A NOTE ON ELEMENTARY DERIVATIONS

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Abstract. Let $R$ be a UFD containing a field of characteristic 0, and $B_m = R[Y_1, \ldots, Y_m]$ be a polynomial ring over $R$. It was conjectured in [5] that if $D$ is an $R$-elementary monomial derivation of $B_3$ such that $\ker D$ is a finitely generated $R$-algebra then the generators of $\ker D$ can be chosen to be linear in the $Y_i$’s. In this paper, we prove that this does not hold for $B_4$. We also investigate $R$-elementary derivations $D$ of $B_m$ satisfying one or the other of the following conditions:

(i) $D$ is standard.
(ii) $\ker D$ is generated over $R$ by linear constants.
(iii) $D$ is fix-point-free.
(iv) $\ker D$ is finitely generated as an $R$-algebra.
(v) $D$ is surjective.
(vi) The rank of $D$ is strictly less than $m$.


Key words: Derivations, Hilbert fourteenth problem.
1. Introduction. In this paper, unless otherwise noted, $k$ is a field of characteristic 0, $R$ is a UFD containing $k$ and $B$ is an $R$-algebra which is a polynomial ring in a finite number of variables over $R$. If $m$ is a positive integer, then $R^m$ means the polynomial ring in $m$ variables over $R$. If $B \cong R^m$, then a coordinate system of $B$ over $R$ is an element $(Y_1, \ldots, Y_m) \in B^m$ satisfying $B = R[Y_1, \ldots, Y_m]$. Recall that a derivation $D : B \to B$ is an additive map satisfying $D(xy) = D(x)y + xD(y)$ for all $x, y \in B$. If $D(R) = \{0\}$, then we say that $D$ is an $R$-derivation of $B$. $D$ is called locally nilpotent if for every $x \in B$, there exists $n \geq 0$ such that $D^n(x) = 0$.

Definition 1.1. If $B = R^m$, then an $R$-derivation $D : B \to B$ is called $R$-elementary if there exists a coordinate system $(Y_1, \ldots, Y_m)$ of $B$ over $R$ such that $DY_i \in R$ for all $i$. In this case we have:

$$D = \sum_{i=1}^{m} a_i \frac{\partial}{\partial Y_i} \quad \text{(where } a_i \in R).$$

Definition 1.2. Let $C = k[N]$. A derivation $D : C \to C$ is elementary if, for some integers $m, n \geq 0$ such that $m + n = N$, there exists a coordinate system $(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ of $C$ satisfying:

$$k[X_1, \ldots, X_n] \subseteq \ker D \quad \text{and} \quad \forall i, \quad DY_i \in k[X_1, \ldots, X_n].$$

In this case, $D$ is $k[X_1, \ldots, X_n]$-elementary:

$$D = \sum_{i=1}^{m} a_i \frac{\partial}{\partial Y_i} \quad \text{(where } a_i \in k[X_1, \ldots, X_n]).$$

An immediate consequence of the above definition is that all elementary derivations are locally nilpotent.

Definition 1.3. A derivation $D : B \to B$ is called irreducible if the only principal ideal of $B$ containing $D(B)$ is $B$ itself. A locally nilpotent derivation $D$ is called fix-point-free if the ideal of $B$ generated by the image of $D$ is equal to $B$. A slice of $D$ is an element $s \in B$ such that $D(s) = 1$.

It is clear that any surjective locally nilpotent derivation of $B$ admits a slice. The converse is also true: if $s$ is a slice of a locally nilpotent derivation $D$
of $B$ and $y \in B$, let

$$x = \sum_{k=0}^{\infty} (-1)^k \frac{s^{k+1}}{(k+1)!} D^k(y)$$

then $x \in B$ since $D$ is locally nilpotent and it is easy to verify that $D(x) = y$.

Knowing that a locally nilpotent derivation of a polynomial algebra admits a slice helps to understand the kernel of the derivation. More precisely, the following is a well known fact (see [8]).

**Proposition 1.1.** If $D : C \to C$ is a locally nilpotent $R$-derivation of an $R$-algebra $C$ with a slice $s$, then

2. The map

$$\zeta : C \to C$$

$$x \mapsto \sum_{i \geq 0} \frac{1}{i!} (-s)^i D^i(x)$$

is a homomorphism of $R$-algebras with image equal to $\ker D$. In particular, if $C = R[Y_1, \ldots, Y_m]$ then

$$\ker D = R[\zeta(Y_1), \ldots, \zeta(Y_m)].$$

$R$-derivations of $B$ can be classified according to their rank:

**Definition 1.4.** The rank of an $R$-derivation $D$ of $B$ is defined to be the least integer $s$ ($0 \leq s \leq n$) for which there exists a coordinate system $(X_1, \ldots, X_n)$ of $B$ over $R$ satisfying $R[X_1, \ldots, X_{n-s}] \subseteq \ker D$. In other words, rank $D$ is the least number of partial derivatives of $B$ needed to express $D$.

Clearly, the rank of $D$ is zero if and only if $D$ is the zero derivation.

**Definition 1.5.** Let $B = R[Y_1, \ldots, Y_m]$ and consider an $R$-elementary derivation

$$D = \sum_{i=1}^{m} a_i \partial_i : B \to B$$

where $a_i \in R$ and $\partial_i = \partial/\partial Y_i$ for all $i$.

1. Any element of $\ker D$ of the form

$$r_1 Y_1 + \cdots + r_m Y_m$$

(\text{where } r_i \in R)

is said to be a linear constant of $D$. 


2. Given $i, j \in \{1, \ldots, m\}$, define $L_{ij} = \frac{a_i}{g_{ij}} Y_j - \frac{a_j}{g_{ij}} Y_i$ where:

$$g_{ij} = \begin{cases} 
\gcd(a_i, a_j) & \text{if } a_i \neq 0 \text{ or } a_j \neq 0 \\
1 & \text{if } a_i = 0 = a_j.
\end{cases}$$

It is clear that $L_{ij} \in \ker D$, $L_{ii} = 0$ and $L_{ji} = -L_{ij}$ (for all $i, j$). We call the elements $L_{ij}$ the standard linear constants of $D$.

3. If $\ker D$ is generated as an $R$-algebra by the standard linear constants, we say that $D$ is a standard derivation.

This paper investigates $R$-elementary derivations $D : R[m] \to R[m]$ satisfying one or the other of the following conditions:

(i) $D$ is standard.

(ii) $\ker D$ is generated over $R$ by linear constants.

(iii) $D$ is fix-point-free.

(iv) $\ker D$ is finitely generated as an $R$-algebra.

(v) $D$ is surjective.

(vi) $\text{Rank } D < m$.

Studying the finite generation of the kernel of derivations of polynomial rings is closely related to the famous fourteenth’s problem of Hilbert, that can be stated as follows

If $L$ is a subfield of $k(X_1, ..., X_n)$ (the quotient field of $k[n]$), is $L \cap k[X_1, ..., X_n]$ a finitely generated $k$-algebra?

Deveney and Finston ([3]) used a counterexample to Hilbert’s fourteenth problem found by Roberts in 1990 ([6]) to prove that the kernel of the elementary derivation

$$D = X_1^{t+1} \frac{\partial}{\partial Y_1} + X_2^{t+1} \frac{\partial}{\partial Y_2} + X_3^{t+1} \frac{\partial}{\partial Y_3} + (X_1 X_2 X_3)^t \frac{\partial}{\partial Y_4}$$

of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]$ is not finitely generated as a $k$-algebra for any $t \geq 2$.

