ON EXTENSION OF FUNCTORS ONTO THE KLEISLI CATEGORY OF THE INCLUSION HYPERSPACE MONAD

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ABSTRACT. It is proved that there exists no extension of any non-trivial weakly normal functor of finite degree onto the Kleisli category of the inclusion hyperspace monad.

0. In this note we consider the problem of extension of endofunctors in the category of compacta onto the Kleisli category of the inclusion hyperspace monad. The main result states that there is no such extension for weakly normal functors of finite degree. Note that the class of normal functors of finite degree is introduced by E.V. Shchepin [7] and contains, in particular, the finite power functors $(-)^n$ and $G$-symmetric power functors $\text{SP}_G^n$. Related results devoted to extensions of functors onto the Kleisli categories of some monads in the category of compacta can be found in [10, 11, 8], see also [12, 13].

1. A triple $T = (T, \eta, \mu)$ is said to be a monad on a category $\mathcal{C}$ if $T$ is an endofunctor acting in $\mathcal{C}$ and $\eta: 1_{\mathcal{C}} \to T$, $\mu: T^2 \to T$ are natural transformations satisfying $\mu \circ \eta T = \mu \circ T \eta = 1_T$, $\mu \circ \mu T = \mu \circ T \mu$ [2].

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Recall the definition of Kleisli category $C_T$ of the monad $T$ [4]. The objects of both categories $C$ and $C_T$ are the same, and $C_T(X, Y) = C(X, TY)$. The composition $g * f$ of morphisms $f \in C_T(X, Y)$, $g \in C_T(Y, Z)$ is defined by the formula: $g * f = \mu Z \circ T g \circ f$.

Denote by $I: C \to C_T$ the functor being the identity on objects and satisfying $If = \eta Y \circ f \in C_T(X, Y)$, $f \in C(X, Y)$.

A functor $F: C_T \to C_T$ is called an extension of the functor $F: C \to C$ onto the Kleisli category $C_T$ of the monad $T$ if $FI = IF$.

In the sequel we shall consider only the category $Comp$ of compacta (= compact Hausdorff spaces) and their continuous maps.

Recall the construction of the inclusion hyperspace monad $G$. For a compactum $X$ by $\text{exp} X$ we denote the set of nonempty closed subsets of $X$ endowed with the Vietoris topology. A base of this topology consists of the sets of the form

$$\langle U_1, \ldots, U_n \rangle = \bigg\{ A \in \text{exp} X \mid A \subset \bigcup_{i=1}^{n} U_i, \ A \cap U_i \neq \emptyset \text{ for each } i = 1, \ldots, n \bigg\},$$

where $U_1, \ldots, U_n$ are open in $X$. A set $A \in \text{exp}^2 X$ is said to be an inclusion hyperspace if $[A \in A, A \subset B \Rightarrow B \in A]$ (see [1]). Denote by $GX$ the set of inclusion hyperspaces with the inherited from $\text{exp}^2 X$ topology.

For a map $f: X \to Y$ the map $Gf: GX \to GY$ is defined by the formula: $Gf(A) = \{ B \in \text{exp} Y \mid B \supset f(A) \text{ for some } A \in A \}$, $A \in GX$. Obviously, we obtain a functor $G: Comp \to Comp$ (the inclusion hyperspace functor).

Setting $\eta X(x) = \{ A \in \text{exp} X \mid x \in A \}$, $\mu X = \bigcup\{ \bigcap A \mid A \in A \}$, $x \in X$, $A \in G^2 X$, we obtain the monad $G = (G, \eta, \mu)$ [6].

The following definition is due to E. Shchepin [7]. A functor $F: Comp \to Comp$ is called weakly normal if it is continuous, monomorphic, epimorphic, preserves weight, intersections, singletons and empty set. A monad is said to be weakly normal if so is its functorial part. See, e.g., [3] for a detailed exposition of properties of weakly normal monads.

Note that the monad $G$ is weakly normal [5].

