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FULL EXPOSITION OF SPECHT'S PROBLEM*

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In honor of Yuri Bakhturin's birthday

Abstract. This paper combines [15], [16], [17], and [18] to provide a detailed sketch of Belov's solution of Specht's problem for affine algebras over an arbitrary commutative Noetherian ring, together with a discussion of the general setting of Specht's problem in universal algebra and some applications to the structure of T-ideals. Some illustrative examples are collected along the way.

Specht's problem is whether every set of polynomial identities of an algebra is **finitely based**, i.e., is a consequence of a finite number of identities. It can be asked more generally for all classes of algebras, and actually was first asked for groups by B.H. Neumann [70], affirmatively answered long ago for finite groups by Oates and Powell [72]. The first counterexample is due to Olshan-

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skii [73], with explicit examples of infinite systems of group identities given by Adian [1], Kleiman [55] and Vaughan-Lee [95]. Kleiman showed that the system $(x_1^2 \cdots x_n^2)^4 = 1$, $n = 1, 2, \ldots$, is not finitely based. An account of group identities is given in [71].

0.1. Specht problem for associative algebras. Specht [91] raised the question for associative algebras of characteristic 0. Already in the 1960s, Latyshev [59, 60, 61] verified Specht's problem for the Grassmann algebra and related algebras, including any algebra satisfying the identity $[[[x, y], z], t]$. In 1973 Krakowski and Regev [56] proved that the T-ideal of the Grassmann algebra is generated by the single identity $[[x, y], z]$; a quick elementary proof using a lemma of Latyshev is given in [14, Lemma 3.43]. Kruse [58] and L'vov [66] found an affirmative answer for finite rings; specific generating identities for matrices over finite fields were given in [32, 33, 68]. In the 1970s, using a combinatorial approach of Higman [39], Latyshev [62, 63], and independently Genov [31] and Popov [75], solved Specht's problem for algebras of characteristic 0 satisfying an identity not satisfied by 2×2 matrices and an identity not satisfied by the tensor square of the Grassmann algebra. Belov, Borisenko, and Latyshev [13] proved that each variety generated by monomial algebras is generated by a single automata algebra.

An equivalent formulation to Specht's problem, for arbitrary T-ideals, is whether the countably generated free algebra $C\{x\}$ satisfies the ACC (ascending chain condition) on T-ideals. Kemer obtained a positive solution for characteristic 0 in 1988 and 1990, cf. [48], thereby enabling him to apply techniques of Noetherian theory to T-ideals of PIs; we indicate how this works at the end of this paper. Aljadeff and Belov [2] (also cf. Sviridova [92]) have proved a graded version of Kemer's theorem.

Specht's problem has counterexamples in characteristic p, discovered by Belov [10, 11], and later by Grishin [36] and Shchigolev [88], in 1999 and 2000. All such examples are infinitely generated. Thus, in positive characteristic, one could hope only for a positive result for affine PI-algebras. In 1991, Kemer [47] proved this over infinite fields, and in 2002 Belov, in his second dissertation, in Russian, extended the theory to cover affine PI-algebras over arbitrary commutative Noetherian rings. There is no hope for such a result over a non-Noetherian commutative base ring C, since any chain of ideals of C can be viewed as a chain of T-ideals.

0.2. Specht's problem for nonassociative algebras. Since the notion of algebraic variety is appropriate to any class of universal algebras, one can pose Specht's problem for arbitrary classes of algebraic varieties, in particular for classes of nonassociative algebras. Let us review the known results concerning familiar classes of nonassociative algebras. As Drensky has pointed out to us, it is very easy to construct counterexamples in any characteristic, such as the system $((x_1x_2)(x_3x_4)x_5 \cdots x_{n-2})(x_{n-1}x_n) = 0$, $n = 6, 7, \dots$ (where the parentheses are left normed and $x_1x_2x_3 = (x_1x_2)x_3$).

As with many famous problems in algebra, Lie algebras have served as a bridge from groups to associative algebras. Already in 1970, Vaughan-Lee [94] proved that the T-ideal $\mathrm{id}(gl_2)$ over an infinite field of characteristic 2 is not finitely based; see also [23]. Drensky [24] then found a finite dimensional Lie algebra over an infinite field of arbitrary characteristic $p > 0$ whose PIs are not finitely based; a reasonably straightforward demonstration is given in [19, Example 5.1]. Other interesting examples include Drensky [27] and Kleiman (unpublished).