To prove that the invariant subalgebras of some derivations in this paper are finitely generated we will use the following tool we proved in [5].

**Proposition 1.2** ([5, Lemma 2.2]). Let $E \subseteq A_0 \subseteq A \subseteq C$ be integral domains, where $E$ is a UFD. Suppose that some element $d$ of $E\{0\}$ satisfies:

• \((A_0)_d = A_d\)

• \(pC \cap A_0 = pA_0 \) for each prime divisor \(p\) of \(d\), \((\text{in } E)\) then \(A_0 = A\).

Using our notations, \(E\) plays the role of \(R\), \(A\) plays the role of \(\ker D\), \(A_0\) is a subalgebra of \(\ker D\) (which is a candidate for \(\ker D\)) and \(C\) plays the role of \(B\).

2. Unimodular rows and variables. Recall that an element \(F \in B \cong \mathbb{R}^{|m|}\) is called a variable of \(B\) over \(R\) if there exists a coordinate system \((F, F_2, \ldots, F_m)\) of \(B\) over \(R\).

Given an element \(F\) of \(B\), it is desirable to know if \(F\) is a variable over \(R\). That question seems to be hard in general. In this section, we give a necessary and sufficient condition for a linear form to be a variable.

**Definition 2.1.** Let \(A\) be a ring and \(n\) a positive integer. An element \((a_1, \ldots, a_n)\) of \(A^n\) is called a unimodular row of length \(n\) over \(A\) if \(a_1b_1 + \ldots + a_nb_n = 1\) for some \(b_1, \ldots, b_n \in A\). A unimodular row over \(A\) is called extendible if it is the first row of an invertible matrix over \(A\). The ring \(A\) is called Hermite if every unimodular row over \(A\) is extendible.

It is well known that Hermite rings include:

1. polynomial rings over a field
2. Formal power series over a field
3. Laurent polynomials over a field
4. Any PID
5. Any complex Banach Algebra with a contractible maximal ideal space.

A well-known example of a non Hermite ring is the following.

**Example 2.1.** (M. Hochster, [4]) Let \(R = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1) = \mathbb{R}[x, y, z]\) \((x, y, z\) are the images of \(X, Y, Z\) in \(R\) respectively), then \((x, y, z)\) is a unimodular row over \(R\) which is not extendible. So \(R\) is not Hermite.

Clearly any extendible unimodular row is unimodular. The converse holds in case of length 2 by the following (obvious) proposition.
Proposition 2.1. If $A$ is an arbitrary ring (commutative with identity), then any unimodular row of length $\leq 2$ over $A$ is extendible.

We relate now the notion of a “linear variable” with that of “extendible unimodular row”. First, a lemma.

Lemma 2.1. Let $E$ be a domain, and $V = E[X_1, \ldots, X_n]$ be a polynomial ring in $n$ variables over $E$. If $\gamma = (F_1, \ldots, F_n)$ is a coordinate system of $V$ over $E$, then the determinant of the matrix

$$A = \left( \frac{\partial F_i}{\partial X_j} \right)_{1 \leq i, j \leq n}$$

is a unit of $E$.

Proposition 2.2. Let $A$ be a domain, $(a_1, \ldots, a_n) \in A^n$ and $B = A[Y_1, \ldots, Y_n] = A^n$. Then the following conditions are equivalent:

1. The linear form $a_1 Y_1 + \cdots + a_n Y_n$ is a variable of $B$ over $A$
2. $(a_1, \ldots, a_n)$ is an extendible unimodular row of $B$ over $A$.

Proof. Assume first that $F = a_1 Y_1 + \cdots + a_n Y_n$ is a variable of $B$ over $A$, then $B = A[F, F_2, \ldots, F_n]$ for some elements $F_2, \ldots, F_n$ of $B$. By Lemma 2.1, where

$$\det(\mathcal{M}) \in R^n$$

(1)

$$(with F = F_1). Sending all the variables to 0 in $\mathcal{M}$ gives a matrix with entries in $R$ and first row equal to $(a_1, \ldots, a_n)$. Relation (1) shows that the determinant of this matrix is a unit in $A$ and hence $(a_1, \ldots, a_n)$ is an extendible unimodular row of $B$ over $A$.

For the converse, suppose that $\mathcal{M}$ is an invertible matrix with entries in $A$ and first row equal to $(a_1, \ldots, a_n)$. Let $(F_2, \ldots, F_n) \in B^{n-1}$ be such that

$$\mathcal{M}^{-1} \begin{bmatrix} F \\ F_2 \\ \vdots \\ F_n \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}.$$
This implies that $A[F,F_2,\ldots,F_n] \supseteq A[Y_1,\ldots,Y_n]$. Since the other inclusion is clear, $B = A[F,F_2,\ldots,F_n]$ and $F$ is then a variable of $B$ over $A$. □

3. Homogeneous derivations.

**Definition 3.1.** Let $C = \bigoplus_i C_i$ be a $\mathbb{Z}$-graded or an $\mathbb{N}$-graded ring. A derivation $D : C \to C$ is called homogeneous of degree $n$ if there exists an integer $n$ such that $D(C_i) \subseteq C_{i+n}$ for all $i$.

Consider the natural $\mathbb{N}$-grading on $B = R[Y_1,\ldots,Y_m]$ where the degree of each element of $R$ is zero and the degree of each of the variables in one. Every $R$-elementary derivation on $B$ is then homogeneous of degree $-1$.

The following proposition will be used later in this paper.

**Proposition 3.1.** Let $B = R[Y_1,\ldots,Y_m]$ equipped with the natural $\mathbb{N}$-grading. If $D$ is a homogeneous derivation of $B$ that annihilates a variable of $B$ over $R$, then $D$ annihilates a variable of $B$ over $R$ which is a linear form in the $Y_i$’s (over $R$).

**Proof.** Suppose that $F \in \ker D$ is a variable of $B$ over $R$. Without loss of generality, one can assume that the homogeneous part of degree 0 of $F$ is zero. Write

$$F = F_{(1)} + F_{(2)} + \ldots + F_{(d)}$$

where $d$ is the degree of $F$ and $F_{(i)}$ is the homogeneous part of $F$ of degree $i$. Choose $F_2,\ldots,F_m \in B$ such that $B = R[F,F_2,\ldots,F_m]$ and let

$$\mathcal{M} = \left(\frac{\partial F_i}{\partial Y_j}\right)_{1 \leq i,j \leq n}$$

(with $F = F_1$). Then $\mathcal{M}$ is invertible by Lemma 2.1. Setting all the $Y_i$’s equal to zero in $\mathcal{M}$ gives an element of $\text{GL}_m(R)$ whose first row is $(\alpha_1,\alpha_2,\ldots,\alpha_m)$ where

$$F_{(1)} = \alpha_1 Y_1 + \alpha_2 Y_2 + \cdots + \alpha_m Y_m.$$ 

Proposition 2.2 shows that $F_{(1)}$ is a variable of $B$ over $R$. On the other hand, the fact that $D$ is homogeneous implies that each of the homogeneous components of $F$ are in $\ker D$. In particular $F_{(1)} \in \ker D$. □

4. Standard derivations. We consider first the simple case of $R$-elementary derivations in dimension 2 ($R$ is a UFD containing a field $k$).

