi\to+\infty}(e^{i\xi A_1}h_1, e^{i\xi A_1}h_2). \tag{39}$$

According to (38) and (39) we obtain

$$\langle y_h(x_1,\dots,x_n), y_h(x_1,\dots,x_n)\rangle_{\tilde{H}} = (h,h)_{\hat{H}}.$$

Then the mapping $T : \hat{H} \to \tilde{H}$, defined by the equality

$$Th = y_h(x_1,\dots,x_n),$$

where $y_h(x_1,\ldots,x_n)$ is a solution of the problem (36), (37), is an isometric mapping of $\hat{H}$ onto $\tilde{H}$ because

$$\langle Th_1, Th_2\rangle = \langle y_{h_1}(x), y_{h_2}(x)\rangle = (h_1,h_2), \quad (x\in\mathbb{R}^n).$$

Now we consider the operators $\tilde{\Phi}=\Phi T^{-1}$ and $\tilde{A}_1 = TA_1T^{-1}$. Then $\tilde{A}_1 y_h(x) = y_{A_1h}(x)$ ($h\in\hat{H}$, $x\in\mathbb{R}^n$) and $\tilde{A}_1 = -i\dfrac{\partial}{\partial x_1}$ in $\tilde{H}$. The colligation $(A_1,\hat{H},\Phi,E,\sigma_1)$ is unitary equivalent to the colligation $(\tilde{A}_1,\tilde{H},\tilde{\Phi},E,\sigma_1)$. As $\hat{H}=\operatorname*{span}_{t\in\mathbb{R}}\{e^{itA_1}\Phi^*(E)\}$, so $\tilde{H}=\operatorname*{span}_{t\in\mathbb{R}}\{e^{it\tilde{A}_1}\tilde{\Phi}^*(E)\}$. But

$$\begin{aligned}\operatorname*{span}_{t\in\mathbb{R}}\{e^{it\tilde{A}_1}\tilde{\Phi}^*(E)\} &= \operatorname*{span}_{t\in\mathbb{R}}\{e^{it\tilde{A}_1}T\Phi^*(E)\}\\ &= \operatorname*{span}_{t\in\mathbb{R}}\{e^{it\tilde{A}_1}V(x_1,\ldots,x_n)(E)\}\\ &= \operatorname*{span}_{t\in\mathbb{R}}\{V(x_1+t,x_2,\ldots,x_n)(E)\},\end{aligned}$$

because $\dfrac{\partial}{\partial t}(e^{it\tilde{A}_1}V(x_1,\ldots,x_n)) = \dfrac{\partial}{\partial x_1}(e^{it\tilde{A}_1}V(x_1,\ldots,x_n))$, i.e.

$$\tilde{H}=\operatorname*{span}_{t\in\mathbb{R}}\{V(x_1+t,x_2,\ldots,x_n)(E)\}.$$

We introduce the operators $\tilde{A}_2=-i\dfrac{\partial}{\partial x_2},\ldots,\tilde{A}_n=-i\dfrac{\partial}{\partial x_n}$ in the space $\tilde{H}$. Using Lemma 2 we obtain that the operator-function

$$e^{it\tilde{A}_1}V(x_1,\ldots,x_n)=V(x_1+t,x_2,\ldots,x_n)$$

is a solution of the equations (34). Then $\tilde{H}$ is an invariant space with respect to the operators $\tilde{A}_k$, $k=2,3,\ldots,n$. From the equations (36) it follows that the operators $\tilde{A}_k$, $k=2,3,\ldots,n$, satisfy the equalities

$$\tilde{A}_k=\sigma_1^{-1}\sigma_k\tilde{A}_1-\sigma_1^{-1}\tilde{\gamma}_{k1}$$

and $\tilde{A}_k$ are linear bounded operators in $\tilde{H}$.

