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EXTENDED MAXIMUM PRINCIPLES

M. M. Al-Mahameed

Communicated by V. Petkov

ABSTRACT. In this paper we introduce some new results concerning the maximum principles for second order linear elliptic partial differential equations defined on a noncompact Riemannian manifold.

1. Introduction. We extend the classical generalized maximum principle which holds in relatively compact subdomains to a generalized maximum principle which holds in any domain, also we prove existence, uniqueness and integral representation for solutions of the nonhomogeneous equation.

Let us first introduce some terminology and results which we need in this paper.

Let $P$ be a linear, second order, elliptic operator defined on a noncompact, connected, $C^3$-smooth Riemannian manifold $\Omega$ of dimension $d$. Here $P$ is an elliptic operator with real, Hölder continuous coefficients which in any coordinate system $(U; x_1, \ldots, x_d)$ has the form

$$P(x, \partial_x) = -\sum_{i,j=1}^{d} a_{ij}(x) \partial_i \partial_j + \sum_{i=1}^{d} b_i(x) \partial_i + c(x),$$

2000 Mathematics Subject Classification: 35B50, 35L15.

Key words: Maximum principles, elliptic equations.
where $\partial_i = \partial / \partial x_i$. We assume that for every $x \in \Omega$ the real quadratic form

$$
(1.2) \quad \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d
$$

is positive definite.

We denote the cone of all positive (classical) solutions of the elliptic equation $Pu = 0$ in $\Omega$ by $C_P(\Omega)$. In case that the coefficients of $P$ are smooth enough, we denote by $P^*$ the formal adjoint of $P$.

**Definition 1.1.** (i) If $C_P(\Omega) \neq \emptyset$ and $P$ has a positive Green function in $\Omega$, then $P$ is said to be a subcritical operator in $\Omega$.

(ii) If $C_P(\Omega) \neq \emptyset$ and $P$ does not have a positive Green function in $\Omega$, then $P$ is said to be a critical operator in $\Omega$.

(iii) If $C_P(\Omega) = \emptyset$, i.e. the elliptic equation $Pu = 0$ does not have a positive solution in $\Omega$, then $P$ is said to be a supercritical operator in $\Omega$.

**Remark 1.2.** For Schrödinger equation

$$(\Delta + V)u = 0 \quad \text{in} \quad \Omega$$

where $\Omega$ is a connected open set in $\mathbb{R}^n$ and $V$ is a real-valued function belonging to $L_{p,\text{loc}}(\Omega)$ with $p > \frac{n}{2}$ for $n \geq 2$ and $p = 1$ for $n = 1$, M. Murata [3] introduced the following classification: (i) $V$ is subcritical if $-\Delta + V$ has a positive Green function; (ii) $V$ is critical if $-\Delta + V \geq 0$ (nonnegative); and (iii) $V$ is supercritical if $-\Delta + V$ is not nonnegative. Definition 1.1 is due to [4].

Let $(\{\Omega_k\})_{k=1}^{\infty}$ be an exhaustion of $\Omega$, i.e. a sequence of smooth, relatively compact domains such that $\Omega_1 \neq \emptyset$, $\Omega_k \subset \Omega_{k+1}$ and $\bigcup_{k=1}^{\infty} \Omega_k = \Omega$. Assume that $C_P(\Omega) \neq \emptyset$. Then for every $k \geq 1$ the Dirichlet Green function $G_P^{\Omega_k}(x, y)$ exists and is positive. By the generalized maximum principle, $\left\{G_P^{\Omega_k}(x, y)\right\}_{k=1}^{\infty}$ is an increasing sequence which, by Harnak inequality, converges uniformly in any compact subdomain of $\Omega$ either to $G_P^\Omega(x, y)$, the positive minimal Green function of $P$ in $\Omega$ and $P$ is said to be a subcritical operator in $\Omega$, or to infinity and in this case $P$ is critical in $\Omega$. The operator $P$ is said to be supercritical in $\Omega$ if $C_P(\Omega) = \emptyset$.

It follows that $P$ is critical (resp. subcritical) in $\Omega$ if and only if $P^*$ is critical (resp. subcritical) in $\Omega$. Furthermore, if $P$ is critical in $\Omega$, then $C_P(\Omega)$ is a one dimensional cone and any positive supersolution of the equation $Pu = 0$ in $\Omega$ is a solution. In this case $\phi \in C_P(\Omega)$ in called a ground state of $P$ in $\Omega$.