Bahturin and Olshanskii [8] established the finite basis property for finite Lie rings, see also [29] for a concrete example of a basis of identities of a finite Lie algebra. Iltyakov [41, 43] verified Specht's conjecture for any affine Lie algebra of characteristic 0 satisfying a "Capelli system of identities," defined below. Other positive Lie results are given in [57, 98].

Specht's problem still is open for affine Lie algebras in general. Since representability plays such an important role, one expects a positive solution for affine linear Lie algebras. Iltyakov [41] verified Specht's conjecture for affine alternative algebras of characteristic 0. Medvedev [69] and Pchelintsev [74] gave non-affine counterexamples in characteristics 2 and 3, respectively. Badeev [7] has counterexamples for Specht's conjecture on commutative Moufang loops.

These counterexamples are based on Shestakov's discovery [89] that many counterexamples in nonassociative algebras can be obtained from the nonassociative analog of Kemer's use of the Grassmann envelope (in the associative case), to pass from affine superalgebras to non-affine algebras. This idea also has been used in other situations besides Specht's question.

For Jordan algebras, an extra subtlety arises from the existence of identities of special Jordan algebras. Vais and Zelmanov [93] verified Specht's conjecture for affine Jordan algebras of characteristic 0 satisfying a non-special identity.

0.3. The objective of this paper. Belov's results are given in Isvestija [12], but the article does not contain all the details. Over the last two years, Belov, Rowen, and Vishne have presented the proof in full detail in a series of four papers [15, 16, 17, 18] (the last of which has not yet appeared), totalling about 150 journal pages. Our goal in this exposition is to make these articles more palatable by presenting a detailed overview of the proof (over an arbitrary commutative Noetherian base ring C), given in Theorem 10.9, together with its main ideas, and also giving an indication of which parts are routine and which parts required innovations. One of the main themes is the translation from combinatoric questions (of explicit evaluations of polynomials) to geometric notions such as quivers. This deep interplay between combinatoric algebra and representation theory, as well as further details of the history of Specht's question is discussed at length in the introduction of [12] and in [90].

The proof of Specht's problem (in the affine case) is divided into several cases: First we handle the case that the base ring $C = F$ is a field, which could be finite. Then one reduces from the case of C a general Noetherian ring to the case where C is an integral domain. Ironically, this reduction is much easier than the first case, relying on straightforward results from commutative Noetherian ring theory. We conclude the proof by passing to the field of fractions; the main concern in this step is handling torsion in the localization procedure.

Although our proof is based on applying quivers defined with respect to matrix algebras, Kemer [48] proved in characteristic 0 that every prime variety can be represented as the Grassmann envelope of a simple superalgebra. Thus, we can describe its identities in terms of quivers whose vertices correspond to algebras of T-prime varieties (as characterized by Kemer). This approach provides all known (non-affine) counterexamples to Specht's problem in characteristic p, where the glued blocks correspond to Grassmann algebras, cf. [19]. It would be interesting to develop the parallel theory of quivers for non-affine algebras, in order to understand this situation better.

1. Background. A **polynomial identity** (PI) of an algebra A over a base ring C is a noncommutative polynomial with coefficients in C, which vanishes identically for any substitution in A. We write $\mathrm{id}(A)$ for the set of PIs of an algebra A. We use [14] as a general reference for PIs, and in particular for Specht's problem. Other relevant references are [7, 6, 24, 35, 43, 67, 76, 82, 81]. The set $\mathrm{id}(A)$ can be viewed as an ideal of the free associative algebra $C\{x\}$, where x stands for the countable set of variables $x_0, x_1, \dots$, closed under all algebra homomorphisms $C\{x\} \to C\{x\}$. Such an ideal I of $C\{x\}$ is called a T-**ideal**. In general, the **T-ideal** of a polynomial **in an algebra** A is the ideal generated by all substitutions of the polynomial in A.

Conversely, for any T-ideal I of $C\{x\}$, each element of I is a PI of the quotient algebra $C\{x\}/I$, and $C\{x\}/I$ is **relatively free**, in the sense that for any PI-algebra A with $\mathrm{id}(A) \supseteq I$, and any $a_1, a_2 \dots \in A$, there is a natural homomorphism $C\{x\}/I \to A$ sending $x_i \mapsto a_i$ for $i = 1, 2, \dots$.