We shall prove that the next equalities

$$\tilde{\gamma}_{k1}=\gamma_{k1}+i(\sigma_k\Phi\Phi^*\sigma_1-\sigma_1\Phi\Phi^*\sigma_k),\quad k=2,3,\ldots,n,$$

are valid onto E.

According to Corollary 1 we have

$$W^{-1}(x_1,\dots,x_n,z)=I+iR^*(x_1,\dots,x_n,z)(x_1\sigma_1+\cdots+x_n\sigma_n) \tag{40}$$

$((l_1,\dots,l_n)\in\Gamma(\sigma,\gamma)\cap\mathbb{R}^n$, $(x_1,\dots,x_n,z)\in K_a)$. Then for $x=1$, $x_2=x_3=\cdots=x_n=0$, $z=l_1$, $(l_1,\dots,l_n)\in\Gamma(\sigma,\gamma)\cap\mathbb{R}^n$, using the condition 2), we obtain

$$(l_k\sigma_1-l_1\sigma_k+\tilde{\gamma}_{k1})W(1,0,\dots,0,l_1)_{|M(l,\sigma,\gamma)} = 0,$$

$$(l_k\sigma_1-l_1\sigma_k+\gamma_{k1})W^{-1}(1,0,\dots,0,l_1)_{|M(l,\sigma,\tilde{\gamma})} = 0.$$

Hence

$$\begin{aligned}&(l_k\sigma_1-l_1\sigma_k+\tilde{\gamma}_{k1})W(1,0,\dots,0,l_1)\\ &=(W^{-1})^*(1,0,\dots,0,l_1)(l_k\sigma_1-l_1\sigma_k+\gamma_{k1})\end{aligned} \tag{41}$$

$(k=2,3,\dots,n)$ onto $M(l,\sigma,\gamma)$ with fixed $(l_1,\dots,l_n)\in\Gamma(\sigma,\gamma)\cap\mathbb{R}^n$. From (35) and (40) it follows that the equalities (41) take the form

$$\begin{aligned}&(l_k\sigma_1-l_1\sigma_k+\tilde{\gamma}_{k1})(I-i\Phi(A_1-l_1I)^{-1}\Phi^*\sigma_1)\\ &=(I-i\sigma_1\Phi(A_1-l_1I)^{-1}\Phi^*)(l_k\sigma_1-l_1\sigma_k+\gamma_{k1})\end{aligned} \tag{42}$$

$(k=2,3,\dots,n)$ onto $M(l,\sigma,\gamma)$, $(l_1,\dots,l_n)\in\Gamma(\sigma,\gamma)\cap\mathbb{R}^n$.

If $l_1\in\mathbb{R}$ and $l_1\notin\sigma(A_1)$ then all roots of the equations

$$\det(l_k\sigma_1-l_1\sigma_k+\gamma_{k1})=0,\qquad k=2,3,\dots,n, \tag{43}$$

are real because $\sigma_1>0$. Let $l_k^{(1)}(l_1),\dots,l_k^{(m_k)}(l_1)$ $(k=2,3,\dots,n)$ be all different roots of the equations (43). The corresponding subspaces $\ker(l_k^{(s_k)}(l_1)\sigma_1-l_1\sigma_k+\gamma_{k1})$, $k=2,3,\dots,n$, $s_k=1,2,\dots,m_k$, are orthogonal with respect to the positive operator σ_1. For any fixed $\mu=l_k^{(s_k)}(l_1)$ the equality (42) take the form

$$\begin{aligned}&\tilde{\gamma}_{k1}-\gamma_{k1}=i(\mu\sigma_1-l_1\sigma_k+\tilde{\gamma}_{k1})\Phi(A_1-l_1I)^{-1}\Phi^*\sigma_1\\ &-i\sigma_1\Phi(A_1-l_1I)^{-1}\Phi^*(\mu\sigma_1-l_1\sigma_k+\gamma_{k1})\end{aligned} \tag{44}$$

onto $M(l,\sigma,\gamma)$ where $l=(l_1,l_2^{(s_2)}(l_1),\dots,l_n^{(s_n)}(l_1))$. According to Lemma 2 it follows that the equality

$$\Phi e^{ix_1A_1}(A_1\Phi^*\sigma_k-\Phi^*\gamma_{k1})\sigma_1^{-1}=\sigma_1^{-1}(\sigma_k\Phi A_1-\tilde{\gamma}_{k1}\Phi)e^{ix_1A_1}\Phi^* \tag{45}$$

is true onto E.