We fix a reference point $x_0 \in \Omega_1$. From time to time, we consider the convex set

$$
K_P(\Omega) := \{u \in C_P(\Omega) : u(x_0) = 1\}
$$
of all normalized positive solutions.

**Remark 1.3.** We would like to point out that the criticality theory, and in particular the results of this paper, are also valid for the class of weak solutions of elliptic equations in divergence form and also for the class of strong solutions of strongly elliptic equations with locally bounded coefficients. Nevertheless, for the sake of clarity, we prefer to present our results only for the class of classical solutions.

Subcriticality is a stable property in the following sense. If $P$ is subcritical in $\Omega$ and $V \in C_0^\infty(\Omega)$ is a real function, then there exist $\epsilon > 0$ such that $P - \mu V$ is subcritical, for all $|\mu| < \epsilon$ [3, 4]. On the other hand, if $P$ is critical in $\Omega$ and $V \in C^\alpha(\Omega)$ is a nonzero, nonnegative function, then for every $\epsilon > 0$ the operator $P + \epsilon V$ is subcritical and $P - \epsilon V$ is supercritical in $\Omega$.

We associate to $\Omega$ a fixed exhaustion $\{\Omega_n\}_{n=1}^\infty$. For every $k \geq 1$, we denote $\Omega_k^* = \Omega \setminus \overline{\Omega_k}$ and for every $k > k_0$ we denote by $\Omega_{k,k_0}$ the ‘annulus’ $\Omega_k \setminus \overline{\Omega_{k_0}}$. Let $f, g \in C(\Omega)$ be positive functions. We say that $f$ is equivalent to $g$ on $\Omega$ and use the notation $f \approx g$, if there exists a positive constant $C$ such that

$$C^{-1} g(x) \leq f(x) \leq C g(x) \quad \text{for all} \quad x \in \Omega.$$ 

We denote by $f^+$ (resp. $f^-$) the positive (resp. negative) part of a function $f$. So, $f = f^+ - f^-$. By 1, we denote the constant function on $\Omega$ taking at any point $x \in \Omega$ the value 1.

Let $B$ be a Banach space and $B^*$ its dual. If $T$ is a (bounded) operator in $B$ we denote by $T^*$ its adjoint. The range and the kernel of $T$ are denoted by $R(T)$ and $N(T)$. For every $f \in B$ and $g^* \in B^*$ we use the notation $\langle g^*, f \rangle := g^*(f)$. We denote the spectrum of an operator $T$ acting on $B$ by $\sigma(T)$.

**Definition 1.4.** Let $P$ be an elliptic operator defined on $\Omega$. A function $u \in C(\Omega_n^*)$ is said to be a positive solution of the operator $P$ of minimal growth in a neighborhood of infinity in $\Omega$ if $u$ satisfies the following two conditions:

(i) The function $u$ is a positive solution of the equation $Pu = 0$ in $\Omega_n^*$;

(ii) If $v$ is a continuous function on $\overline{\Omega_k^*}$ for some $k > n$ which is a positive solution of the equation $Pu = 0$ in $\Omega_k^*$, and $u \leq v$ on $\partial \Omega_k$, then $u \leq v$ on $\Omega_k^*$.

**Definition 1.5.** Let $P_i$, $i = 1, 2$ be two subcritical operators in $\Omega$. We say that the Green functions $G_{P_1}^\Omega(x,y)$ and $G_{P_2}^\Omega(x,y)$ are equivalent (resp. semi-equivalent) if $G_{P_1}^\Omega \approx G_{P_2}^\Omega$ on $\Omega \times \Omega \setminus \{(x,x) : x \in \Omega\}$ (resp. $G_{P_1}^\Omega(\cdot,y_0) \approx G_{P_2}^\Omega(\cdot,y_0)$ on $\Omega \setminus \{y_0\}$ for some fixed $y_0 \in \Omega$).

Our maximum principles are of Phragmén-Lindelöf type and state that if $Pu \geq 0$ in $\Omega$, $P$ is subcritical in $\Omega$ and $u$ satisfies some growth condition near infinity in $\Omega$, then $u \geq 0$ (see Theorems 1.6 and 2.6). As a consequence, we prove
existence and uniqueness theorems for suitable solutions of the nonhomogeneous equation (see Theorems 1.8 and 2.8).