Two algebras are called **PI-equivalent** if they satisfy the same PIs, and the major question in PI-theory is to classify PI-equivalence classes of algebras. For any polynomial $f(x_1, x_2, \dots) \in C\{x\}$, define $\deg_i f$ to be the maximal degree with respect to x_i of its monomials. For example, we write $[a, b]$ for the additive commutator $ab - ba$. All identities of commutative algebras of characteristic 0 are determined by the single identity $[x, y] = xy - yx$ of degree 2.

1.1. Identities of matrix algebras. Since matrix algebras are so fundamental in the study of associative algebras, one might naturally start with the matrix algebra $\mathrm{M}_n(K)$ over a field K of characteristic 0 and ask to find a finite set of identities that determines $\mathrm{id}(\mathrm{M}_n(K))$. For $n = 1$, this is just the single identity $[x, y]$ for the field K.

Define the **Capelli polynomial**

$$c_k(x_1, \dots, x_k; y_1, \dots, y_k) =$$

$$= \sum_{\pi \in S_k} \mathrm{sgn}(\pi) x_{\pi(1)} y_1 \cdots x_{\pi(k)} y_k$$

of degree $2k$, and the **standard polynomial**

$$s_k(x_1, \dots, x_k) = c_k(x_1, \dots, x_k; 1, \dots, 1) =$$

$$= \sum_{\pi \in S_k} \mathrm{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(k)}$$

of degree k.

Any C-subalgebra of $\mathrm{M}_n(K)$ satisfies the identities c_k for all $k > n^2$. The celebrated **Amitsur-Levitzki** theorem says that $\mathrm{M}_n(K)$ satisfies s_{2n}, and this is its PI of minimal degree.

Razmyslov [79] proved that $\mathrm{id}(\mathrm{M}_2(K))$ is determined by a finite set of identities; Drensky [25] proved that the identities s_4 and the Wagner polynomial $g_2 := [[x, y]^2, z]$ suffice. (The polynomial $g_2 \in \mathrm{id}(\mathrm{M}_2(K))$ since for any 2×2 matrices a and b, the matrix $[a, b]$ has trace 0, implying $[a, b]^2$ is scalar). Although the statement of the assertion is straightforward, the proof is intricate, and utilizes Lie theory, as expounded in [7]. Razmyslov [80] also proved that any variety containing $\mathrm{id}(\mathrm{M}_2(K))$ is determined by a finite set of identities

But an explicit set of generators of $\mathrm{M}_3(\mathbb{Q})$ remains unknown! Ironically, the problem becomes considerably more tractable when one adjoins identities of matrices involving coefficients of the characteristic polynomial; in characteristic 0 it is enough to adjoin the traces, in view of Newton's formulas. Then by results proved independently by Helling [38], Procesi [77], and Razmyslov [78], all T-ideals with traces are consequences of the identities trace(1) $= n$ and the Hamilton-Cayley identity (saying that any matrix satisfies its Hamilton-Cayley polynomial).

1.2. Linearization. A polynomial is **blended** [82, Definition 2.3.15] if each indeterminate appearing nontrivially in the polynomial appears in each of its monomials. Since any polynomial $f(x_1, \dots, x_n)$ can be written as

$$f(0, x_2, \dots, x_n) + (f(x_1, \dots, x_n) - f(0, x_2, \dots, x_n)),$$

one sees (by induction on the number of non-blended indeterminates) that any T-ideal is additively spanned by T-ideals of blended polynomials, cf. [82, Exercise 2.3.7]. Thus, we may assume that f is blended.

Given any (blended) polynomial $f(x_1, \dots, x_m)$, we define the **linearization process** by introducing a new indeterminate x_i' and passing to

$$f(x_1, \dots, x_i + x_i', \dots, x_m) - f(x_1, \dots, x_i, \dots, x_m) - f(x_1, \dots, x_i', \dots, x_m).$$

This process, applied repeatedly to a PI, yields a multilinear identity. Whereas in characteristic 0 the linearization process can be reversed by taking $x_i' = x_i$, this fails in positive characteristic, as exemplified by the Boolean identity $x^2 - x$, whose multilinearization is the identity $x_1 x_2 + x_2 x_1$, satisfied by all commutative algebras of characteristic 2 (which need not be Boolean).