**Proposition 4.1.** Every $R$-elementary derivation of $R^{[2]}$ is standard.
Proof. Let $B = R[Y_1, Y_2] = R^{[2]}$, and $D = a_1 \frac{\partial}{\partial Y_1} + a_2 \frac{\partial}{\partial Y_2}$ an $R$-elementary derivation of $B$. We may clearly assume that $D$ is irreducible; i.e., $a_1$ and $a_2$ are relatively prime in $R$. Using Proposition 1.2, we will show that $\ker D = R[a_1 Y_1 - a_2 Y_2]$.

Let $F = a_1 Y_2 - a_2 Y_1$ and $R_0 = R[F]$. Then, $R_0 \subseteq \ker D$ and $(R_0)_{a_1} = (\ker D)_{a_1}$.

Let $p$ be a prime divisor of $a_1$, and let $x \in pB \cap R_0$; we show that $x \in pR_0$, the inclusion $pR_0 \subseteq pB \cap R_0$ being clear. For this, write $x = \Phi(F)$ for some $\Phi \in R[T] = R^{[1]}$ then the image $\overline{\Phi} \in \overline{R}[T]$ of $\Phi$ (where $\overline{R} = R/pR$) is in the kernel of the epimorphism

$$\alpha : \overline{R}[T] \rightarrow \overline{R}[F]$$

sending $T$ to $\overline{F}$. Since $\overline{F}$ is transcendental over $\overline{R}$, $\alpha$ is an isomorphism. Consequently, $\overline{\Phi} = 0$ and $x \in pR_0$. □

The implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iv)$ above (see the introduction) are true by the definition of standard derivations. By proposition 4.1, the $k[X_1, X_2]$-elementary derivation

$$(2) \quad X_1 \frac{\partial}{\partial Y_1} + X_2 \frac{\partial}{\partial Y_2}$$

of $k[X_1, X_2, Y_1, Y_2]$ is standard. Clearly, this derivation is not fix-point-free and consequently not surjective. This shows that $(i) \Rightarrow (iii)$ and $(i) \Rightarrow (v)$ are false in general. For the implication $(i) \Rightarrow (vi)$, note that the derivation (2) above does not annihilate a variable of $k[X_1, X_2, Y_1, Y_2]$ over $k[X_1, X_2]$. Indeed, if $F \in k[X_1, X_2, Y_1, Y_2]$ is a variable of $k[X_1, X_2, Y_1, Y_2]$ over $k[X_1, X_2]$ such that $D(F) = 0$, then we may assume that $F = \alpha_1 Y_1 + \alpha_2 Y_2$ for some unimodular row $(\alpha_1, \alpha_2)$ over $k[X_1, X_2]$ (Proposition 3.1). But the fact that $D(F) = 0$ implies that

$$X_1 \alpha_1 + X_2 \alpha_2 = 0$$

and hence the ideal generated by $\alpha_1$ and $\alpha_2$ in $k[X_1, X_2]$ is included in the ideal generated by $X_1$ and $X_2$. This contradicts the fact that $(\alpha_1, \alpha_2)$ is a unimodular row. We conclude that the rank of $D$ is 2 and that the implication $(i) \Rightarrow (vi)$ is false.

5. The case where $\ker D$ is generated by linear constants. The following theorem gives a counterexample “of rank $m$” to the implication $(ii) \Rightarrow (i)$ above.
**Theorem 5.1.** The kernel of the elementary derivation

\[ D = (X_1^2 - X_2X_3) \frac{\partial}{\partial Y_1} + (X_2^2 - X_1X_3) \frac{\partial}{\partial Y_2} + (X_3^2 - X_1X_2) \frac{\partial}{\partial Y_3} \]

of \( B = k[X_1, X_2, X_3, Y_1, Y_2, Y_3] \) is generated by two linear constants (in fact it is a polynomial ring in two variables over \( k[X_1, X_2, X_3] \)) but \( D \) is not standard. Moreover the rank of \( D \) over \( k[X_1, X_2, X_3] \) is 3.

**Proof.** Let \( a_1 = X_1^2 - X_2X_3 \), \( a_2 = X_2^2 - X_1X_3 \), \( a_3 = X_3^2 - X_1X_2 \), and let \( R = k[X_1, X_2, X_3] \). Then \( a_1, a_2, a_3 \) are pairwise relatively prime elements of \( R \). Consider the two elements of \( B \)

\[
f = X_3Y_1 + X_1Y_2 + X_2Y_3, \quad g = X_2Y_1 + X_3Y_2 + X_1Y_3
\]

and the usual standard linear constants

\[
L_1 = a_3Y_2 - a_2Y_3 = X_3^2Y_2 - X_1X_2Y_2 - X_2^2Y_3 + X_1X_3Y_3
\]

\[
L_2 = -a_3Y_1 + a_1Y_3 = -X_3^2Y_1 + X_1X_2Y_1 + X_2^2Y_3 - X_2X_3Y_3
\]

\[
L_3 = a_2Y_1 - a_1Y_2 = X_2^2Y_1 - X_1X_3Y_1 - X_2^2Y_2 + X_2X_3Y_2
\]

It is immediate that \( D(f) = D(g) = 0 \) and that the following relations are true

\[
L_1 = -X_2f + X_3g, \quad L_2 = -X_2f + X_1g, \quad L_3 = -X_1f + X_2g.
\]

Let \( R_0 := R[f, g] \), then \( R[L_1, L_2, L_3] \subseteq R_0 \). It is easy to see that \( (R[L_1, L_2, L_3])_{a_3} = (\ker D)_{a_3} \), so \( (R_0)_{a_3} = (\ker D)_{a_3} \). We will show that \( \ker D = R[f, g] \); so, it is enough (Proposition 1.2) to show that \( a_3B \cap R_0 \subseteq a_3R_0 \). Let \( \overline{R} = R/a_3R \) and consider the ring homomorphism

\[
\phi : \overline{R}[T_1, T_2] \rightarrow \overline{R}[\overline{f}, \overline{g}]
\]

sending \( T_1 \) to \( \overline{f} \) and \( T_2 \) to \( \overline{g} \). We claim that \( \phi \) is an isomorphism. Indeed, since the elements \( \overline{f} \) and \( \overline{g} \) are not algebraic over \( \overline{R} \), the transcendence degree of \( \overline{R}[\overline{f}, \overline{g}] \) over \( \overline{R} \) is either one or two. If it is one, then \( \overline{f}, \overline{g} \) are linearly dependent over \( K := \text{qt}(\overline{R}) \) and so there exists an \( \overline{\alpha} \in \text{qt}(\overline{R})^* \) such that \( x_3 = \overline{\alpha}x_2 \), \( x_1 = \overline{\alpha}x_3 \), \( x_2 = \overline{\alpha}x_1 \) (where \( x_i \) is the image of \( X_i \) in \( \overline{R} \)); in particular, \( x_2^2 = x_1x_3 \) in \( \overline{R} \) and so

\[
X_2^2 = X_1X_3 + (X_3^2 - X_1X_2)Y
\]

for some \( Y \in R \). This is absurd. Thus, \( \text{trdeg}_{\overline{R}} \overline{R}[\overline{f}, \overline{g}] = 2 \), and so the height of \( \ker \phi \) is zero. This shows that \( \phi \) is injective, and hence an isomorphism. To finish the proof, consider an element \( x = \Phi(f, g) = a_3b \) of \( a_3B \cap R_0 \) (\( \Phi \in R[T_1, T_2] \) and
$b \in B)$. Then the image $\Phi$ of $\Phi$ in $\mathbb{R}[T_1, T_2]$ is in the kernel of $\phi$, and consequently it is zero, so $\Phi = a_3 h$ for some $h \in R[T_1, T_2]$, and hence $x = \Phi(f, g) \in a_3 R_0$ as desired. We conclude that $\ker D = R[f, g]$.

