ON THE EXPONENTIAL BOUND OF THE CUTOFF RESOLVENT

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ABSTRACT. A simpler proof of a result of Burq [1] is presented.

Let $\mathcal{O} \subset \mathbb{R}^n, n \geq 2$, be a bounded domain with $C^\infty$ boundary $\Gamma$ and connected complement $\Omega = \mathbb{R}^n \setminus \mathcal{O}$. Consider in $\Omega$ the operator

$$\Delta_g := c(x)^2 \sum_{i,j=1}^n \partial_{x_i}(g_{ij}(x)\partial_{x_j}),$$

where $c(x), g_{ij}(x) \in C^\infty(\overline{\Omega}), c(x) \geq c_0 > 0$ and

$$\sum_{i,j=1}^n g_{ij}(x)\xi_i\xi_j \geq C|\xi|^2, \quad \forall (x, \xi) \in T^*\Omega, \quad C > 0.$$
We also suppose that \( c(x) = 1, g_{ij}(x) = \delta_{ij} \) for \( |x| \geq \rho_0 \) for some \( \rho_0 \gg 1 \). Denote by \( G \) the selfadjoint realization of \( \Delta_g \) in the Hilbert space \( H = L^2(\Omega; c(x)^{-2}dx) \) with a domain of definition \( D(G) = \{ u \in H^2(\Omega), Bu|_{\Gamma} = 0 \} \), where either \( B = Id \) (Dirichlet boundary conditions) or \( B = \partial_{\nu} \) (Neumann boundary conditions). Consider the resolvent \( R(\lambda) := (G + \lambda^2)^{-1} : H \to H \) defined for \( \text{Im} \lambda < 0 \), and introduce the cutoff resolvent \( R_\chi(\lambda) := \chi R(\lambda) \chi \), where \( \chi \in C^\infty_0(\mathbb{R}^n) \), \( \chi(x) = 1 \) for \( |x| \leq \rho_0 + 1 \), \( \chi(x) = 0 \) for \( |x| \geq \rho_0 + 2 \). It is well known that \( R_\chi(\lambda) \) extends through the real axis as a meromorphic function the poles of which are called resonances. Using the Carleman estimates proved by Lebeau-Robbiano ([4] in the Dirichlet case and [5] in the Neumann one) Burq has proved the following result

**Theorem** ([1]). There exist constants \( C, C_1, C_2, \gamma > 0 \) so that \( R_\chi(\lambda) \) extends holomorphically to the region
\[
\{ \lambda \in \mathbb{C} : \text{Im} \lambda \leq C_1 e^{-\gamma |\lambda|}, |\text{Re} \lambda| \geq C_2 \}
\]
and satisfies there the estimate
\[
\| R_\chi(\lambda) \|_{\mathcal{L}(H)} \leq C e^{\gamma |\lambda|}.
\]

Furthermore, he applied this theorem to obtain uniform rate of the decay of the local energy. Denote by \( u(t) \) the solution of the equation
\[
\begin{cases}
(\partial_t^2 - \Delta_g) u(t) = 0, \\
Bu|_{\Gamma} = 0, \\
u(0) = f_1, \partial_t u(0) = f_2.
\end{cases}
\]
Given any compact \( K \subset \overline{\Omega} \) and any \( m > 0 \), set
\[
p_m(t) = \sup \left\{ \frac{\| \nabla_x u \|_{L^2(K)} + \| \partial_t u \|_{L^2(K)}}{\| \nabla_x f_1 \|_{H^m(K)} + \| f_2 \|_{H^m(K)}}, (0,0) \neq (f_1, f_2) \in [C^\infty(\overline{\Omega})]^2, \text{supp } f_j \subset K \right\}.
\]
Burq derived from (1) the following bounds
\[
p_m(t) \leq C_m (\log t)^{-m} \quad \text{for} \quad t \geq 2.
\]
Note that another method allowing to derive (2) from (1) is presented in [6, Section 3].
On the exponential bound of the cutoff resolvent