Recall also that for a weakly normal functor the support $\text{supp} (a)$ of a point $a \in FX$ is defined [7]: $\text{supp} (a) = \bigcap \{ A \in \text{exp} X \mid FA \ni a \}$ (here $FA$ is identified with $Fi(A)$, where $i: A \to X$ is the natural imbedding). We say that the degree of functor $F$ is $\leq n$ (briefly, $^o F \leq n$) if $|\text{supp} (a)| \leq n$ for every
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$a \in FX, X \in \text{Comp}$. If, moreover, \(\circ F \not\leq n - 1\), we say that the degree of \(F\) equals \(n, \circ F = n\).

2. Now we prove the following result.

**Theorem.** No weakly normal functor of finite degree \(n > 1\) extends onto the Kleisli category of the inclusion hyperspace monad.

**Proof.** To the contrary, assume that a weakly normal functor \(F: \text{Comp} \to \text{Comp}, \circ F = n > 1\), extends onto \(\text{Comp}_G\). From results of J. Vinarék [9] it follows that there exists a natural transformation \(\xi: FG \to GF\) with

(1) \(\xi \circ F \eta = \eta \circ F\),

(2) \(\xi \circ F \mu = \mu \circ G \xi \circ \xi G\).

Let \(a \in F\{1, \ldots, n\}\) be a point with \(\text{supp} a = \{1, \ldots, n\}\).

Let \(r: \{1, \ldots, n + 1\} \to \{1, \ldots, n\}\) be the retraction \(r(n + 1) = n\). Since \(\circ F = n\), the preimage \((Fr)^{-1}(a)\) of the point \(a\) consists of two points, namely, \(a\) and the copy of \(a\) in \(F\{1, \ldots, n - 1, n + 1\}\).

For every compactum \(K\) and distinct points \(t_1, t_2, t_3 \in K\) denote by \(t(t_1, t_2, t_3)\) the inclusion hyperspace \(\{A \in \exp K \mid |A \cap \{t_1, t_2, t_3\}| \geq 2\}\) and by \(a(t_1, t_2)\) the following point: \(a(t_1, t_2) = F f(a)\), where \(f: \{1, \ldots, n\} \to K \sqcup \{1, \ldots, n - 2\}\) is the embedding such that \(f|\{1, \ldots, n - 2\} = \text{id}, f(n - 1) = t_1, f(n - 2) = t_2\).

Set \(K = \{x_1, x_2, x_3, y_1, y_2, y_3\}, Z = K \sqcup \{1, \ldots, n - 2\}\), and suppose all points \(x_i, y_j\) be distinct.

We shall write \(\overline{x}, \overline{y}, a_{ij}\) instead of \(t(x_1, x_2, x_3), t(y_1, y_2, y_3), a(x_i, y_j)\) respectively. And we shall often identify the points \(i\) and \(\eta Z(i), i = 1, \ldots, n - 2\).

**Claim 1.** The following equality holds: \(\xi Z(a(\overline{x}, \eta Z(y_1))) = t(a_{11}, a_{21}, a_{31})\).

Denote by \(A\) the left part of the previous equality. Consider the retraction \(g: Z \to Z \setminus \{x_2\}\) such that \(g(x_2) = x_1\). Then \(Gg(\overline{x}) = \eta Z(x_1), Gg(\eta Z(y_1)) = \eta Z(y_1)\). Therefore,

\[GFg(A) = \xi Z \circ FGg(a(\overline{x}, \eta Z(y_1))) = \xi Z \circ F \eta Z(a_{11}) = \eta FZ(a_{11}).\]
Hence, the preimage \((Fg)^{-1}(a_{11}) = \{a_{11}, a_{21}\}\) belongs to \(A\), because \(A\) is an inclusion hyperspace. Since \(x\) is invariant with respect to all automorphisms of \(\{x_1, x_2, x_3\}\), we see that \(\{a_{11}, a_{31}\}\) and \(\{a_{21}, a_{31}\}\) also belong to \(A\).