Next we prove that $D$ is not standard. To see this, it is enough to notice that $f$ is homogeneous of degree 2 in the $X_i$’s and the $Y_j$’s while each standard linear constant is homogeneous of degree 3. In other words, $f \in D \setminus R[L_1, L_2, L_3]$ where $L_1, L_2, L_3$ are the standard linear constants of $D$.

We finish by proving that the rank of $D$ over $k[X_1, X_2, X_3]$ is 3. Suppose on the contrary that $\text{rank } D < 3$, then $D$ annihilates a variable $F$ of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3]$ over $k[X_1, X_2, X_3]$. By Proposition 3.1, we may assume that $F = \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3$ for some unimodular row $(\alpha_1, \alpha_2, \alpha_3)$ of $k[X_1, X_2, X_3]$. Since $D(F) = 0$, we have

$$ (X^2_1 - X_2 X_3) \alpha_1 + (X^2_2 - X_1 X_3) \alpha_2 + (X^2_3 - X_1 X_2) \alpha_3 = 0. $$

Sending the variables $X_2, X_3$ to 0 in (3) simultaneously shows that $\alpha_1(X_1, 0, 0) = 0$, so $\alpha_1 \in (X_1, X_2, X_3) k[X_1, X_2, X_3]$; similarly, $\alpha_2, \alpha_3 \in (X_1, X_2, X_3) k[X_1, X_2, X_3]$ and this contradicts the fact that $1 \in (\alpha_1, \alpha_2, \alpha_3) k[X_1, X_2, X_3]$. □

**Remark 5.1.** The main result in [5] treats the case of elementary derivations $D = \sum_{i=1}^3 a_i \frac{\partial}{\partial Y_i}$ of $R[Y_1, Y_2, Y_3]$ where for some $i \in \{1, 2, 3\}$, $R/pR$ is a UFD for every prime divisor $p$ of $a_i$. With the notation of Theorem 5.1, each $a_i$ is prime and $R/a_i R$ is not a UFD.

**Remark 5.2.** The above theorem shows that the condition “fix-point-free” of Theorem 6.1 below is not superfluous. The Theorem also gives an example of a derivation satisfying condition (ii) above but neither of the conditions (iii), (v) and (vi) (clearly, $D$ is not fix-point-free and hence not surjective).

The above theorem can be used to construct counterexamples to the implication $(ii) \implies (i)$ of derivations $D$ satisfying “rank $D < n$”. First some notations. Let $m$ and $n$ be two positive integers such that $m < n$, $B_n = R[Y_1, \ldots, Y_n]$, $B_m = R[Y_1, \ldots, Y_m]$. Let $D = \sum_{i=1}^m a_i \frac{\partial}{\partial Y_i}$ be an $R$-elementary derivation of $B_m$.

**Proposition 5.1.** $D$ is standard as an $R$-elementary derivation of $B_m$ if and only if it is standard as an $R$-elementary derivation of $B_n$.

**Proof.** Consider $D$ as a derivation of $B_n$. The following two facts finish the proof:

- The standard linear constants of $D$ are the $L_{ij}$’s (as defined above) with $1 \leq i < j \leq m$ and $Y_{m+1}, \ldots, Y_n$. 


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\[ \ker D = C[Y_{m+1}, \ldots, Y_n] \] where \( C \) is the kernel of \( D \) as a derivation of \( B_m \). \( \square \)

We prove next that the implication \((ii) \implies (iv)\) is true in the case of a noetherian ring. Namely, we have the following proposition.

**Proposition 5.2.** Let \( R \) be a noetherian domain of characteristic zero, \( B = R[Y_1, \ldots, Y_m] \) and \( D = \sum_{i=1}^{m} a_i \frac{\partial}{\partial Y_i} \) an \( R \)-elementary derivation of \( B \). If \( \ker D \) is generated over \( R \) by linear forms, then it is a finitely generated \( R \)-algebra.

**Proof.** Let \( M \) be the set of all linear constants of \( D \), then clearly \( M \) is an \( R \)-module. If \( D = \sum_{i=1}^{m} a_i \frac{\partial}{\partial Y_i} \) where \( a_i \in R \), then it is clear that \( M \) is isomorphic as an \( R \)-module to the submodule

\[ N = \left\{ (\alpha_1, \ldots, \alpha_m) \in R^m; \begin{pmatrix} a_1 & \ldots & a_m \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = 0 \right\} \]

of \( R^m \). Since \( R \) is noetherian, \( R^m \) is noetherian and \( N \) is finitely generated \( R \)-module. \( \square \)

6. **Fix-point-free \( R \)-elementary derivations.** Let \( C \) be an integral domain containing \( \mathbb{Q} \), and let \( D : C \to C \) be a locally nilpotent derivation. It is well-known that there is an associated \( \mathbf{G}_a \)-action, \( \alpha : \mathbf{G}_a \times \text{Spec} \ C \to \text{Spec} \ C \), and it turns out that the set of fixed points of \( \alpha \) is the closed subset \( V(I) \) of \( \text{Spec} \ C \), where \( I \) denotes the ideal \((DC)\) of \( C \) generated by \( DC \) (the image of \( D \)). In particular, \( \alpha \) is fix-point-free if and only if \((DC) = C \). This motivates the definition of fix-point-free derivation given in Definition 1.3.

Obviously, if a derivation of \( B \) admits a slice then it is fix-point-free. It is well-known that the converse is not true in general. The following proposition proves, among other things, that the converse holds for elementary derivations.

**Proposition 6.1.** Let \( R \) be a domain containing \( \mathbb{Q} \). If \( B = R[Y_1, \ldots, Y_m] = R^m \), and \( D : B \to B \) an \( R \)-elementary derivation, then:

1. If \( D \) is fix-point-free, then it admits a slice. Moreover, \( \ker D \) can be generated by \( m \) linear constants.

2. If \( D \) is fix-point-free and \( R \) is Hermite, then there exists a coordinate system \((Z_1, \ldots, Z_m)\) of \( B \) over \( R \) related to \((Y_1, \ldots, Y_m)\) by a linear change of variables, such that \( D = \partial / \partial Z_m \).
Proof. Write \( D = \sum_{i=1}^{m} a_i \partial_i \) where \( a_i \in R \) and \( \partial_i = \partial/\partial Y_i \). If \( D \) is fix-point-free then \( 1 \in (DY_1, \ldots, DY_m) \) so \( \sum_{i=1}^{m} a_i r_i = 1 \) for some \( (r_1, \ldots, r_m) \in R^m \). Consequently, \( s = \sum_{i=1}^{m} r_i Y_i \) is a slice of \( D \) and by Proposition 1.1, \( B = A[s] = A^{[1]} \) where \( A = \ker D \). Also, Proposition 1.1 shows that \( \ker D = R[\zeta(Y_1), \ldots, \zeta(Y_m)] \) where \( \zeta \) is the homomorphism of \( R \)-algebras:

\[
\zeta : B \rightarrow B \quad x \mapsto \sum_{i \geq 0} \frac{1}{i!} (-s)^i D^i(x)
\]

In particular, each \( \zeta(Y_i) \) is a linear constant.