Now we introduce the linear bounded commuting operators $A_k : \hat{H} \to \hat{H}$, $k = 2, 3, \ldots, n$, by the equalities $A_k = T^{-1}\tilde{A}_k T$ and we obtain

$$A_k\Phi^* g = T^{-1}\tilde{A}_k T\Phi^* g = T^{-1}\tilde{A}_k y_{\Phi^* g}(x) = T^{-1}\tilde{A}_k V(x) g$$
$$= T^{-1}(\tilde{A}_k V\sigma_k - V\gamma_{k1})\sigma_1^{-1} g = (A_1\Phi^*\sigma_k - \Phi^*\gamma_{k1})\sigma_1^{-1} g \tag{46}$$

for any $g \in E$, $k = 2, 3, \ldots, n$. Then from (45) and (46) we have

$$\Phi A_k - \sigma_1^{-1}(\sigma_k\Phi A_1 - \tilde{\gamma}_{k1}\Phi)e^{ix_1 A_1}\Phi^* = 0$$

onto E for $x_1 \in \mathbb{R}$, hence

$$\Phi A_k = \sigma_1^{-1}(\sigma_k\Phi A_1 - \tilde{\gamma}_{k1}\Phi) \tag{47}$$

onto $\hat{H} = \operatorname*{span}_{t\in\mathbb{R}}\{e^{itA_1}\Phi^*(E)\}$,

$$A_k\Phi^* = (A_1\Phi^*\sigma_k - \Phi^*\gamma_{k1})\sigma_1^{-1} \tag{48}$$

onto the space E. After transformations the relations (47) and (48) take the form

$$\sigma_k\Phi\Phi^*\sigma_1 - \sigma_1\Phi(A_k - \mu I)(A_1 - l_1 I)^{-1}\Phi^*\sigma_1$$
$$= (\mu\sigma_1 - l_1\sigma_k + \tilde{\gamma}_{k1})\Phi(A_1 - l_1 I)^{-1}\Phi^*\sigma_1, \tag{49}$$

$$\sigma_1\Phi\Phi^*\sigma_k - \sigma_1\Phi(A_1 - l_1 I)^{-1}(A_k - \mu I)\Phi^*\sigma_1$$
$$= \sigma_1\Phi(A_1 - l_1 I)^{-1}\Phi^*(\mu\sigma_1 - l_1\sigma_k + \tilde{\gamma}_{k1}) \tag{50}$$

onto E for all $k = 2, 3, \ldots, n$. From (49), (50) and (44) it follows that

$$\sigma_1\Phi\Phi^*\sigma_k - \sigma_k\Phi\Phi^*\sigma_1 = i(\tilde{\gamma}_{k1} - \gamma_{k1}), \quad k = 2, 3, \ldots, n \tag{51}$$

onto $M(l, \sigma, \gamma)$, $\mu = l_k^{(s_k)}(l_1)$, $s_k = 1, 2, \ldots, m_k$. It is easy to see that the equality (51) is true onto the space E because the subspaces

$$\ker(l_k^{(s_k)}(l_1)\sigma_1 - l_1\sigma_k + \gamma_{k1}), \quad k = 2, 3, \ldots, n, \quad s_k = 1, 2, \ldots, m_k,$$

are orthogonal with respect to the positive operator σ_1.