To end the proof of this claim, it is sufficient to show that \(\{a_{11}\} \notin A\). It is easy. To the contrary, let \(\{a_{11}\}\) belong to \(A\). Then \(\{a_{31}\}\) and \(\{a_{31}, a_{12}\}\) also belong to \(A\).

In similar manner one can prove the following fact.

Claim 2. The inclusion hyperspace \(B = \xi Z(a(x, y))\) contains the set \(\{a_{11}, a_{12}, a_{21}, a_{22}\}\).

Claim 3. The inclusion hyperspace \(B\) contains the set \(\{a_{11}, a_{12}, a_{22}, a_{31}\}\).

Consider the following points \(\mathcal{M}, \mathcal{N} \in G^2Z\):

\[
\mathcal{M} = t(x, \eta Z(x_1), \eta Z(x_2)), \quad \mathcal{N} = t(\eta Z(y_1), \eta Z(y_2), y).
\]

Since \(\mu Z(\mathcal{M}) = \overline{x}, \mu Z(\mathcal{N}) = \overline{y}\), we obtain \(\xi Z \circ F \mu Z(a(\mathcal{M}, \mathcal{N})) = \xi Z(a(\overline{x}, \overline{y})) = B\). Therefore

\[
B = \mu FZ \circ G \xi Z \circ \xi GZ(a(\mathcal{M}, \mathcal{N})).
\]

It is easy to see that

\[
L = \left\{a(x, \eta Z(y_1)), a(x, \eta Z(y_2)), a(\eta Z(x_1), \eta Z(y_1)), a(\eta Z(x_1), \eta Z(y_2))\right\} \in \xi GZ(a(\mathcal{M}, \mathcal{N})).
\]

Then \(\bigcap \xi Z(L) \subset B\). Now, using the equality \(\xi Z(a(\eta Z(x_i), \eta Z(y_j))) = \xi Z \circ F \eta Z(a_{ij}) = \eta FZ(a_{ij})\) and Claim 1, we obtain that \(\{a_{11}, a_{31}, a_{12}, a_{22}\} \in B\).

Now consider the point \(\mathcal{R} = a(t(\overline{x}, \eta Z(x_2), \eta Z(x_3)), \eta GZ(\overline{y})) \in FG^2Z\). Since \(\mu Z \circ \eta GZ = 1_{GZ}\), we have \(\xi Z \circ F \mu Z(\mathcal{R}) = B\), and thus

\[
B = \mu FZ \circ G \xi Z \circ \xi GZ(\mathcal{R}).
\]

Then the equality

\[
\xi GZ(\mathcal{R}) = t\left(a(\overline{x}, \overline{y}), a(\eta Z(x_2), \overline{y}), a(\eta Z(x_3), \overline{y})\right)
\]
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and Claim 3 imply the set \{a_{11}, a_{12}, a_{22}, a_{31}\} must belong to at least two of the following inclusion hyperspaces \(\xi Z(a(x, y)), \xi Z(a(\eta Z(x_2), y)), \xi Z(a(\eta Z(x_3), y))\). But it is easy to verify that this set belongs neither \(\xi Z(a(\eta Z(x_2), y))\) nor \(\xi Z(a(\eta Z(x_3), y))\). Contradiction. \(\square\)

3. Ending this note, we formulate the following questions

Problem 1. Let \(T\) be a weakly normal monad and there exists a weakly normal functor \(F\), \(1 <^\circ F \leq n\), which extends onto the Kleisli category of monad \(T\). Is there an extension of the power functor \((-)^2\) onto \(\text{Comp}_T\)?

Problem 2. Let \(T\) be a weakly normal monad such that the power functor \((-)^2\) extends onto \(\text{Comp}_T\). Is there an extension of the power functor \((-)^n\), \(n \geq 3\), onto \(\text{Comp}_T\)?

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