If \( R \) is a Hermite ring, then \( (r_1 \ldots r_m) \) is extendible, i.e., it is the first row of a matrix \( U \in \text{Gl}_m(R) \) and it follows that \( s \) is a variable of \( B \) over \( R \) by Proposition 2.2. Also, Proposition 2.2 shows that \( \ker D = R[\zeta(Y_1), \ldots, \zeta(Y_m)] \) by a linear change of variables. \( \square \)

Remark 6.1. Proposition 6.1 shows in particular that if \( D : B \rightarrow B \) is fix-point-free elementary derivation of \( B \), then \( D \) is surjective (since it has a slice) and \( \ker D \) is finitely generated over \( R \) by \( m \) linear constants.

Remark 6.2. In the above proposition, \( R \) needs not to be a UFD. It suffices that \( R \) is any domain containing the rationals.

We prove next that “fix-point-free” implies “standard” in the easy case where the image under \( D \) of one of the \( Y_i \)'s is a unit. Namely:

**Proposition 6.2.** Let \( R \supseteq \mathbb{Q} \) be a UFD, \( B = R[Y_1, \ldots, Y_m] = R^{[m]} \) and \( D : B \rightarrow B \) an \( R \)-elementary derivation. If \( DY_i \in R^* \) for some \( i \), then \( \ker D \) is generated by \( m - 1 \) standard linear constants.

**Proof.** We may assume that \( DY_1 \in R^* \). Define \( s = (DY_1)^{-1}Y_1 \), then \( s \) is a slice of \( D \) and consequently the map \( B \xrightarrow{\zeta} B \) defined by \( \xi(x) = \sum_{j \geq 0} \frac{1}{j!} (-s)^j D^j(x) \) is a homomorphism of \( R \)-algebras with image equal to \( \ker D \).

Thus \( \ker D = R[\zeta(Y_1), \ldots, \zeta(Y_m)] \) and we are done since \( \zeta(Y_j) = Y_j - (DY_j)s = L_{1,j} \) for each \( j \). \( \square \)
We prove now the main result of this section.

**Theorem 6.1.** Let $R \supseteq \mathbb{Q}$ be a UFD, $B = R[Y_1, \ldots, Y_m] = R^m$ and $D : B \to B$ an $R$-elementary derivation. If $D$ is fix-point-free, then it is standard.

**Proof.** By Proposition 6.1,

$$\ker D = R[\xi(Y_1), \ldots, \xi(Y_m)],$$

where each $\xi(Y_i) = Y_i - a_i s$ is a linear constant. We obtain:

(4) \quad $\ker D$ is generated as an $R$-algebra by $m$ linear constants.

So it suffices to show that each linear constant is a linear combination (over $R$) of the standard linear constants. In other words, we have to show that the $R$-module $T(D)$ is trivial, where:

- $\text{LC}(D) = \text{set of linear constants of } D \text{ (an } R\text{-submodule of } \ker D\text{),}$
- $\text{SLC}(D) = R\text{-submodule of } \text{LC}(D) \text{ generated by the standard linear constants,}$
- $T(D) = \text{LC}(D) / \text{SLC}(D).$

Let $\mathfrak{m}$ be a maximal ideal of $R$ and consider the derivation $D_{\mathfrak{m}} : B_{\mathfrak{m}} \to B_{\mathfrak{m}}$ obtained by localization at the set $R \setminus \mathfrak{m}$. Now $R_{\mathfrak{m}}$ is a UFD, $B_{\mathfrak{m}} = R_{\mathfrak{m}}[Y_1, \ldots, Y_m] = R_{\mathfrak{m}}^m$ and $D_{\mathfrak{m}} = \sum_{i=1}^{m} a_i \partial_i$ is an $R_{\mathfrak{m}}$-elementary derivation. Since $D$ is fix-point-free, we have $(a_1, \ldots, a_m)R \nsubseteq \mathfrak{m}$ so, for some $i$, $a_i$ is a unit of $R_{\mathfrak{m}}$. By Proposition 6.2, $D_{\mathfrak{m}}$ is standard, so $T(D_{\mathfrak{m}}) = 0$. It is immediate that $\text{LC}(D_{\mathfrak{m}}) = \text{LC}(D)$ and $\text{SLC}(D_{\mathfrak{m}}) = \text{SLC}(D)$, so $T(D_{\mathfrak{m}}) = T(D)$ and we have shown:

$$T(D)_{\mathfrak{m}} = 0 \quad \text{for all maximal ideals } \mathfrak{m} \text{ of } R.$$

We conclude that $T(D) = 0$ and the result follows. □

So far we have shown that the implications $(iii) \implies (i)$, $(iii) \implies (ii)$, $(iii) \implies (iv)$ and $(iii) \implies (v)$ are all true. By Proposition 6.1, we also know that $(iii) \implies (vi)$ is true in the case of Hermite rings. In this case, we can actually say a lot more: the rank of the derivation is one and hence it is “conjugate to a partial derivative”.

If $R$ is not Hermite, we don’t know if $(iii) \implies (vi)$ is true or not. However, the following gives an example of a fix-point-free elementary derivation which is not “conjugate to a partial derivative” of $B$. 
Proposition 6.3. Let $R = \mathbb{R}[x, y, z]$ be as in Example 2.1 above, and let $B = R[Y_1, Y_2, Y_3] \cong R^3$. Let $D = x\frac{\partial}{\partial Y_1} + y\frac{\partial}{\partial Y_2} + z\frac{\partial}{\partial Y_3}$. Then $D$ is fix-point-free $R$-elementary derivation of $B$ satisfying $\text{rank } \tilde{D} \geq 2$.

Proof. Let $s = xY_1 + yY_2 + zY_3 \in B$, then $D(s) = x^2 + y^2 + z^2 = 1$ in $R$, and $s$ is then a slice of $D$. In particular $D$ is fix-point-free, and $B = A[s] \cong A[1]$ where $A = \ker D$. We prove next that $\text{rank } D \geq 2$. Clearly $\text{rank } D \neq 0$, so it suffices to show that $\text{rank } D = 1$. Assume that $\text{rank } D = 1$, then one can find a coordinate system $(F, G, H)$ of $B$ over $R$ such that $D = \Phi(F, G, H) \frac{\partial}{\partial H}$ for some $\Phi \in R^3$. Clearly, $A = R[F, G]$ and so $B = A[s] = \mathbb{R}[F, G, s]$. Thus, $s$ is a variable of $B$ over $R$. By Proposition 2.2, $(x, y, z)$ is an extendible unimodular row. This is a contradiction (see Example 2.1) \qed

7. The case where ker $D$ is finitely generated as an $R$-algebra.

It was conjectured in [5] that if $D$ is an $R$-elementary monomial derivation of $R[Y_1, Y_2, Y_3]$ such that ker $D$ is a finitely generated $R$-algebra then the generators of ker $D$ can be chosen to be linear in the $Y_i$’s. In this section we prove that this is not always the case. Theorem 7.1 gives a counterexample to the implications $(iv) \implies (i), (iv) \implies (ii), (iv) \implies (iii)$.