The purpose of the present note is to give another proof of how the Carleman estimates of Lebeau-Robbiano imply (1). The first observation is that Theorem follows easily from the bound

\[ \| R_{\chi}(\lambda) \|_{\mathcal{L}(H)} \leq \tilde{C} e^{\gamma |\lambda|}, \quad \lambda \in \mathbb{R}, \ |\lambda| \gg 1, \]

(e.g. see [2, Corollary 3.1]). In fact, it suffices to prove (3) for \( \lambda \gg 1 \) as the case \( \lambda \ll -1 \) can be treated similarly. So, in what follows \( \lambda \) will be real, \( \lambda \gg 1 \).

Consider the Helmholtz equation

\[
\begin{cases}
(\Delta g + \lambda^2)u = v & \text{in } \Omega, \\
Bu = 0 & \text{on } \Gamma, \\
u - \lambda - \text{outgoing},
\end{cases}
\]

where \( v \in C^\infty(\Omega), \) \( \text{supp} \, v \subset \Omega_{a_0} := \{ x \in \Omega : |x| < a_0 \} \), where \( a_0 \gg 1 \) is taken so that the support of the perturbation is contained in \( \Omega_{a_0} \). Clearly, (3) is equivalent to the estimate

\[ \| u \|_{L^2(\Omega_{a_0})} \leq C e^{\gamma \lambda} \| v \|_{L^2(\Omega)}. \]

Take \( a > a_0 \) to be fixed later on and denote \( S = \{ x \in \mathbb{R}^n : |x| = a \} \). Define the Neumann operator \( N(\lambda) : H^1(S) \to L^2(S) \) by \( N(\lambda)g := \lambda^{-1} \partial_{\nu'} w \big|_S \), where \( w \) solves the equation

\[
\begin{cases}
(\Delta + \lambda^2)w = 0 & \text{in } |x| > a, \\
w = g & \text{on } S, \\
w - \lambda - \text{outgoing}.
\end{cases}
\]

Here \( \Delta \) denotes the free Laplacian and \( \nu' \) denotes the outer unit normal to \( S \). It is well known that for strictly convex \( S \) we have the bound

\[ \| N(\lambda) \|_{\mathcal{L}(H^1(S), L^2(S))} \leq C \]

with a constant \( C > 0 \) independent of \( \lambda \) (e.g. see [3, Corollary 3.3]). Hereafter, given a domain \( K \), \( H^s(K) \) will denote the Sobolev space equipped with the semiclassical norm \( \| f \|_{H^s(K)} := \| \Lambda_s f \|_{L^2(K)} \), where \( \Lambda_s \) is a \( \lambda - \Psi DO \) on \( K \) with principal symbol \( (|\xi|^2 + 1)^{s/2} \).

Clearly, \( u \) and \( v \) satisfy the equation

\[
\begin{cases}
(\Delta g + \lambda^2)u = v & \text{in } \Omega, \\
Bu = 0 & \text{on } \Gamma, \\
\lambda^{-1} \partial_{\nu'} u_\big|_S + N(\lambda)f = 0,
\end{cases}
\]
where \( f = u|_S \) and \( \nu = -\nu' \) denotes the inner unit normal to \( S \). By Green’s formula we have

\[
-\text{Im} \langle N(\lambda)f, f \rangle_{L^2(S)} = -\text{Im} \langle u, e^{-2\nu} \rangle_{L^2(\Omega_{a_0})}
\]

\[
\leq e^{-\beta \lambda} \|u\|_{L^2(\Omega_{a_0})}^2 + e^{\beta \lambda} \|v\|_{L^2(\Omega)}^2,
\]

\( \forall \beta \). Given any \( X > 0 \) take a function \( \rho_X(t) \in C_0^\infty(\mathbb{R}), \ 0 \leq \rho_X(t) \leq 1, \ \rho_X(t) = 1 \) for \( |t| \leq X \), \( \rho_X(t) = 0 \) for \( |t| \geq X + 1 \). Denote by \( \Delta_S \) the Laplace-Beltrami operator on \( S \). We need the following