Let $P$ be a subcritical operator in $\Omega$ and let $\phi \in C(\Omega)$ be a positive function such that $\phi$ is a solution of the equation $Pu = 0$ in $\Omega^*_1$ which has a minimal growth in a neighborhood of infinity in $\Omega$.

We denote by $B$ the real ordered Banach space

$$B = \{ u \in C(\Omega) / |u(x)| \leq c\phi(x) \text{ for some } c > 0 \text{ and all } x \in \Omega \}$$

equipped with the norm

$$\|u\|_B = \inf \{ c > 0 / |u(x)| \leq c\phi(x) \text{ } \forall x \in \Omega \}.$$ 

The ordering on $B$ is the natural pointwise ordering of functions. For the purpose of spectral theory, we consider also the canonical complexification of $B$ without changing our notation.

In [6] the following maximum principle for supersolutions in $B$ is proved.

**Theorem 1.6.** Let $P$ be a subcritical operator in $\Omega$ and let $\phi \in C(\Omega)$ be a positive function such that $\phi$ is a solution of the equation $Pu = 0$ in $\Omega^*_1$ which has a minimal growth in a neighborhood of infinity in $\Omega$. Suppose that $v \in B$ satisfies the equation $Pv = f \geq 0$ in $\Omega$, where $f \in C^\alpha(\Omega)$. Then $v \geq 0$ in $\Omega$.

In the critical case we have (see [6])

**Proposition 1.7.** Let $P$ be a critical operator in $\Omega$ and let $\phi_0$ be a ground state of the operator $P$ in $\Omega$. Suppose that $Pu \geq 0$ in $\Omega$ and that for some $C > 0$, $u \geq -C\phi_0$ in $\Omega^*_1$. Then $u = C_1\phi_0$, where $C_1$ is a real constant.

The maximum principle (Theorem 1.6) implies the following theorem concerning the existence, uniqueness and integral representation for solutions of the nonhomogeneous equation [6].

**Theorem 1.8.** Let $P$ be a subcritical operator in $\Omega$ and let $\phi$ be a positive solution of the equation $Pu = 0$ in $\Omega^*_1$ which has a minimal growth in a neighborhood of infinity in $\Omega$.

(i) Let $f \in C^\alpha(\Omega)$, $0 < \alpha \leq 1$, be a real function such that

$$\int_{\Omega} G_\Omega^\alpha(x,y) |f(y)| \, dy \leq C\phi(x)$$

for all $x \in \Omega^*_2$. Then there exists a unique solution $u \in B$ of the equation $Pu = f$ in $\Omega$. Moreover,

$$u(x) = \int_{\Omega} G_\Omega^\alpha(x,y) f(y) \, dy.$$
(ii) Suppose that \( f \in C^\alpha(\Omega), \ 0 < \alpha \leq 1, \) and \( f \geq 0. \) Then \( f \) satisfies estimate (1.3) if and only if there exists a solution \( u \in B \) of the equation \( Pu = f \) in \( \Omega. \) In this case, \( u \) is the minimal nonnegative solution of the equation \( Pv = f \) in \( \Omega. \)

2. The results and their proofs. We present now a new version of the maximum principle which extends Theorem 1.6 and is valid for solutions which don’t grow too fast.

We denote by \( \beta = \beta^K \) the real ordered Banach space.

\[
\beta = \beta^K := \{ f \in C(\Omega)/ |f(x)| \leq cu(x) \ \text{for some fixed} \ c > 0 \ \text{and} \ u \in K_p(\Omega), \]

(2.1) equipped with the norm

\[
\|f\|_\beta := \inf_{u \in K_p(\Omega)} \inf \{ c > 0 / |f(x)| \leq cu(x) \ \forall x \in \Omega \}.
\]

The ordering on \( \beta \) is the natural pointwise ordering of functions. For the purpose of spectral theory, we consider also the canonical complexification of \( \beta \) without changing our notation. Clearly, \( B \subset \beta \) and there exist \( C > 0 \) such that \( \|f\|_\beta \leq C \|f\|_B \) for all \( f \in B. \)