Now we shall show that

$$(A_k - A_k^*)/i = \Phi^*\sigma_k\Phi, \quad k = 2, 3, \ldots, n,$$

onto $\hat{H}$. According to (47), (48) and (51) it follows that

$$\left(\frac{A_k - A_k^*}{i} - \Phi^*\sigma_k\Phi\right)\Phi^*\sigma_1(E) = 0,$$

hence

(52) $$(A_k - A_k^*)/i = \Phi^*\sigma_k\Phi, \quad k = 2, 3, \ldots, n,$$

onto $\Phi^*(E)$. But the equality

(53) $$((A_k - A_k^*)/i - \Phi^*\sigma_k\Phi)A_1^m = (A_1^*)^m((A_k - A_k^*)/i - \Phi^*\sigma_k\Phi)$$

is hold for all $m \in \mathbb{N}$. Then the relations (52) and (53) show that

$$((A_k - A_k^*)/i - \Phi^*\sigma_k\Phi)A_1^m\Phi^*(E)$$

$$= (A_1^*)^m((A_k - A_k^*)/i - \Phi^*\sigma_k\Phi)\Phi^*(E) = 0.$$

Consequently we obtain

$$((A_k - A_k^*)/i - \Phi^*\sigma_k\Phi)A_1^m\Phi^*(E) = 0, \quad m \in \mathbb{N}_0,$$

i.e. $(A_k - A_k^*)/i = \Phi^*\sigma_k\Phi$ onto $\hat{H}$ for all $k = 2, 3, \ldots, n$. But $(A_1 - A_1^*)/i = \Phi^*\sigma_1\Phi$ because $(A_1, \hat{H}, \Phi, E, \sigma_1)$ is a colligation. Hence the set

$$X = (A_1, \ldots, A_n, \hat{H}, \Phi, E, \sigma_1, \ldots, \sigma_n, \{\gamma_{k1}\}_2^n, \{\tilde{\gamma}_{k1}\}_2^n)$$

is a commutative regular colligation. Then the operator-function

$$S(x_1, \ldots, x_n, z) = I - i\Phi(x_1A_1 + \cdots + x_nA_n - zI)^{-1}\Phi^*(x_1\sigma_1 + \cdots + x_n\sigma_n),$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $z \in \mathbb{C}^n$, is a complete characteristic function of the colligation X. But the operator-function

$$S(1, 0, \ldots, 0, z) = I - i\Phi(A_1 - zI)\Phi^*\sigma_1$$

is the characteristic operator-function of the single operator colligation $(A_1, \hat{H}, \Phi, E, \sigma_1)$ and from (35) we have

(54) $$S(1, 0, \ldots, 0, z) = W(1, 0, \ldots, 0, z)$$

onto E.

According to Theorem 3 and Theorem 4 the complete characteristic function of the commutative regular colligation and the linear manifold $M(l,\sigma,\gamma)$ are consonant with respect to the algebraic curve $\Gamma(\sigma,\gamma)$ and the operator-function $S(x_1,\dots,x_n,z)$ is defined uniquely by its trunk $\hat{S}(x_1,\dots,x_n,z)$ that coincides with the joint characteristic function of the colligation. Now using Corollary 2, Corollary 3 and the equality (54) it follows that

$$S(x_1,\dots,x_n,z) = W(x_1,\dots,x_n,z).$$

Consequently there exists a commutative regular colligation

$$X = (A_1,\dots,A_n,\hat{H},\Phi,E,\sigma_1,\dots,\sigma_n,\{\gamma_{k1}\}_2^n,\{\tilde{\gamma}_{k1}\}_2^n)$$

with a complete characteristic function that coincides with the given operator-function $W(x_1,\dots,x_n,z)$ belonging to the class $\Omega_\Gamma(\sigma_1,\dots,\sigma_n)$. The proof is completed. $\square$

It is important to note that the inverse assertion of Theorem 6 can be easily obtained after simple checking of the conditions 1), 2), 3) of Definition 6.

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