Theorem 7.1. The kernel of the derivation

$$D = X_1^2 \frac{\partial}{\partial Y_1} + X_2^2 \frac{\partial}{\partial Y_2} + X_3^2 \frac{\partial}{\partial Y_3} + X_2X_3 \frac{\partial}{\partial Y_4}$$

of $k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4] \cong k^7$ is a finitely generated $k[X_1, X_2, X_3]$-algebra which cannot be generated over $k[X_1, X_2, X_3]$ by linear forms in the $Y_i$’s.

To that end we will use Proposition 1.2 and the elimination theory of Groebner bases. Regarding Groebner bases, $S$-polynomials and Buchberger’s criteria, the reader may refer to ([1]).

Consider the following elements of ker $D$

$$L_{12} = X_1^2Y_2 - X_3^2Y_1$$
$$L_{14} = X_1^2Y_4 - X_2X_3Y_1$$
$$L_{34} = X_3Y_4 - X_2Y_3$$
$$f = X_1^2Y_4^2 - X_2^2Y_2Y_3 + X_3^2Y_1Y_2 + X_2^3Y_1Y_3 - 2X_2X_3Y_1Y_4.$$ 

We will prove that ker $D = k[X_1, X_2, X_3, f, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}]$. For this, let $k[X, Y, T]$ denote the polynomial ring

$$k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4, T_1, T_2, T_3, T_4, T_{12}, T_{13}, T_{14}, T_{24}, T_{34}]$$
in 16 variables and let $I$ be the ideal of $k[X,Y,T]$ generated by the elements

$$T_1 - X_1, T_2 - X_2, T_3 - X_3, T_4 - f, T_{12} - L_{12}, T_{13} - L_{13},$$
$$T_{14} - L_{14}, T_{24} - L_{24}, T_{34} - L_{34}, X_1.$$

The next lemma gives a Groebner basis for the ideal $I$. The elements of this basis will be used in computing the generators of $\ker D$. The proof of the lemma is left to the reader.

**Lemma 7.1.** A Groebner basis for $I$ with respect to the lexicographic order on $k[X,Y,T]$ with

$$X_1 > X_2 > X_3 > Y_1 > \ldots > Y_4 > T_1 > \ldots > T_4 > T_{12} > T_{13} > T_{14} > T_{24} > T_{34}$$

is given by the elements

$g_1 = -T_2 + X_2$
$g_2 = -T_3 + X_3$
$g_3 = X_1$
$g_4 = Y_1 T_2^2 + T_{12}$
$g_5 = Y_1 T_3^2 + T_{13}$
$g_6 = Y_1 T_2 T_3 + T_{14}$
$g_7 = T_1$
$g_8 = -Y_4 T_2 + T_{24} + T_3 Y_2$
$g_9 = Y_3 T_2 - Y_4 T_3 + T_{34}$
$g_{10} = Y_2 T_{13} + Y_3 T_{12} - 2Y_4 T_{14} + T_4$
$g_{11} = -T_3 T_{12} + T_{14} T_2$
$g_{12} = T_2 T_{13} - T_3 T_{14}$
$g_{13} = T_4 + Y_1 T_3 T_{24} + Y_3 T_{12} - Y_4 T_{14}$
$g_{14} = -Y_2 T_{14} + Y_1 T_3 T_{24} + Y_4 T_{12}$
$g_{15} = Y_1 T_2 T_{34} - Y_3 T_{12} + Y_4 T_{14}$
$g_{16} = -Y_3 T_{14} + Y_1 T_3 T_{34} + Y_4 T_{13}$
$g_{17} = T_3 Y_3 T_{12} - T_3 Y_4 T_{14} + T_{14} T_{34}$
$g_{18} = Y_3 T_{12} T_{34} + Y_3 T_{14} T_{24} - Y_4 T_{13} T_{24} - Y_4 T_{14} T_{34} + T_4 T_{34}$
$g_{19} = -T_{14}^2 + T_{12} T_{13}$
$g_{20} = -T_{14} T_{34} + T_3 T_4 - T_{13} T_{24}$
$g_{21} = T_2 T_4 - T_{14} T_{24} - T_{12} T_{34}$
$g_{22} = -T_{13} Y_4 T_3 + T_{13} T_{34} + Y_3 T_3 T_{14}$
$g_{23} = Y_1 T_{24}^2 - Y_2 Y_3 T_{12} - Y_2 T_4 + Y_4^2 T_{12}$
$g_{24} = Y_1 T_{24} T_{34} + Y_3 Y_2 T_{14} + Y_3 T_4 - Y_4^2 T_{14}$
$g_{25} = T_{14}^2 Y_2 - 2Y_4 T_{14} T_{12} + T_4 T_{12} + Y_3 T_{12}^2$
Clearly, $Γ$ is a coordinate system of $\text{k}$.

Indeed, let $Γ = \{(1, \ldots , 16)\}$ be the 16-tuple

\[
\begin{align*}
g_{26} & = Y_1T_{34}^1 + Y_2^2T_{12} - 2Y_3Y_4T_{14} + Y_4^2T_{13}, \\
g_{27} & = T_{34}Y_2T_{14} - T_{34}Y_4T_{12} - T_{24}Y_3T_{12} + T_{24}Y_4T_{14}, \\
g_{28} & = T_{13}Y_3T_{14}T_{24} + Y_3T_{34}T_{14}^2 - Y_4T_{2}^2T_{14} - T_{13}Y_4T_{14}T_{34} + T_{13}T_{4}T_{34}.
\end{align*}
\]

We prove next that $\ker D = k[X_1, X_2, X_3, f, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}]$.

Let $k[T]$ and $k[X, Y]$ denote respectively the polynomial rings $k[T_1, T_2, T_3, T_4, T_{12}, T_{13}, T_{14}, T_{24}, T_{34}]$ and $k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]$. Let $A_0 = k[X_1, X_2, X_3, f, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}]$, then $A_0 \subseteq \ker D$ and $(A_0)_{X_i} = (\ker D)_{X_i}$ for $i = 1, 2, 3$. By Proposition 1.2, it is enough to show that $X_1k[X, Y] \cap A_0 \subseteq X_1A_0$ (the other inclusion being obvious). So let $x \in X_1k[X, Y] \cap A_0$ and choose $z \in k[X, Y]$, $Φ \in k[T]$ such that $x = Φ(X_1, X_2, X_3, f, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}) = X_1z$. This means that $Φ$ is in the kernel of the homomorphism

\[
\theta : k[T] \xrightarrow{ψ} A_0 \hookrightarrow k[X, Y] \xrightarrow{π} k[X, Y]/(X_1)
\]

where $π$ is the canonical epimorphism and $ψ$ sends $T_i$ to $X_i$, $i = 1, 2, 3$, $T_4$ to $f$ and $T_{jk}$ to $L_{jk}$. Also, consider the homomorphism

\[
κ : k[X, Y, T] \xrightarrow{σ} k[X, Y] \xrightarrow{π} k[X, Y]/(X_1)
\]

where $σ$ is the homomorphism sending $X_i$ to $X_i$, $Y_i$ to $Y_i$ ($i = 1, 2, 3, 4$), $T_i$ to $X_i$ ($i = 1, 2, 3$), $T_4$ to $f$, and $T_{ij}$ to $L_{ij}$. It is clear that $θ$ is the restriction of $κ$ to $k[T]$ and hence

\[
ker θ = ker κ \cap k[T].
\]