Recall that \( \phi \in C(\Omega) \) is a fixed positive function such that \( \phi \) is a solution of the equation \( Pu = 0 \) in \( \Omega^+ \) which has a minimal growth in a neighborhood of infinity in \( \Omega. \) We consider also the set

\[
B^{K,\phi}_{\infty} := \left\{ f / f \in C(\Omega_x) \ \text{for some} \ N \in \mathbb{N}, \ \text{and} \ \forall \epsilon > 0 \ \exists n \geq N, \ C_\epsilon \geq 0, \ u_\epsilon \in K_p(\Omega) \ \text{such that} \ |f(x)| \leq cu_\epsilon(x) + C_\epsilon \phi(x) \ \text{in} \ \Omega^n \right\}.
\]

Remark 2.1. (i) Note that \( B^{K,\phi}_{\infty} \cap C(\Omega) \) is a closed subspace of \( \beta. \) We denote this Banach subspace by \( \beta_0 = \beta^K. \) Clearly, \( B \subset \beta_0. \)

(ii) Consider the following closed subspace of \( \beta \)

\[
A^K := \left\{ f \in C(\Omega) / \lim_{n \to \infty} \inf_{u \in K_p(\Omega), x \in \Omega^n} \sup \{|f(x)|/u(x)| = 0 \right\},
\]

(2.2) which contains functions that grow slower than functions in \( K_p(\Omega). \) Clearly, \( A^K \subset \beta_0 \) and if \( B \subset A^K, \) then \( A^K = \beta_0. \) There are examples where \( B \not\subset A^K \) [2]. Note that A. Ancona proved recently that if \( P \) is symmetric with respect to a Riemannian measure \( \sigma, \) then \( B \subset A^K \) [2].
We first prove some lemmas which will imply the maximum principle in $\beta_0$.

**Lemma 2.2.** Let $P$ be a critical or subcritical operator in $\Omega$. Suppose that $v \in C(\overline{\Omega}_2^*) \cap B_{K,\phi}^{K,\phi}$ is a solution of the equation $Pu = 0$ in $\Omega_2^*$ such that $v = 0$ in $\partial \Omega_2$. Then $v = 0$.

**Proof.** Suppose that $v(x_1) > 0$ for some $x_1 \in \Omega_2^*$ and let $\epsilon > 0$. By definition, there exist $n \in \mathbb{N}$, $C_\epsilon \geq 0$ and $u_\epsilon \in K_p(\Omega)$ such that $|v(x)| \leq \epsilon u_\epsilon + C_\epsilon \phi$ in $\Omega_n^*$. By the generalized maximum principle $|v(x)| \leq \epsilon u_\epsilon(x) + C_\epsilon \phi(x)$ in $\Omega_2^*$. Let

$$C_{0,\epsilon} = \inf \{ C \geq 0/\epsilon u_\epsilon + C \phi(x) - v(x) \geq 0 \text{ in } \Omega_2^* \}.$$ 

It follows that $\epsilon u_\epsilon + C_{0,\epsilon} \phi(x) - v(x) > 0$ in $\Omega_2^*$. Since $\phi$ is a positive solution of the equation $Pu = 0$ in $\Omega_2^*$ which has a minimal growth in a neighborhood of infinity in $\Omega$, it follows that there exist $\delta = \delta(\epsilon) > 0$ such that $\epsilon u_\epsilon + C_{0,\epsilon} \phi(x) - v(x) > \delta \phi(x)$ in $\Omega_2^*$. By the minimality of $C_{0,\epsilon}$, we infer that for every $\epsilon > 0$ $C_{0,\epsilon} = 0$. Hence, $\epsilon u_\epsilon - v(x) > 0$ in $\Omega_2^*$ for every $\epsilon > 0$, contradicting the hypothesis that $v(x_1) > 0$. $\square$

**Lemma 2.3.** Let $P$ be a critical or subcritical operator in $\Omega$. Suppose that $v \in C(\overline{\Omega}_2) \cap B_{K,\phi}^{K,\phi}$ is a solution of the equation $Pu = 0$ in $\Omega_2^*$ such that $v \geq 0$ on $\partial \Omega_2$. Then $v \geq 0$ and $v/\phi$ is bounded in $\Omega_2^*$. Moreover, if $v > 0$ on $\partial \Omega_2$, then $v$ is a positive solution of minimal growth in a neighborhood of infinity in $\Omega$.