**Lemma.** For every \( X > 0 \) there exists \( \gamma_0 = \gamma_0(X) \geq 0 \) so that

\[
-\text{Im} \langle N(\lambda)f, f \rangle_{L^2(S)} \geq e^{-\gamma_0 \lambda} \|\rho_X(\lambda^{-1}\sqrt{-\Delta_S})f\|_{L^2(S)}^2.
\]

**Proof.** Without loss of generality we may suppose that \( S \) is of radius 1. It is well known that the outgoing Neumann operator can be expressed in terms of the Hankel functions of second type, \( H_\nu^{(2)}(z) \). Let \( \{\mu_j\} \) be the eigenvalues of \( \sqrt{-\Delta_S} \) repeated according to multiplicity. We have the identities

\[
-\text{Im} \langle N(\lambda)f, f \rangle_{L^2(S)} = \sum \text{Im} \left( \frac{h_\nu'(\lambda)}{h_\nu(\lambda)} \right) \alpha_j^2,
\]

\[
\|\rho_X(\lambda^{-1}\sqrt{-\Delta_S})f\|_{L^2(S)}^2 = \sum \rho_X^2(\lambda^{-1}\mu_j) \alpha_j^2;
\]

where \( \{\alpha_j\} \) are such that

\[
\|f\|_{L^2(S)}^2 = \sum \alpha_j^2,
\]

and \( h_\nu(z) = z^{1/2}H_\nu^{(2)}(z), \ \nu = \sqrt{\mu_j^2 + (\frac{n}{2} - 1)^2} \), satisfies the equation

\[
h_\nu''(z) = \left( \frac{\nu^2 - 1/4}{z^2} - 1 \right) h_\nu(z).
\]

For real \( z > 0 \), set \( \psi_\nu(z) = -\text{Im} \frac{h_\nu'(z)}{h_\nu(z)}, \ \eta_\nu(z) = -\text{Re} \frac{h_\nu'(z)}{h_\nu(z)} \). In view of (10) we have

\[
\psi_\nu'(z) = \text{Im} \left( \left( \frac{h_\nu'(z)}{h_\nu(z)} \right)^2 - \frac{h_\nu''(z)}{h_\nu(z)} \right) = 2\eta_\nu \psi_\nu.
\]
This implies
\[ \frac{d}{dz} \left\{ \psi_{\nu}(\nu z) \exp \left( -2\nu \int_{z_0}^z \eta_{\nu}(\nu y) dy \right) \right\} = 0, \]
and hence
\[ (12) \quad \psi_{\nu}(\nu z) = \psi_{\nu}(\nu z_0) \exp \left( 2\nu \int_{z_0}^z \eta_{\nu}(\nu y) dy \right). \]

Fix \( z_0 = 2 \). We are going to show that for \( \nu \geq \nu_0 \gg 1 \) we have: \( \forall \delta > 0, \exists c = c(\delta) \geq 0 \) so that
\[ (13) \quad \psi_{\nu}(\nu z) \geq e^{-c\nu}, \quad \forall z \geq \delta, \]
and
\[ (14) \quad \psi_{\nu}(z) > 0, \quad \forall z > 0. \]

By Olver’s expansions
\[ \psi_{\nu}(\nu z_0) = \sqrt{\frac{z_0^2 - 1}{z_0} + O(\nu^{-1})}. \]