We claim that $\ker κ$ is the ideal $I$ (considered above) of $k[X, Y, T]$ generated by the elements

\[
X_1, T_1 - X_1, T_2 - X_2, T_3 - X_3, T_4 - f, T_{12} - L_{12}, T_{13} - L_{13},
\]

\[
T_{14} - L_{14}, T_{24} - L_{24}, T_{34} - L_{34}.
\]

Indeed, let $Γ = (γ_1, \ldots , γ_{16})$ be the 16-tuple

\[
(X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4, T_1 - X_1, T_2 - X_2, T_3 - X_3, T_4 - f,
\]

\[
T_{12} - L_{12}, T_{13} - L_{13}, T_{14} - L_{14}, T_{24} - L_{24}, T_{34} - L_{34}).
\]

Clearly, $Γ$ is a coordinate system of $k[X, Y, T]$, that is

\[
k[X, Y, T] = k[γ_1, \ldots , γ_{16}].
\]
The domain and codomain of $\kappa$ are respectively $k[\Gamma]$ and $k[\gamma_1, \ldots, \gamma_7]/(\gamma_1)$ and $\kappa$ is defined by

$$\kappa(\gamma_i) = \begin{cases} 
0, & \text{if } i = 1 \text{ or } i > 7 \\
\gamma_i + (\gamma_i), & \text{if } 2 \leq i \leq 7.
\end{cases}$$

So we have

$$\ker \kappa = \langle \gamma_1, \gamma_8, \gamma_9, \ldots, \gamma_{16} \rangle = I,$$

and the claim is proved.

Using the elimination theory, we know that the set $\Sigma = \{g_7, g_{11}, g_{12}, g_{19}, g_{20}, g_{21}\}$ generates the ideal $I \cap k[T]$ of $k[T]$. Hence,

$$\Phi = \sum \xi_i h_i(T)$$

(6)

where $\xi_i \in k[T]$ and $h_i \in \{g_7, g_{11}, g_{12}, g_{19}, g_{20}, g_{21}\}$. On the other hand, one can easily verify the following identities:

$$
\begin{align*}
\psi(g_7) &= X_1 \\
\psi(g_{11}) &= -X_3 L_{12} + X_2 L_{14} = X_1^{-2} L_{24} \\
\psi(g_{12}) &= -X_3 L'_{14} + X_2 L_{13} = -X_1^2 L_{34} \\
\psi(g_{19}) &= -L_1^2 + L_{12} L_{13} = X_1^2 f \\
\psi(g_{20}) &= -L_{14} L_{34} + X_3 f - L_{13} L_{24} = 0 \\
\psi(g_{21}) &= X_2 f - L_{14} L_{24} - L_{12} L_{34} = 0.
\end{align*}
$$

This means that $x = \Phi(X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}, f) \in X_1 A_0,$

and consequently

$$\ker D = k[X_1, X_2, X_3, f, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}].$$

The next two lemmas show that $\ker D$ cannot be generated over $k[X_1, X_2, X_3]$ by linear forms in the $Y_i$’s.

**Lemma 7.2.** With the above notation, if $L$ is an element of $\ker D$ of the form

$$L = \alpha_1 Y_1 + \cdots + \alpha_4 Y_4$$

for some $\alpha_1, \ldots, \alpha_4 \in k[X_1, X_2, X_3]$, then

$$L \in k[X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}].$$
Proof. If $L$ is a linear form in the $Y_i$’s over $k[X_1, X_2, X_3]$ in ker $D$, then $L$ has the form

$$L = \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 + \alpha_4 Y_4$$

where $\alpha_i \in k[X_1, X_2, X_3] \; i \in \{1, 2, 3, 4\}$. Since $L \in \text{ker} \; D$, we have

$$\alpha_1 X_1^2 + \alpha_2 X_2^2 + \alpha_3 X_3^2 + \alpha_4 X_2 X_3 = 0. \quad (7)$$

Let $\phi = \alpha_1 X_1^2 + \alpha_2 X_2^2 + \alpha_3 X_3^2$, then equation (7) shows that both $X_2$ and $X_3$ are divisors of $\phi$. Taking equation (7) modulo $X_2$ gives that

$$X_2^2 \alpha_{12} + X_3^2 \alpha_{32} = 0 \quad (8)$$

where $\alpha_{12} = \alpha_1 |_{X_2=0}$ and $\alpha_{32} = \alpha_3 |_{X_2=0}$. Since $X_1$ and $X_3$ are relatively prime, equation (8) implies that $\alpha_1 = -X_3^2 \beta_{32} + X_2 \beta_1$ and $\alpha_3 = X_1^2 \beta_{32} + X_2 \beta_3$ for some $\beta_1, \beta_3 \in k[X_1, X_2, X_3]$ and $\beta_{32}$ in $k[X_1, X_3]$. After simplification we find

$$\phi = X_1^2 X_2 \beta_1 + X_2 X_3^2 \beta_3 + \alpha_2 X_2^2. \quad (9)$$

Since $X_3$ is a divisor of $\phi$, equation (9) implies that

$$X_2^2 X_3 \beta_1 |_{X_3=0} + X_2^2 \alpha_2 |_{X_3=0} = 0.$$

Consequently, $\alpha_2 = X_1^2 u + X_3 v$ and $\beta_1 = -X_2 u + X_3 w$ for some $u \in k[X_1, X_2]$ and $v, w \in k[X_1, X_2, X_3]$. Replacing these values of $\alpha_2$ and $\beta_1$ in the expression (9) of $\phi$, we get

$$\phi = X_2 X_3 (X_1^2 w + X_3 \beta_3 + X_2 v)$$

and consequently $\alpha_4 = -\phi / (X_2 X_3) = -(X_1^2 w + X_3 \beta_3 + X_2 v)$. Hence,

$$\begin{align*}
\alpha_1 &= -X_3^2 u - X_3^2 \beta_{32} + X_2 X_3 w \\
\alpha_2 &= X_1^2 u + X_3 v \\
\alpha_3 &= X_1^2 \beta_{32} + X_2 \beta_3 \\
\alpha_4 &= -(X_1^2 w + X_3 \beta_3 + X_2 v)
\end{align*}$$

and so

$$L = \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3 + \alpha_4 Y_4$$

$$= u(X_1^2 Y_2 - X_3^2 Y_1) + \beta_{32} (X_1^2 Y_3 - X_3^2 Y_1)$$

$$+ v(X_3 Y_2 - X_2 Y_3) - w(X_1^2 Y_4 - X_2 X_3 Y_1)$$

$$+ \beta_3 (X_2 Y_3 - X_3 Y_2)$$

$$\in k[X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}].$$
Lemma 7.3. With the above notation,

\[ f \notin k[X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}] . \]