1.3. Kemer's solution of Specht's problem in characteristic 0. One of Kemer's main results was:

Theorem 1.1 ([14, Theorem 4.66]). *Every affine PI-algebra of characteristic* 0 *is PI-equivalent to a finite dimensional (f.d.) algebra.*

The proof of Theorem 1.1 relies heavily on the classical structure of f.d. algebras, in particular **Wedderburn's Principal Theorem** which states that over an algebraically closed field one can write any f.d. algebra $A = S \oplus J$, where J is the radical of A (which is nilpotent) and $S \cong A/J$ is semisimple.

Definition 1.2. *A* ***semisimple substitution*** (*into an algebra* A) *is a substitution into an element of* S*, and a* ***radical substitution*** *is a substitution into an element of* J*. A* ***pure substitution*** *is a substitution into an element of* $S \cup J$*, i.e., either semisimple or radical.*

When working with a multilinear polynomial f, one can decompose any evaluation of f into sums of evaluations with pure substitutions. Two key ingredients of the proof of Theorem 1.1 are Kemer's Lemmas, given in [14, Chapter 4], which describe the dimension of the semisimple part and the nilpotence index of J in terms of such evaluations of polynomials on A. Kemer's Second Lemma, which involves "μ-Kemer polynomials," is particularly intricate.

Kemer then uses these combinatoric ideas to provide an inductive argument which takes an arbitrary T-ideal of an affine algebra towards the T-ideal of a f.d. algebra, and concludes the proof of Theorem 1.1. From then on, the solution of Specht's problem for affine algebras of characteristic 0 is not difficult, cf. [14, Theorem 4.69].

To pass to the non-affine case, Kemer first proves the analog to Theorem 1.1 for superalgebras, and then passes to arbitrary non-affine algebras via the Grassmann envelope and the representation theory of the symmetric group in characteristic 0. This latter argument uses properties of Young diagrams that do not hold in characteristic p. Kemer's use of the Grassmann envelope is highly significant, since it also points to the counterexamples in positive characteristic, as explained in Shestakov [89], leading as well to the nonassociative counterexamples described above. A more detailed discussion of the connection between superalgebras and representation theory is given in [12]. Working in the Grassmann algebra, Grishin [36] and Shchigolev [88] constructed infinitely based T-spaces.

1.4. Kemer's solution of Specht's problem for affine PI-algebras over an infinite field. In light of the previous discussion, one could hope for a solution of Specht's problem for affine PI-algebras over an infinite field of arbitrary characteristic, by working with polynomials which are not quite multilinear.

1.4.1. Quasi-linearization. To handle characteristic $p > 0$, Kemer [47] took a closer look at the linearization process.

Definition 1.3. *A function f is i-**quasi-linear** on A if*

$$f(\dots, a_i + a_i', \dots) = f(\dots, a_i, \dots) + f(\dots, a_i', \dots)$$

*for all $a_i, a_i' \in A$; f is A-**quasi-linear** if f is i-quasi-linear on A for all i.*

Definition 1.4. *Suppose $f(x_1, x_2, \dots) \in C\{x\}$ has degree d_i in x_i. The i-**partial linearization** of f is*

$$(1) \qquad \Delta_i f := f(x_1, x_2, \dots, x_{i,1} + \dots + x_{i,d_i}, \dots) - \sum_{j=1}^{d_i} f(x_1, x_2, \dots, x_{i,j}, \dots)$$

where the substitutions were made in the i component, and $x_{1,1}, \dots, x_{1,d_i}$ are new variables.

Remark 1.5. Applying (1) lowers $\deg_i f$. When $\Delta_i f = 0$, then f is i-quasi-linear, so we apply (1) at most $\deg_i f$ times repeatedly, if necessary, to each x_i in turn, to obtain a polynomial that is A-quasi-linear. (Quasi-linear polynomials are used heavily by Kemer in [47], but obtained slightly differently in (1) via homogeneous components of partial linearizations.)