Clearly, this together with (12) imply (14). To prove (13) we will first consider the case when \( z \geq 2 \). Again by Olver’s expansions
\[ \eta_{\nu}(\nu z) = \frac{4z^2 - 3}{2z(z^2 - 1)} \nu^{-1} + O(\nu^{-2}), \]
uniformly for \( z \geq 2 \), and hence \( \eta_{\nu}(\nu z) > 0 \). This together with (12) yield
\[ \psi_{\nu}(\nu z) \geq \psi_{\nu}(\nu z_0) \geq \text{Const} > 0, \]
which proves (13) in this case. Furthermore, still by Olver’s expansions we have \( \eta_{\nu}(\nu z) = O(1) \) uniformly in \( \delta \leq z \leq 2 \). Hence, by (12), for \( \delta \leq z \leq 2 \),
\[ \psi_{\nu}(\nu z) \geq \psi_{\nu}(\nu z_0) \exp \left( -2\nu \int_{\delta}^{2} |\eta_{\nu}(\nu y)| dy \right) \]
\[ \geq \psi_{\nu}(\nu z_0) \exp (-C\nu), \quad C > 0, \]
which implies (13) in this case.

Let now \( 1/2 < \nu \leq \nu_0 \). Using the well known asymptotics of the Hankel functions as \( z \to +\infty \), \( \nu > 1/2 \) fixed, we get
\[ (15) \quad \psi_{\nu}(z) = 1 + O(z^{-1}), \quad 1/2 < \nu \leq \nu_0. \]
Since $\nu = O(\lambda)$ on supp $\rho \chi$, it is easy to see that (7) follows from (8) and (9) combined with (13), (14) and (15).

Let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ for $|x| \leq a_0 + 2$, $\chi = 0$ for $|x| \geq a_0 + 3$. Applying the Carleman estimates of Lebeau-Robbiano [4], [5] to the function $\chi u$ leads to

$$
\int_{\Omega_{a_0+2}} (|u|^2 + |\lambda^{-1}\nabla u|^2) \, dx
\leq e^{2\gamma_1 \lambda} \int_{a_0+2 \leq |x| \leq a_0+3} (|u|^2 + |\lambda^{-1}\nabla u|^2) \, dx + e^{2\gamma_1 \lambda} \|v\|^2_{L^2(\Omega)},
$$

with some $\gamma_1 > 0$. To eliminate the first term in the RHS of (16) we will use the Carleman estimates up to $S$. Set $P = -\lambda^{-2}\Delta - 1$. If $\varphi \in C^\infty(\Omega_a)$, then $P \varphi := e^{\lambda \varphi} P e^{-\lambda \varphi}$ is again a $\lambda - \Psi DO$ with principal symbol $p_\varphi(x, \xi) = p(x, \xi + i\nabla_x \varphi)$, $p$ being the principal symbol of $P$ considered as a $\lambda - \Psi DO$. We will construct a real-valued $C^\infty$ function $\varphi$ defined in a neighbourhood of $a_0 \leq |x| \leq a$ such that $\nabla \varphi \neq 0$ on $a_0 \leq |x| \leq a$, $\varphi = -1$ on $|x| = a_0$, $\varphi \geq \gamma_1 + 1$ on $a_0 + 2 \leq |x| \leq a_0 + 3$ and satisfying the condition

$$
p_\varphi(x, \xi) = 0 \Rightarrow \{\text{Re} \, p_\varphi, \text{Im} \, p_\varphi\} > 0.
$$

We will be looking for $\varphi$ in the form $\varphi(r)$, $r = |x|$. It is easy to see that (17) is equivalent to

$$
\varphi'(\varphi'' \varphi + \frac{1+\varphi'^2}{r}) > 0 \quad \text{for} \quad a_0 \leq r \leq a.
$$

Given any constant $C > 2(a_0 + 3)$, it is easy to check that the function $\varphi'(r) = \sqrt{\frac{C}{r}} - 1$ satisfies (18) with $a = C/2$. Define $\varphi(r)$ as follows

