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A SIMPLE EXAMPLE OF LOCALIZED PARAMETRIC RESONANCE FOR THE WAVE EQUATION

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Dedication. The problem studied here was suggested to us by V. Petkov. Since the beginning of our careers, we have benefitted from his insights in partial differential equations and mathematical physics. In his writings and many discussions, the conjuction of deep analysis and specially interesting problems has been a source inspiration for us.

Abstract. We construct solutions growing exponentially in time of the equation

$$u_{tt} - (a(t,x)u_x)_x = 0, \qquad (t,x) \in \mathbb{R}^{1+1}_{t,x},$$

with $a > 0$ periodic in t and constant outside a compact in x. The function a is discontinuous representing a well which oscillates in time. The novelty is that by a careful geometric optics construction, all reflected and transmitted phases are linear functions of (t,x) rendering the example elementary.

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1. Introduction. For hyperbolic equations with x-independent coefficients

$$u_{tt} \;-\; \partial_x\big(a(t)\partial_x u\big) \;=\; 0\,, \qquad x \in \mathbb{R}, \qquad 0 < a \in C^\infty_{\text{periodic}}(\mathbb{R})\,,$$

solutions which grow exponentially in time were constructed in ([3] and [1]). The method works as well for

$$u_{tt} - u_{xx} + p(t)u \;=\; 0\,, \qquad 0 < p \in C^\infty_{\text{periodic}}(\mathbb{R})\,.$$

In this paper we achieve exponential growth with positive coefficient $a(t,x)$ periodic in time and constant outside a compact set of x.

Exponentially growing solutions *on bounded intervals* and with moving obstacles are constructed in [5, 10, 11, 9, 8]. The dynamics of circle maps plays a key role in the elegant description of the asymptotics as $t \to \infty$ ([8, 9]).

Constructions on unbounded domains are more difficult as energy tends to radiate to infinity. It tends to leave the domain where the periodic variations can be amplifying. The examples we know use a ray, trapped by reflection or refraction [4, 6, 2, 12]. The paper [12] gives a definitive analysis for rays trapped by reflection using fine tools from microlocal analysis. [2] has no boundaries but a ray trapped by refraction. This paper uses a ray trapped by reflection.

This research was started in response to the challenge of constructing a positive compactly supported and periodic in time potential $q(t,x)$ so that the equation $u_{tt} - u_{xx} + q\,u = 0$, has exponentially growing solutions. Hélas, as Vesselin is so fond of saying, for this problem as well as the problem with *smooth* $a(t,x)$, rays move steadily either to the right or the left. They are never trapped. In a forthcoming paper with V. Petkov, we construct such q in all dimensions.

In this note we present examples with rays trapped by reflection. The equation has the form

$$L\,u := u_{tt} - \partial_x(a(t,x)\partial_x u) = 0\,,$$

with $1 \le a \le c^2$ and a periodic in t and equal to c^2 outside a compact set in x. We construct simple examples of solutions which grow exponentially in time. The amplification occurs at reflections at discontinuities of a. A wave is reflected back and forth inside a moving well of width one defined by a. The innovation compared to the general analyses is that by careful choices all the phases encountered are linear functions of t, x. This renders the construction as elementary and explicit as possible.

The paper is organized as follows. In Section 2 we recall the reflection and transmission of plane waves at a stationary boundary. The succeeding section treats a boundary moving at constant subsonic speed. Section 4 constructs localized wave packets which locally resemble the plane waves. The final Section 5, constructs growing solutions from successive reflection. The existence of solutions with exponential growth at $t \to \infty$ follows from the Banach-Steinhaus Theorem. The precise Theorem is given in Section 5.