**Proof.** Denote by $w_k, k > 2$ the solutions of the following Dirichlet problems

$$Pu = 0 \text{ in } \Omega_{k,2},$$

$$u = v \text{ on } \partial \Omega_2,$$

$$u = 0 \text{ on } \partial \Omega_k.$$

By the generalized maximum principle, the sequence $\{w_k\}_{k > 2}$ is a nondecreasing bounded sequence of nonnegative solutions which converges to a function $w \leq C_{1,2}$. Recall that $v \neq 0$ on $\partial \Omega_2$. It follows that $w$ is a nonzero, nonnegative solution of the equation $Pu = 0$ in $\Omega_2^*$ (in fact, $w > 0$ on at least one component of $\Omega_2^*$). Moreover, if $v > 0$ on $\partial \Omega_2$, then $w$ is a positive solution of minimal growth in a neighborhood of infinity in $\Omega$.

Let $v_k, k > 2$, be the solutions of the Dirichlet problems

$$Pu = 0 \text{ in } \Omega_{k,2},$$

$$u = 0 \text{ on } \partial \Omega_2,$$

$$u = v \text{ on } \partial \Omega_k.$$
Then \( v = w_k + v_k \) and therefore, the sequence \( \{v_k\}_{k>2} \) converges to a function \( v_0 \). On the other hand, since \( v \in B_{\infty}^{K,\phi} \), it follows that for every \( \epsilon > 0 \) there exist \( n \in \mathbb{N} \), \( C_\epsilon \geq 0 \) and \( u_\epsilon \in K_p(\Omega) \) such that \(|v(x)| \leq \epsilon u_\epsilon + C_\epsilon \phi \) in \( \Omega^*_n \). By the generalized maximum principle, for every \( k \geq n \) we have \(|v_k(x)| \leq \epsilon u_\epsilon + C_\epsilon \phi \) in \( \Omega^*_n \). Hence, \(|v_0(x)| \leq \epsilon u_\epsilon + C_\epsilon \phi \) in \( \Omega^*_2 \). Lemma 2.2 implies that \( v_0 = 0 \). Hence, \( v = w \geq 0 \) and \( v \) is a nonnegative solution such that \( v/\phi \) is bounded in \( \Omega^*_2 \). □

**Lemma 2.4.** Let \( P \) be a critical or subcritical operator in \( \Omega \). Let \( f \in C^0(\overline{\Omega}_2) \) be a nonnegative function. Suppose that \( v \in C(\overline{\Omega}_2) \cap B_{\infty}^{K,\phi} \) is a solution of the equation \( Pu = f \) in \( \Omega_2^* \) such that \( v \geq 0 \) on \( \partial \Omega_2 \). Then \( v \geq 0 \). Moreover,

\[
(2.3) \quad v(x) = h(x) + \int_{\Omega_2^*} G_{P}^{\Omega_2^*}(x,y)f(y)dy,
\]

where \( h \in C(\overline{\Omega}_2^*) \) is a nonnegative solution of the equation \( Pu = 0 \) in \( \Omega_2^* \) which is bounded by \( C\phi \) (for some constant \( C > 0 \)) and satisfies the boundary condition \( h = v \) on \( \partial \Omega_2 \). In particular, \( \int_{\Omega_2^*} G_{P}^{\Omega_2^*}(x,y)f(y)dy < \infty \).

**Proof.** Since \(|v| = v^+ + v^-\), it follows that \( v^\pm \in B_{\infty}^{K,\phi} \). Let \( w_{k,\pm} \), \( k \geq 2 \) be the nonnegative solutions of the Dirichlet problems

\[
P v = f_\pm \quad \text{in} \quad \Omega_k^*,
\]

\[
u = v_\pm \quad \text{on} \quad \partial \Omega_k,
\]

By the generalized maximum principle and the definition of \( B_{\infty}^{K,\phi} \) it follows that for every \( \epsilon > 0 \) there exist \( n \in \mathbb{N} \), \( C_\epsilon \geq 0 \) and \( u_\epsilon \in K_p(\Omega) \) such that for every \( k \geq n \) we have \( 0 \leq w_k, -(x) \leq \epsilon u_\epsilon + C_\epsilon \phi \) in \( \Omega^*_n \). By a standard elliptic argument, the sequence \( \{w_k\}_{k} \) has a converging subsequence to a nonnegative solution of the equation \( Pu = 0 \) in \( \Omega_2^* \) which takes the value zero on \( \partial \Omega_2 \). Since any such solution is in \( B_{\infty}^{K,\phi} \), it follows from Lemma 2.2 that it is the zero solution and \( \lim_{k \to \infty} w_{k,-} = 0 \).