$$
\varphi(r) = -1 + \int_{a_0}^{r} \sqrt{Ct^{-1} - 1} \, dt.
$$

Clearly, if we take $C \geq C_1(a_0, \gamma_1)$ we can arrange $\varphi(a_0 + 2) \geq \gamma_1 + 1$ and hence $\varphi(r) \geq \gamma_1 + 1$ for $a_0 + 2 \leq r \leq a$. Fix $C = \max\{2(a_0 + 3), C_1(a_0, \gamma_1)\}$ and $a = C/2$. Since $\varphi(a_0) = -1$, there exist $a_0 < a_1 < a_2 < a_0 + 1$ so that $\varphi(r) < 0$ for $a_1 \leq r \leq a_2$. Choose a function $\chi_1 \in C_0^\infty(\mathbb{R}^n)$, $\chi_1 = 0$ for $|x| \leq a_1$, $\chi_1 = 1$ for $|x| \geq a_2$. We would like to apply the Carleman estimates up to $S$ to the function $\chi_1 u$. Set $w = e^{\lambda \varphi} \chi_1 u$. We are going to prove the estimate

$$
\|w\|_{H^1(a_0 \leq |x| \leq a)} + \|w_s\|_{H^1(S)} 
\leq O(\lambda^{1/2}) \|P \varphi w\|_{L^2(a_0 \leq |x| \leq a)} + O(1) \|\text{Op}_\lambda(\eta) w|_s\|_{L^2(S)},
$$

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Lebeau-Robbiano [4], in view of (5), we have \( \lambda \forall \rho \in (20) + O \) where \( \eta \) estimated from above by the LHS of (16) times a factor \( e \) estimated from above by the LHS of (20) times a factor \( e \) of (19) we will complete the proof of (4). Since \( P_\varphi w = -\lambda^2 e^{\lambda \varphi}[\Delta, \chi_1]u \) and \( w|_S = e^{\varphi(a)\lambda} f \), (19) implies

\[
\int_{a_2 \leq |x| \leq a} (|u|^2 + |\lambda^{-1} \nabla u|^2) e^{2\lambda \varphi} dx \leq \int_{a_1 \leq |x| \leq a_2} (|u|^2 + |\lambda^{-1} \nabla u|^2) e^{2\lambda \varphi} dx + O(1) e^{2\lambda \varphi(a)} \| \text{Op}_\lambda(\eta) f \|^2_{L^2(S)} - e^{2\lambda \varphi(a)} \| f \|^2_{L^2(S)}.
\]

Since \( \gamma_1 < \varphi \) on \( a_0 + 2 \leq |x| \leq a_0 + 3 \), the first term in the RHS of (16) is estimated from above by the LHS of (20) times a factor \( e^{-\delta_1 \lambda}, \delta_1 > 0 \). On the other hand, since \( \varphi < 0 \) on \( a_1 \leq |x| \leq a_2 \), the first term in the RHS of (20) is estimated from above by the LHS of (16) times a factor \( e^{-\delta_2 \lambda}, \delta_2 > 0 \). Therefore, we have

\[
e^{-2\gamma_2 \lambda} \| u \|^2_{L^2(\Omega_{a_0+2})} + \| f \|^2_{L^2(S)} \leq e^{2\gamma_3 \lambda} \| v \|^2_{L^2(\Omega)} + O(1) \| \text{Op}_\lambda(\eta) f \|^2_{L^2(S)},
\]

with some constants \( \gamma_2 \) and \( \gamma_3 \). On the other hand, taking \( \eta(x', \xi') = \rho_X(\sqrt{r_0(x', \xi')}) \), applying (7) with \( X = \sqrt{3} \) and combining with (6) give

\[
\| \text{Op}_\lambda(\eta) f \|^2_{L^2(S)} \leq o(1) \| f \|^2_{L^2(S)} + e^{-(\beta - \gamma_0) \lambda} \| u \|^2_{L^2(\Omega_{a_0})} + e^{(\beta + \gamma_0) \lambda} \| v \|^2_{L^2(\Omega)},
\]

\( \forall \beta \). Clearly, taking \( \beta > 2\gamma_2 + \gamma_0 \), (4) follows from (21) and (22).