2. Reflection at a stationary interface. Consider

$$a(x) = \begin{pmatrix} 1 & \text{for} & x < 0 \\ c^2 & \text{for} & x > 0 \end{pmatrix}, \qquad c \geq 1.$$

Seek solutions with reflected and transmitted waves,

$$u = \begin{pmatrix} e^{i(t-x)/\varepsilon} + Re^{i(t+x)/\varepsilon} & \text{for} & x < 0 \\ Te^{i(t-x/c)/\varepsilon} & \text{for} & x > 0 \end{pmatrix}, \qquad R, T \in \mathbb{C}.$$

In order that the differential equation be satisfied in the sense of distributions it is necessary and sufficient that the differential equation be satisfied in $\{x \neq 0\}$ and that at the interface one has

$$[u] = 0, \qquad \text{and} \qquad [a\, u_x] = 0,$$

where the square brackets indicate the jump across the interface from $x < 0$ to $x > 0$. The continuity of u holds if and only if

$$1 + R = T. \tag{1}$$

The continuity of au_x holds if and only if

$$-1 + R = -cT. \tag{2}$$

Eliminate T from these equations to find the reflection coefficient then use (1) to find T,

$$R = \frac{1-c}{1+c}, \qquad T = \frac{2}{1+c}. \tag{3}$$

Note that $R = 0$ when $c = 1$ in which case a is constant. Note also that as $c \to +\infty$ the reflection coefficient tends to -1 which is the reflection coefficient for the Dirichlet problem. The Dirichlet problem is the correct limiting boundary value problem for sources localized in the region $x<0$.

The differential energy law for real solutions in $x<0$ and in $x>0$ is

$$\partial_t \mathcal{E} - \partial_x (u_t\, a\, u_x) = 0, \qquad \mathcal{E} := \frac{u_t^2}{2} + \frac{a(x)\, u_x^2}{2}.$$

The transmission condition guarantees that the flux $u_t\, a\, u_x$ is continuous across the interface so the differential law holds in all of space time and there is conservation of energy.

For plane waves the repartition of the incoming energy among the reflected and transmitted waves is expressed by the identity

$$1 = R^2 + c^2\, T^2.$$

3. Reflection at a moving interface. Study a which is discontinuous across a line in space time which is not stationary,

$$a = \begin{pmatrix} 1 & \text{for} & x < -\sigma t \\ c^2 & \text{for} & x > -\sigma t \end{pmatrix}, \qquad 1 > \sigma \geq 0.$$

Consider an incoming plane wave

$$e^{i(x-t)/\varepsilon}.$$

The reflected phase defined in $x < -\sigma t$ must be eikonal and must be equal to the incoming phase along $x = -\sigma t$. The equal trace condition implies that the phase is linear on $x = -\sigma t$ and therefore globally linear. It must have the form $A(x+t)/\varepsilon$ with $A \in \mathbb{R} \setminus 0$. Equality on $x = -\sigma t$ holds if and only if

$$(4) \qquad t + \sigma t = A(t - \sigma t), \qquad \Longleftrightarrow \qquad A = \frac{1+\sigma}{1-\sigma}.$$

The phase for the transmitted wave is defined in $x > -\sigma t$ and is eikonal with $a = c^2$. It must be equal to the two other phases along the interface. It has the form

$$B(t - x/c)/\varepsilon.$$

Equality on the interface holds if and only if

$$B\left(t + \frac{\sigma}{c}\,t\right) = t + \sigma t, \qquad \Longleftrightarrow \qquad B = \frac{1+\sigma}{1+\sigma/c}. \tag{5}$$

Both A and B are larger than one for $\sigma \in]0,1[$. Both of them are equal to 1 when $\sigma = 0$ which recovers the case of Section 2.

Seek a solution with reflection and transmission of the form

$$u = \begin{pmatrix} e^{i(t-x)/\varepsilon} + Re^{i(t+x)A/\varepsilon} & \text{for} \quad x < -\sigma t \\ Te^{i(t-x/c)B/\varepsilon} & \text{for} \quad x > -\sigma t \end{pmatrix}. \tag{6}$$

Continuity holds on the interface if and only if (1) holds.