Formally, this process is slightly stronger than that given in [17], but yields the following nice result:

Proposition 1.6 ([18, Corollary 2.13]). *For any polynomial f which is not an identity of A, the T-ideal generated by f contains an A-quasi-linear non-identity for which the degree in each indeterminate is a p-power.*

Lemma 1.7 ([18, Lemma 2.16]). *Suppose x_1 has some specialization $x_1 \mapsto \sum \overline{x_{1,j}}$ where the $\overline{x_{1,j}}$ are substitutions of different components. (For example, some of them might be semisimple and others radical.) Then all specializations involving "mixing" the $\overline{x_{1,j}}$ occur in $\Delta f(\overline{x_{1,1}}, \dots, \overline{x_{1,d_i}}, \overline{x_2}, \dots)$.*

If f were linear in x_1 then we could separate these into distinct specializations of f. But when f is non-linear in x_1, we need to turn to Lemma 1.7.

In [47], the definition of quasi-linear also included homogeneity, which can be obtained automatically over infinite fields. Since we are working over finite fields, we say instead that a function f is i-**quasi-homogeneous** of degree s_i on A if

$$f(\dots, \alpha a_i, \dots) = \alpha^{s_i} f(\dots, a_i, \dots)$$

for all $\alpha \in F, a_i \in A$; $f(x_1, \dots, x_t; y_1, \dots, y_m)$ is A-**quasi-homogeneous** of degree s on A, if f is i-quasi-homogeneous on A of degree s_i for all $1 \le i \le t$, with $s = s_1 \cdots s_t$.

Lemma 1.8.([18, Lemma 2.18]). *Given any T-ideal $\mathcal{I}$ and any polynomial $f \in \mathcal{I}$ which is a non-identity of A, we can obtain an A-quasi-homogeneous non-identity in $\mathcal{I}$.*

2. Outline of the proof for affine algebras over an arbitrary commutative Noetherian base ring. The detailed proof of Theorem 1.1 in [14] takes about 120 pages, and one should expect the proof over an arbitrary field to be even more complicated. The main innovation is the use of full quivers to control certain nonzero evaluations of a polynomial, whose careful study (including "hiking") occupies most of the proof. The use of full quivers simplifies some of the computations, but still requires a very careful analysis.

Let C be a commutative Noetherian ring. A faithful C-algebra A is called **weakly representable** if it is embeddable as an C-subalgebra of $\mathrm{M}_n(K)$ for a suitable commutative faithful C-algebra K. A is called **representable** if K can be taken to be a field. The following well-known observation enables us to pass from weakly representable to representable.

Remark 2.1. If A is an affine, weakly representable C-algebra, then K can be taken to be Noetherian [14, Prop. 1.76]. If moreover C is an affine domain, then A is representable, in view of Anan'in's theorem [3].

(Note: The shorter proof in [14] has a gap, which is corrected in [85] in the important case that A is finite over its center; this is the only case needed in [14].)

An immediate consequence of Theorem 1.1 is that every relatively free affine PI-algebra over a field of characteristic 0 is representable. We call a T-ideal **representable** if it is the ideal of identities of a representable algebra.

Lemma 2.2. *Let C be a Noetherian integral domain. Every T-ideal of identities of a C-torsion-free affine PI-algebra contains a representable T-ideal.*

Proof. The field case is [14, Cor. 4.9]. Thus C may be assumed infinite, and we can replace C by its field of fractions without changing the given T-ideal. □

Lemma 2.2 is needed to get started in proving Thoerem 7.3. We need a stronger statement for proving Theorem 9.9 (for general Noetherian base rings), but encounter some technical difficulties and thus defer the argument to the proof itself.

Unless otherwise indicated, from now on we assume that F is a field of characteristic p and of order q, a power of p. Thus, the **Frobenius map** $\varphi_q : a \mapsto a^q$ is an F-algebra endomorphism. (The theory also works in characteristic 0, without the Frobenius map.) Let K be a field containing the algebraic closure of F. The algebra

$$A = \left\{ \begin{pmatrix} F & K[\lambda] \\ 0 & K[\lambda] \end{pmatrix} : \alpha \in F \right\}$$

is obviously representable but not PI-equivalent to a finite dimensional F-algebra. Indeed, its center being of some q-power q' implies that A satisfies some PI of the form

$$[x_1, x_2]^{2q'} - [x_1, x_2]^2.$$

Thus, Kemer's Theorem 1.1 cannot be extended to algebras over finite fields. But the algebra A is representable, so we see that varieties may contain representable algebras even when they do not contain f.d. algebras, and we work instead with representable algebras.