**Proof of (19).** Since \( \partial_x \varphi|_S = -1 \), the boundary conditions on \( S \) become \( \lambda^{-1} \partial_x w|_S = -(N(\lambda) + 1)f_1 \), where \( f_1 := w|_S \). By the Carleman estimates of Lebeau-Robbiano [4], in view of (5), we have

\[
\| w \|_{H^1(a_0 \leq |x| \leq a)} \leq O(\lambda^{1/2}) \| P_\varphi w \|_{L^2(a_0 \leq |x| \leq a)} + O(1) \| f_1 \|_{H^1(S)}.
\]

It is easy to see that (19) would follow from (23) and the estimate

\[
\| \text{Op}_\lambda (1 - \eta) f_1 \|_{H^1(S)} \leq O(\lambda^{1/2}) \| P_\varphi w \|_{L^2(a_0 \leq |x| \leq a)} + o(1) \| w \|_{H^1(a_0 \leq |x| \leq a)} + o(1) \| f_1 \|_{H^1(S)}.
\]

To prove (24) we will use that \( 1 - \eta \) is supported in the elliptic region of the corresponding boundary value problem. Clearly, it suffices to prove (24) locally and then conclude by a partition of the unity on \( S \). Given a \( x_0 \in S \) take a small neighbourhood in \( \mathbb{R}^n \), \( V \), of \( x_0 \), and denote \( U = V \cap S \), \( V_+ = V \cap \{|x| < a\} \). Take in
we have
\[ p = \xi_n^2 + r(x, \xi') - 1 = \xi_n^2 + r_0(x, \xi') - 1 + O(x_n|\xi'|)^2, \]
\[ \text{Re } p_\varphi = \xi_n^2 + r(x, \xi') - 1 - (\varphi_x')^2 = \xi_n^2 + r_0(x, \xi') - 2 + O(x_n(|\xi'|^2 + 1)), \]
\[ \text{Im } p_\varphi = 2\varphi_x' n_x n = -2\xi_n(1 + O(x_n)), \]
where \( r_0(x', \xi') \) is the principal symbol of \(-\Delta_S \) written in the coordinates \((x', \xi') \in T^*U \). Hence, the restriction of \( p_\varphi = 0 \) on \( T^*S \) is given by \( r_0 = 2 \). In what follows \( \| \cdot \|_s \) and \( \| \cdot \|_{s,+} \) will denote the norms in \( H^s(\mathbb{R}^{n-1}) \) and \( H^s(\mathbb{R}^{n-1} \times \mathbb{R}^+) \), respectively, while \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle_+ \) will denote the scalar products in \( L^2(\mathbb{R}^{n-1}) \) and \( L^2(\mathbb{R}^{n-1} \times \mathbb{R}^+) \), respectively. By \( \mathcal{D}_{cl}^{s,k} \) we will denote the space of \( \lambda - \Psi DO's \) with symbols \( a \sim \lambda^k \sum \lambda^{-j} a_j \) with \( a_j \) independent of \( \lambda \) satisfying
\[ |\partial_x^\alpha \partial_\xi^\beta a_j| \leq C_{\alpha\beta}(1 + |\xi|)^{s-j-|\beta|}. \]
We will also denote \( \mathcal{D}_j := (i\lambda)^{-1}\partial_{x_j}, \mathcal{D} = (\mathcal{D}', \mathcal{D}_n) \). Let \( \phi(t) \in C_0^\infty(\mathbb{R}), \phi = 1 \) for \( |t| \leq \delta/2, \phi = 0 \) for \( |t| \geq \delta \). Let also \( \zeta(x') \in C_0^\infty(U), \zeta = 1 \) in a small neighbourhood of \( x_0 \in U \). Set
\[ g = \text{Op}_\lambda((1 - \eta)|\xi'|)|\phi(x_n)\zeta(x')w, \quad h := g|_{x_n=0} = \text{Op}_\lambda((1 - \eta)|\xi'|)|\zeta(x')f_1. \]
We have
\[ i\mathcal{D}_n g|_{x_n=0} = -(N(\lambda) + 1)h + [N(\lambda), \text{Op}_\lambda((1 - \eta)|\xi'|)|\zeta(x')]f_1. \]
Since \( N(\lambda) \) has a parametrix of class \( L_{cl}^{1,0} \) on \( \text{supp}(1 - \eta) \) with principal symbol \(-\sqrt{t_0} - 1 \), we have that the commutator above (which will be denoted by \( A \)) is of class \( L_{cl}^{1,-1} \). Let \( P_{\varphi}^* \) be the formal adjoint to \( P_\varphi \) and denote \( Q_1 = \frac{P_\varphi + P_{\varphi}^*}{2}, \)
\( Q_2 = \frac{P_\varphi - P_{\varphi}^*}{2i} \) with principal symbols \( \text{Re } p_\varphi \) and \( \text{Im } p_\varphi \), respectively. Using the identities
\[ \int_0^\infty \mathcal{D}_n g \cdot \mathcal{F} dx_n = \int_0^\infty |\mathcal{D}_n g|^2 dx_n + i\lambda^{-1}\mathcal{D}_n g|_{x_n=0} \cdot \mathcal{F}|_{x_n=0}, \]
\[ \text{Im } \langle Q_2 g, g \rangle_+ = -\lambda^{-1}\|h\|_0^2 + e(g), \]
where
\[ |e(g)| \leq o(1)\|g\|_{1,+}^2, \]
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it is easy to get