Considering a right triangle with hypotenuse of length ds on the line $x = -\sigma t$ and height $|dt| = ds/\sqrt{1+\sigma^2}$ and base $|dx| = \sigma ds/\sqrt{1+\sigma^2}$ yields

$$Lu = \left([a\,u_x]\,\frac{1}{\sqrt{1+\sigma^2}} + [u_t]\frac{\sigma}{\sqrt{1+\sigma^2}}\right) ds,$$

where ds is arclength measure along the interface. The differential equation is satisfied in the sense of distributions if and only if

$$[a\,u_x] + [u_t]\sigma = 0 \qquad \text{on} \qquad x = -\sigma t. \tag{7}$$

Since

$$d(u_t\,dx + a\,u_x\,dt) = (u_{tt} - (a\,u_x)_x)\,dt \wedge dx,$$

the differential equation asserts that

$$u_t\,dx + a\,u_x\,dt$$

is an exact form. The jump relation is the standard Rankine-Hugoniot condition.

Compute

$$[a\,u_x] = a\,u_x|_{0+} - a\,u_x|_{0-} = \frac{i}{\varepsilon}\left(\left(-c^2\,\frac{TB}{c}\right) - (-1+RA)\right) = \frac{i}{\varepsilon}(-cTB + 1 - RA),$$

$$[u_t] = u_t(0+) - u_t(0-) = \frac{i}{\varepsilon}((TB) - (1+AR)).$$

The jump relation (7) holds if and only if

$$\begin{aligned} 0 &= (-cTB + 1 - RA) + \sigma((TB) - (1 + AR)) \\ &= (-c + \sigma)BT \;-\; (1+\sigma)AR \;+\; (1-\sigma). \end{aligned} \tag{8}$$

Eliminating T from (1) and (8) yields the reflection coefficient

$$R = \frac{B(\sigma - c) + 1 - \sigma}{A(1+\sigma) + B(c - \sigma)}\,. \tag{9}$$

When $\sigma = 0$ one has $A = B = 1$ and this reduces to (3). For any σ, R vanishes when $c = \sigma + (1-\sigma)/B$. When $c \to +\infty$, $R \to -1$.

For large c, the $-c$ terms are dominant and the reflection coefficient is close to -1. The frequency of the reflected wave is larger than that of the incident wave by the factor $A > 1$ independent of c. This is the Doppler effect. For c large the energy density of the reflected wave is greater than that of the incident wave. This, as in [12], is the motor for amplification.

The differential equation is equivalent to

$$d\big(u_t\, dx + a u_x\, dt\big) = 0,$$

while conservation of energy is equivalent to

$$d\big(\mathcal{E}\, dx + u_t\, a\, u_x\, dt\big) = 0.$$

The jump relation for the first is (7) while the jump relation for the second is

$$[u_t\, a\, u_x] + [\mathcal{E}]\,\sigma = 0, \qquad \text{on} \qquad x = -\sigma t.$$

When $\sigma \neq 0$, this jump relation is not a consequence of (7). The energy is not conserved. That is a good thing as our aim is to amplify the energy.

4. Wave packets. To produce spatially localized solutions which display reflection and amplification consider an incoming solution of WKB type in $x < -\sigma t$

$$u_I \sim e^{i(t-x)/\varepsilon}\big(A_0(t,x) + \varepsilon A_1(t,x) + \cdots\big) \qquad \text{in} \qquad x < -\sigma t.$$

The amplitudes A_j are constructed with support in a compact tube of rays. The meaning of the above is that one chooses by Borel's theorem a smooth function A supported in this tube and with Taylor expansion in ε

$$A(\varepsilon, t, x) \sim A_0(t, x) + \varepsilon A_1(t, x) + \cdots .$$

The infinitely accurate approximate solution is then defined to be equal to $e^{i(t-x)/\varepsilon} A(\varepsilon, t, x)$. It is uniquely determined up to $O(\varepsilon^\infty)$.

The leading profile must satisfy

$$(\partial_t + \partial_x) A_0 = 0, \qquad \text{in} \qquad x < -\sigma t. \tag{10}$$

Choose $A_0|_{t=-1/2}$ supported near $x = -1/2$ so the wave arrives at the interface near time $t = 0$.