Proof. If \( f \in k[X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}] \), we can choose a polynomial \( \Phi \) in

\[ E := k[X_1, X_2, X_3, U_1, U_2, U_3, U_4, U_5] \]

such that

\[ (10) \quad f = \Phi(X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}). \]

Consider the \( \mathbb{N}^2 \)-grading on \( k[X, Y] \) defined by declaring \( k \subseteq k[X, Y]_{(0,0)} \) and \( \deg(X_i) = (1, 0), \deg(Y_j) = (0, 1) \) for \( i \in \{1, 2, 3\} \) and \( j \in \{1, 2, 3, 4\} \). Also define a similar \( \mathbb{N}^2 \)-grading on \( E \) by \( k \subseteq E_{(0,0)} \) and \( \deg(X_i) = (1, 0), \deg(U_j) = (2, 1) \) for \( j \in \{1, 2, 3\} \), and \( \deg(U_4) = \deg(U_5) = (1, 1) \). Write

\[ \Phi = \Phi_{d_1} + \Phi_{d_2} + \cdots + \Phi_{d_r} \]

where \( \Phi_{d_i} \) is the homogeneous component of \( \Phi \) of degree \( d_i \in \mathbb{N}^2 \). Since the elements \( L_{12}, L_{13}, L_{14}, L_{24}, L_{34} \) are all homogeneous with respect to the \( \mathbb{N}^2 \)-grading on \( k[X, Y] \) defined above, it is easy to check that

\[ \Phi_{d_i}(X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}) \]

is either zero or homogeneous of degree \( d_i \), for all \( i \in \{1, \ldots, r\} \). Also, since \( f \) is a homogeneous element of degree \( (2, 2) \) of \( k[X, Y] \), equation (10) implies that

\[ f = \Phi_{(2,2)}(X_1, X_2, X_3, L_{12}, L_{13}, L_{14}, L_{24}, L_{34}) \]

and this can only happen if

\[ (11) \quad f = aL_{24}^2 + bL_{34}^2 + cL_{24}L_{34} \]

for some \( a, b, c \in k \). Indeed, a homogeneous element of degree \( (2, 2) \) of \( E \) can only be a linear combination of \( U_4^2, U_5^2 \) and \( U_4U_5 \) because of the degrees of the \( X_i \)'s and the \( U_i \)'s defined above.

Now equation (11) implies that \( f \in k[X_2, X_3, Y_2, Y_3, Y_4] \), which is absurd. \( \square \)

Theorem 7.1 is now a direct consequence of the above two lemmas.
8. The property of being elementary. Let $B = R^{[m]}$, where $R$ is a UFD containing the rationals; given an irreducible locally nilpotent derivation $D$ of $B$, can we determine whether $D$ is $R$-elementary? (That is, can we decide whether there exists a coordinate system $(Y_1, \ldots, Y_m)$ of $B$ over $R$ satisfying $DY_i \in R$ for all $i$?)

An answer in general seems to be hard. The present section answers the question in the case where $R$ is a PID and $m = 2$.

We start with two well known facts:

**Proposition 8.1 ([2]).** Let $R$ be a UFD containing $\mathbb{Q}$ and let $D \neq 0$ be a locally nilpotent $R$-derivation of $B = R[Y_1, Y_2] \cong R^{[2]}$. Then there exists $P \in B$ and $\alpha \in \ker D$ such that $\ker D = R[P]$ and $D = \alpha \left( P Y_2 \frac{\partial}{\partial Y_1} - P Y_1 \frac{\partial}{\partial Y_2} \right)$.

**Proposition 8.2 ([7]).** Let $R$ be a $\mathbb{Q}$-algebra, let $P \in B = R[Y_1, Y_2] \cong R^{[2]}$ and define $\Delta_P = P Y_2 \frac{\partial}{\partial Y_1} - P Y_1 \frac{\partial}{\partial Y_2} : B \to B$. Then the following are equivalent.

1. $P$ is a variable of $B$ over $R$
2. $D$ is locally nilpotent, has a slice and $\ker D = R[P]$.

**Lemma 8.1.** Let $R$ be PID containing $\mathbb{Q}$, $B = R^{[m]}$ and $D : B \to B$ an irreducible $R$-derivation. The following are equivalent:

1. $D$ is $R$-elementary
2. $D = \partial / \partial Z_1$ for some coordinate system $(Z_1, \ldots, Z_m)$ of $B$ over $R$.

**Proof.** If $D$ is $R$-elementary, then there exists a coordinate system $(Y_1, \ldots, Y_m)$ of $B$ over $R$ satisfying $DY_i \in R$ for all $i$. Let $a_i = DY_i$ for each $i$. Since $R$ is a PID, $(a_1, \ldots, a_m)B$ is a principal ideal of $B$ and it follows that $(a_1, \ldots, a_m)B = B$ by the irreducibility of $D$; so $D$ is fix-point-free. As $R$ is Hermite (every PID is Hermite), Proposition 6.1 implies that condition (2) holds. The converse is clear. $\square$

**Proposition 8.3.** Let $R$ be PID containing $\mathbb{Q}$, $B = R^{[2]}$ and $D : B \to B$ an irreducible $R$-derivation. The following are equivalent:

1. $D$ is $R$-elementary
2. $D$ is locally nilpotent and fix-point-free.
Proof. By Lemma 8.1, it is clear that (1) implies (2). If (2) holds, let \((Y_1, Y_2)\) be any coordinate system of \(B\) over \(R\); then Propositions 8.1 and 8.2 imply that, for some variable \(P\) of \(B\) over \(R\), we have \(\ker D = R[P]\) and 
\[
D = P Y_2 \frac{\partial}{\partial Y_1} - P Y_1 \frac{\partial}{\partial Y_2}.
\]
Choose \(Q\) such that \(B = R[P, Q]\), then \(D(Q) \in R^*\) and \(D(P) = 0 \in R\), so \(D\) is \(R\)-elementary. □

**Example 8.1.** Choose \(f(X) \in k[X]\) and \(g(X, Y) \in k[X, Y]\) such that
\[
\gcd(f(X), g(X, Y)) = 1
\]
and let \(D\) be the \(k\)-derivation of \(k[X, Y, Z]\) defined by
\[
D(X) = 0, \quad D(Y) = f(X), \quad D(Z) = g(X, Y).
\]
Then \(D\) is an irreducible locally nilpotent \(k[X]\)-derivation of \(k[X, Y, Z]\). By Proposition 8.3, \(D\) is \(k[X]\)-elementary if and only if
\[
(f(X), g(X, Y))k[X, Y] = k[X, Y].
\]

We conclude with the following:

**Proposition 8.4.** If \(R\) is a PID containing \(Q\), then any nonzero \(R\)-elementary derivation of \(B = R[Y_1, \ldots, Y_m]\) is standard.

Proof. Let \(D = \sum_{i=1}^m a_i \frac{\partial}{\partial Y_i}\) be such a derivation of \(B\) \((a_i \in R\) for all \(i\)). Write \(D = \alpha D'\) where \(\alpha \in B\) and \(D' : B \to B\) is an irreducible derivation. Note that \(\alpha D'(Y_i) \in R\) for all \(i\); it follows that \(\alpha \in R\) and that \(D'\) is \(R\)-elementary. By Lemma 8.1, \(D'\) is standard and hence \(D\) is also standard. □

**REFERENCES**