\[ \text{Re} \langle (Q_1 - D_n^2)g, g \rangle_+ + \|D_n g\|_{0,+}^2 = \text{Re} \langle P_\varphi g, g \rangle_+ + \lambda^{-1} \text{Re} \langle N(\lambda)h + Af_1, h \rangle + \epsilon(g) \]

(25)

\[ \leq \varepsilon^{-1} \int_0^\infty \|P_\varphi g(\cdot, x_n)\|_{1,+}^2 \, dx_n + \varepsilon \|g\|_{1,+}^2 + O(\lambda^{-2}) \|f_1\|_{H^1(S)}^2, \]

\[ \forall \varepsilon > 0. \] On the other hand, the principal symbol of \(Q_1 - D_n^2\) is \(\geq C|\xi'|^2\), \(C > 0\), on \(\text{supp}(1 - \eta)\), \(0 \leq x_n \leq \delta\), \(0 < \delta \ll 1\). Therefore, by Gårding’s inequality we get

\[ 0 < C'\|g\|_{1,+}^2 \leq \varepsilon^{-1} \int_0^\infty \|P_\varphi g(\cdot, x_n)\|_{1,+}^2 \, dx_n + \varepsilon \|g\|_{1,+}^2 + O(\lambda^{-2}) \|f_1\|_{H^1(S)}^2, \]

and hence

(26)

\[ \|g\|_{1,+}^2 \leq O(1) \int_0^\infty \|P_\varphi g(\cdot, x_n)\|_{1,+}^2 \, dx_n + O(\lambda^{-2}) \|f_1\|_{H^1(S)}^2. \]

On the other hand,

\[ \|h\|_0^2 = -\int_0^\infty \frac{d}{dx_n} \|g(\cdot, x_n)\|_0^2 \, dx_n \]

\[ = -2\lambda \int_0^\infty \text{Re} \langle g(\cdot, x_n), iD_n g(\cdot, x_n) \rangle \, dx_n \leq O(\lambda) \|g\|_{1,+}^2, \]

which combined with (26) gives

\[ \|h\|_0 \leq O(\lambda^{1/2}) \left( \int_0^\infty \|P_\varphi g(\cdot, x_n)\|_{1,+}^2 \, dx_n \right)^{1/2} + O(\lambda^{-1/2}) \|f_1\|_{H^1(S)} \]

\[ \leq O(\lambda^{1/2}) \|P_\varphi w\|_{0,+} + O(\lambda^{-1/2}) \|w\|_{1,+} + O(\lambda^{-1/2}) \|f_1\|_{H^1(S)}, \]

which in turn implies (24) by making a partition of the unity on \(S\).

REFERENCES