On the other hand, \( w_{k,+} \geq 0 \) and since

\[
w_{k,+} - w_{k,-} = v,
\]

it follows that \( \lim_{k \to \infty} w_{k,+} = v \geq 0 \).
Note that
\[ w_{k,+}(x) = h_k(x) + g_k(x) + \int_{\Omega_{k,2}} G_{P}^{\Omega_{k,2}}(x, y) f(y) dy, \]
where \( h_k \) satisfies
\[
\begin{align*}
Pu &= 0 \quad \text{in} \quad \Omega_{k,2}, \\
u &= v \quad \text{on} \quad \partial\Omega_2, \\
u &= 0 \quad \text{on} \quad \partial\Omega_k,
\end{align*}
\]
and \( g_k \) satisfies
\[
\begin{align*}
Pu &= 0 \quad \text{in} \quad \Omega_{k,2}, \\
u &= 0 \quad \text{on} \quad \partial\Omega_2, \\
u &= v \quad \text{on} \quad \partial\Omega_k.
\end{align*}
\]
Clearly, \( 0 \leq h_k \leq C\phi \), and \( \{h_k\} \) converges to a nonnegative solution \( h \) of the equation \( Pu = 0 \) in \( \Omega_*^2 \), which is bounded by \( C\phi \) and satisfies the boundary condition \( h = v \) on \( \partial\Omega_2 \). On the other hand, for every \( \epsilon > 0 \) there exist \( n \in \mathbb{N} \), \( C_\epsilon \geq 0 \) and \( u_\epsilon \in K_p(\Omega) \) such that for every \( k \geq n \) we have \( 0 \leq g_k(x) \leq \epsilon u_\epsilon + C_\epsilon \phi \) in \( \Omega_{k,2}^* \). The same argument used for \( w_{k,-} \to 0 \) demonstrates now that the sequence \( \{g_k\} \) converges to zero.

Moreover, the sequence \( \left\{ \int_{\Omega_{k,2}} G_{P}^{\Omega_{k,2}}(x, y) f(y) dy \right\} \) is a nondecreasing locally bounded sequence of nonnegative solutions of the equation \( Pu = f \). By monotone convergence, this sequence converges to \( \int_{\Omega_2} G_{P}^{\Omega_2}(x, y) f(y) dy \) and the lemma follows. \( \square \)

**Lemma 2.5.** Let \( P \) be a critical or subcritical operator in \( \Omega \) and let \( \Psi \in C(\partial\Omega_2) \) be a real function. Let \( f \in C^\alpha(\overline{\Omega}_2) \) be a real function such that
\[
\int_{\Omega_2} G_{P}^{\Omega_2}(x, y) |f(y)| dy \leq B_{\infty}^{K,\phi}.
\]
Then there exists a unique solution \( v \in C(\overline{\Omega}_2) \cap B_{\infty}^{K,\phi} \) of the equation \( Pu = f \) in \( \Omega_*^2 \) which satisfies \( v = \Psi \) on \( \partial\Omega_*^2 \). Moreover,
\[
v(x) = h(x) + \int_{\Omega_*^2} G_{P}^{\Omega_*^2}(x, y) f(y) dy,
\]
where \( h(x) \) is a solution of the homogeneous equation \( Pu = 0 \) in \( \Omega_2^* \) which satisfies \( h = \Psi \) on \( \partial \Omega_2^* \) and \( |h(x)| \leq C\phi(x) \) in \( \Omega_2^* \) for some \( C > 0 \).

**Proof.** Let \( h_\pm \) be the unique nonnegative solution of the equation \( Pu = 0 \) in \( \Omega_2^* \) which is bounded by \( C\phi \) for some constant \( C > 0 \), and satisfies \( h_\pm = \Psi^\pm \) on \( \partial \Omega_2^* \). Consider the function

\[
v_\pm(x) = h_\pm(x) + \int_{\Omega_2^*} G_{P,2}^\Omega(x,y)f^\pm(y)dy.
\]

Clearly, \( v_\pm \in C(\overline{\Omega_2}) \cap B_{\infty}^{K,\phi} \) and satisfies the equation \( Pv_\pm = f^\pm \) in \( \Omega_2^* \) and the boundary condition \( v_\pm = \Psi^\pm \) on \( \partial \Omega_2^* \). Therefore, the function \( v = v_+ - v_- \) is a desired solution. The uniqueness follows from Lemma 2.2. \( \square \)

Now we establish the maximum principle for solutions which do not grow too fast at infinity.