Up to errors $O(\varepsilon^\infty)$ the exact solution is the sum of the incoming wave, a reflected wave and a transmitted wave, $u = u_I + u_R + u_T$. The reflected wave is given by

$$u_R \sim e^{i(t+x)/\varepsilon} \big(B_0(t, x) + \varepsilon B_1(t, x) + \cdots \big) \qquad \text{in} \qquad x < -\sigma t,$$

with principal profile satisfying

$$\big(\partial_t - \partial_x\big) B_0 = 0, \qquad \text{in} \quad x < -\sigma t, \qquad B_0 = R\, A_0 \quad \text{on} \quad x = -\sigma t,$$

The transmitted wave is given by

$$u_T \sim e^{i(t-x/c)/\varepsilon} \big(C_0(t, x) + \varepsilon C_1(t, x) + \cdots \big) \qquad \text{in} \qquad x > -\sigma t,$$

with principal profile satisfying

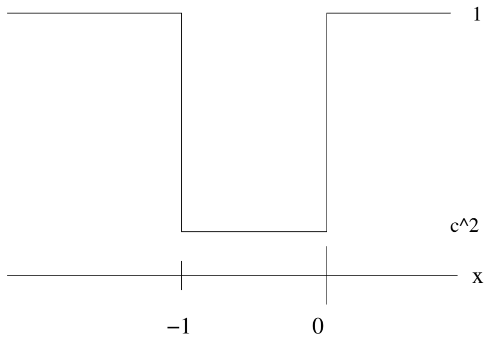
$$\big(\partial_t + c\, \partial_x\big) C_0 = 0, \qquad \text{in} \quad x > -\sigma t, \qquad C_0 = T\, A_0 \quad \text{on} \quad x = -\sigma t,$$

The key relation is $B_0 = RA_0$. The energy density of the reflected wave is $|RA_0/\varepsilon|^2$ and that of the incoming wave is $|A_0/\varepsilon|^2$. *For $R > 1$ there is an amplification of energy.*

5. Resonance. We arrange a pair of moving interfaces so that there is a reflection with energy amplification occuring at times approximately equal to $0, 1, 2, 3, \ldots$.

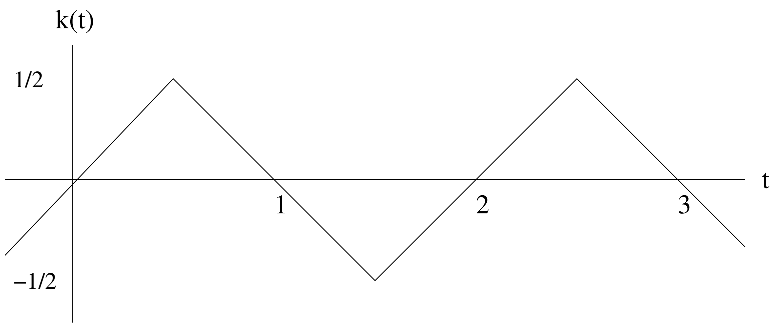
Define

$$g(x) := \begin{pmatrix} 1 & \text{when} & x \in]-1,0[\\ c^2 & \text{when} & x \notin [-1,0] \end{pmatrix}.$$



Graph of $g(x)$

Define $k(t)$ to be the sawtooth function with period 2 and slopes equal to ± 1 sketched in the figure,



Graph of $k(t)$

Define a smooth function $h(t)$ periodic of period 2 so that

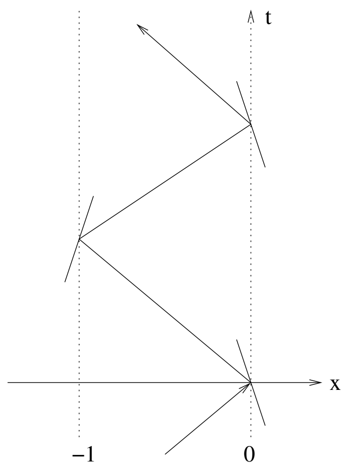
$$h(t) \;=\; k(t) \qquad \text{when} \qquad \text{dist}\big(t\,,\,\mathbb{Z}_{\text{odd}}\big) < \frac{1}{10}.$$