**Theorem 2.6.** Let \( P \) be a subcritical operator in \( \Omega \). Suppose that \( Pv = f \geq 0 \) in \( \Omega \), where \( f \in C^\alpha(\Omega) \) and \( v \in \beta_0 \). Then \( v \geq 0 \) in \( \Omega \).

**Proof.** Suppose that there exists \( x_1 \in \Omega \) such that \( v(x_1) > 0 \). Then there exists a ball \( B_\epsilon = B(x_1, \epsilon) \subset \Omega \) such that \( v > 0 \) in \( B_\epsilon \). Lemma 2.4 implies that \( v \geq 0 \) in \( \Omega \setminus B_\epsilon \) and therefore \( v > 0 \) in \( \Omega \).

So, we may assume that \( v \leq 0 \) in \( \Omega \) and suppose that there exists \( x_1 \in \Omega \) such that \( v(x_1) < 0 \). Then there exists a ball \( B_\epsilon = B(x_1, \epsilon) \subset \Omega \) such that \( v < 0 \) in \( B_\epsilon \). Without loss of generality, we may assume that \( B_\epsilon \subset \Omega_1 \). Let \( u_{\epsilon,k} \) be the solution of the following Dirichlet problem

\[
Pu = 0 \quad \text{in} \quad \Omega_k \setminus B_\epsilon,
\]

\[
u = 0 \quad \text{on} \quad \partial B_\epsilon,
\]

\[
u = v \quad \text{on} \quad \partial \Omega_k.
\]

Using again the generalized maximum principle and Lemma 3.6 it follows that \( \lim_{k \to \infty} u_{\epsilon,k} = 0 \).

Consider also the Dirichlet problem

\[
Pu = 0 \quad \text{in} \quad \Omega_k \setminus B_\epsilon,
\]

\[
u = v \quad \text{on} \quad \partial B_\epsilon,
\]

\[
u = 0 \quad \text{on} \quad \partial \Omega_k,
\]

and denote its negative solution by \( v_{\epsilon,k} \). Set \( v_\epsilon := \lim_{k \to \infty} v_{\epsilon,k} \). Clearly, \( 0 < -v_\epsilon \leq C_\epsilon G_{P,2}^\Omega(\cdot, x_1) \). Using the well known asymptotic behavior of \( G_{P,2}^\Omega(\cdot, x_1) \) near the
pole $x_1$, it follows that there exists $C > 0$ such that for $\epsilon > 0$ small enough $0 < -v_\epsilon \leq Ce^{d_2-2} G_{\Omega}(x_1)$ if $d \geq 3$, and $0 < -v_\epsilon \leq -C(\log \epsilon)^{-1} G_{\Omega}(x_1)$ if $d = 2$. Therefore, $\lim_{\epsilon \to 0} v_\epsilon = 0$.

Finally, let $w_{\epsilon,k}$ be the solution of the Dirichlet problem

$$Pu = f \quad \text{in} \quad \Omega_k \setminus B_\epsilon,$$

$$u = 0 \quad \text{on} \quad \partial B_\epsilon,$$

$$u = 0 \quad \text{on} \quad \partial \Omega_k.$$ 

Then $w_{\epsilon,k} \geq 0$. On the other hand,

$$(2.6) \quad v = u_{\epsilon,k} + v_{\epsilon,k} + w_{\epsilon,k}.$$ 

Letting first $k \to \infty$ and then $\epsilon \to 0$ in equation (2.6), we obtain that

$$v = \lim_{\epsilon \to 0} \left( \lim_{k \to \infty} w_{\epsilon,k} \right) \geq 0$$

contradicting the hypothesis that $v \leq 0, v \neq 0$. □

The next proposition deals with the critical case and extends Proposition 1.7.

**Proposition 2.7.** Let $P$ be a critical operator in $\Omega$ and let $\phi_0$ be a ground state of the operator $P$ in $\Omega$. Let $W \in C_0^\alpha(\Omega)$ be a nonzero, nonnegative function. Suppose that $Pu \geq 0$ ($Pu \leq 0$) in $\Omega$, where $u \in \beta_0^{KP+W}$. Then $u = C_1 \phi_0$, where $C_1$ is a real constant.

**Proof.** Suppose that $Pu \geq 0$. Then there exists $V \in C_0^\alpha(\Omega)$ a nonzero nonnegative function such that $(P + V)u \geq 0$ in $\Omega$. Since the cones $C_{P+W}$ and $C_{P+V}$ are equivalent [4], it follows that $u \in B_0^{KP+V}$. Applying the maximum principle (Theorem 2.6) it follows that $u \geq 0$. Therefore, $u$ is a nonnegative supersolution of the critical operator $P$ and hence $u = c\phi_0$, where $c \geq 0$.

Assume that $Pu = 0$. If $u \geq 0$, then $u = c\phi_0$. Suppose that $u(x_1) < 0$. We may assume that $u < 0$ in $\Omega_1$. Let $V \in C_0^\alpha(\Omega_1)$ be a nonzero, nonnegative function. Using the equivalence of the cones $C_{P+W}$ and $C_{P+V}$, it follows that $u \in B_0^{KP+V}$. Therefore, Lemma 2.3 implies that $u < 0$ in $\Omega_1$. Hence $-u$ is a global nonnegative solution of the critical operator $P$. Thus $u = c\phi_0$, where $c$ is a real constant. □

In the following theorem we prove existence, uniqueness and integral representation for solutions in $\beta_0$ of the nonhomogeneous equation.
Theorem 2.8. Let $P$ be a subcritical operator in $\Omega$.

(i) Let $f \in C^\alpha(\Omega)$, $0 < \alpha \leq 1$, be a real function such that

\begin{equation}
\int_\Omega G_P^\Omega(x,y)|f(y)|dy \in \beta_0.
\end{equation}

Then there exists a unique solution $u \in \beta_0$ of the equation $Pu = f$ in $\Omega$. Moreover,

$$u(x) = \int_\Omega G_P^\Omega(x,y)f(y)dy.$$ 

(ii) Suppose that $f \in C^\alpha(\Omega)$, $0 < \alpha \leq 1$, and $f \geq 0$. Then

$$\int_\Omega G_P^\Omega(x,y)f(y)dy \in \beta_0$$

if and only if there exists a solution $u \in \beta_0$ of the equation $Pu = f$ in $\Omega$. In this case, $u$ is the minimal nonnegative solution of the equation $Pv = f$ in $\Omega$.

Proof. (i) Let

$$u_n(x) = \int_{\Omega_n} G_P^{\Omega_n}(x,y)f(y)dy.$$ 

By the Lebesgue dominated convergence theorem and a standard elliptic argument,

$$u(x) = \lim_{n \to \infty} u_n(x) = \int_\Omega G_P^\Omega(x,y)f(y)dy$$

is a solution of $Pu = f$ in $\Omega$. It follows that $u \in \beta_0$. The uniqueness follows from Theorem 2.6.

(ii) Suppose that $u \in \beta_0$ is a solution of the equation $Pu = f \geq 0$ in $\Omega$. Theorem 2.6 implies that $u \geq 0$. Consider again the sequence

$$u_n(x) = \int_{\Omega_n} G_P^{\Omega_n}(x,y)f(y)dy.$$ 

Clearly, $0 \leq u_n \leq u$ in $\Omega_n$ and therefore,

$$0 \leq w(x) := \int_\Omega G_P^\Omega(x,y)f(y)dy = \lim_{n \to \infty} \int_{\Omega_n} G_P^{\Omega_n}(x,y)f(y)dy \leq u(x) \in \beta_0.$$
Part (i) implies now that \( w = u \in \beta_0 \).

Let \( \bar{u} \geq 0 \) be a solution of the equation \( P v = f \) in \( \Omega \) and consider again the sequence \( \{ u_n \} \). By the generalized maximum principle \( 0 \leq u_n \leq \bar{u} \). Hence, \( u(x) = \lim_{n \to \infty} u_n(x) \leq \bar{u}(x) \) and \( u \) is the minimal nonnegative solution of the equation \( P v = f \) in \( \Omega \). \( \Box \)

**Remark 2.9.** Condition (2.7) holds true if \( f/u \) is a small perturbation of \( P \) in \( \Omega \) for some \( u \in C_P(\Omega) \). In particular, it holds for all \( f \in \beta \) if \( 1 \) is a small perturbation.

**REFERENCES**


*M. M. Al-Mahameed
Math. Dept.
Irbid National University
Irbid Jordan

Received December 27, 2006

E-mail: al_mahameed2000@yahoo.com

Revised May 15, 2007*