Near the odd integers the graph of $h(t)$ has slope equal to -1, and near the even integers the slope is equal to 1. Define

$$a(t,x) \;:=\; g(x+\sigma\, h(t))\,, \qquad \sigma\in]0,1[\,.$$

The function a describes an oscillating well. The well is one unit wide. For times near the integers the sides of the well move linearly. For even integers the well moves to the left, while for odd integers to the right. The linear motions have speed σ.

Choose c large so that the modulus of the amplification factor R computed in Section 4 is strictly greater than 1. Choose the incoming wave with support in a band of width 1/20 centered on the ray $x=t$ and starting at time $t=-1/2$. There are reflections at the sides of the well at times near $0,1,2,\ldots$. Each comes with its doppler shift 1, reflection coefficient R, and amplification R. In the following figure the transmitted rays are not included.



The central multiply reflected ray

Theorem. *Denote by* $S(t)\in \mathrm{Hom}(H^1\times L^2)$ *the map*

$$(u(0),u_t(0)) \quad\mapsto\quad (u(t),u_t(t)).$$

Suppose that c is chosen as above so that $R>1$.
i. *For any* $\mu>0$

$$\lim_{n\to\infty}\ \big\|R^{-n+\mu}\,S(n)\big\|_{\mathrm{Hom}(H^1\times L^2)} \;=\; \infty\,.$$

ii. *For any $\mu > 0$, there is an initial datum $(u(0), u_t(0))$ so that*

$$\lim_{n\to\infty} \left\| R^{-n+\mu}\, S(n)(u(0), u_t(0)) \right\|_{H^1\times L^2} \;=\; \infty\,.$$

Proof. **i.** For any $0 < n \in \mathbb{N}$, on the interval $-1/2 \le t \le n + 1/2$ the solution is described with infinite accuracy in the limit $\varepsilon \to 0$ by the multiply reflected (and transmitted) WKB solutions. The reflection times are approximately equal to $0, 1, 2, \ldots$. Each reflection occurs at a linear interface moving toward the wave. At each reflection there is an amplification of the leading coefficient by $R > 1$.

After each reflection, the interface in smooth fashion reverses path so that about one unit of time later the wave is again reflected by the linearly moving interface which is at the other side of the oscillating well.

For $t = n$ the wave is amplified by reflection at least $n - 1$ times. There is a constant C independent of n so that

$$\|(u(n), u_t(n)\|_{H^1\times L^2} \ge C\, R^{n-1}\, \|(u(0), u_t(0))\|_{H^1\times L^2}.$$

For positive μ as small as one likes, let

$$M_n := R^{-n+\mu}\, S(n), \qquad n = 1, 2, \ldots .$$

The preceding construction shows that

$$\lim_{n\to\infty} \|M_n\| = \infty.$$

which is **i**.

The Banach-Steinhaus Theorem implies that there is an initial datum $(u(0), u_t(0))$ so that

$$\lim_{n\to\infty} \left\| M_n((u(0), u_t(0)) \right\| \;=\; \infty\,,$$

which is the conclusion of **ii.** $\square$

Remarks. **1.** Slightly more is proved. The same assertion is valid if $S(t)$ is the map from Cauchy data supported in $[-1/2-\mu, -1/2+\mu]$ to the value of the solution on the interval $[-1, 0]$.

2. One can create examples with the same flavor for the wave equation

$$u_{tt} - u_{xx} + q(t, x)u = 0,$$

where $q \geq 0$ consists of a pair of delta functions oscillating periodically,

$$q(t,x) = \delta(x + \sigma h(t)) + \delta(x + \sigma h(t) - 1).$$

In both situations, the problem is equivalent to a transmission problem and rays are reflected at the singularities of the coefficients.

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