Remark 2.3. Here are the main steps in our proof:

(1) By [14, Corollary 4.9], the T-ideal of A contains the T-ideal of a finite dimensional algebra, so, replacing A by that algebra, we start with a representable algebra.

(2) The Zariski closure (see Section 3) of a representable algebra is in the same PI variety, but also satisfies the main structure theorems of finite dimensional algebras, including the Wedderburn Principal Theorem that any Zariski closed algebra A can be written as a vector space direct sum of its radical J and the semisimple part $S = A/J$.

(3) Since Zariski closed semisimple algebras S are finite direct sums of matrix algebras, and since the radical is nilpotent, the main difficulty in understanding the structure of an algebra is in the interaction between S and J. We describe this interaction through various identifications of matrix entries in the representation, which we call **gluing**.

(4) Full quivers and pseudo-quivers of representations provide a tool for studying gluing, and evaluating polynomials on Zariski-closed algebras.

(5) A non-identity is transformed into a **characteristic coefficient-absorbing polynomial** by means of **hiking** arising from the pseudo-quiver, to which we attach a Capelli polynomial (which for convenience is taken to be central).

(6) Shirshov's theorem enables us to adjoin characteristic coefficients to obtain integrality, and thus finite generation of algebras as modules over the base ring.

(7) Step (6) enables us to find representable T-ideals inside arbitrary T-ideals, thus providing a procedure to reduce quivers.

(8) Quiver reduction, together with an argument about torsion, yields a geometric technique to apply induction and conclude the proof in the field-theoretic case.

(9) Noetherian induction on the base ring reduces the theorem to the case where the base ring is an integral domain (which must be infinite, in view of (8)).

(10) One can separate the p-torsion according to finitely many prime numbers p, and thereby eliminate p-torsion by means of (8).

(11) The proof for varieties of PI-algebras over an integral domain is concluded by extending the base ring to its field of fractions.

(12) An easy argument enables one to pass to arbitrary varieties (for which none of the coefficients of the identities need be invertible).

The exposition given here follows this outline. Some of the steps of this program can be applied to arbitrary classes of nonassociative algebras, as is indicated at the end of this paper.

3. The Zariski closure of a representable algebra. Assume that A is a faithful, representable algebra over an affine Noetherian integral domain F.

Suppose $K \supseteq F$ is an algebraically closed field, together with the representation $\rho\colon A \to \mathrm{M}_n(K)$ of an F-algebra A. The **Zariski closure** $\rho(A)^{\mathrm{cl}}$ is the closure of $\rho(A)$ with respect to the Zariski topology of $\mathrm{M}_n(K)$.

This definition clearly depends on the choice of the representation ρ. Nevertheless, abusing language slightly, we assume that ρ is a given faithful representation, and we view $A \subseteq \mathrm{M}_n(K)$. We denote its Zariski closure as A^{cl}. Since any K-subalgebra of $\mathrm{M}_n(K)$ is Zariski closed, we can take the Zariski closure of A in any K-subalgebra $B \subseteq \mathrm{M}_n(K)$ containing A. In particular, we can take $B = KA$, the K-subspace of $\mathrm{M}_n(K)$ spanned by A.

Any algebra is PI-equivalent to its Zariski closure. Thus, the PI classification problem reduces to determining the PIs of Zariski closed algebras of relatively free algebras.

Remark 3.1. When the base ring F is finite then it is a field. If F is infinite, then $A^{\mathrm{cl}} = KA$, so the Zariski closed algebras are precisely the f.d. K-subalgebras of $\mathrm{M}_n(K)$, and the theory of this section degenerates to the theory of f.d. algebras. In other words, any Zariski closed algebra contains the field of fractions of F.

Accordingly, we assume from now on that F denotes a field. The innovation in our theory comes when we take the base field F to be finite.

Example 3.2. $|F| = q$ iff F satisfies the Fermat identity $x^q - x$. Thus, every finite field is Zariski closed, and the Zariski closure of any infinite subfield of K is K itself. In this way, the Zariski closure differentiates the finite from the infinite.

Example 3.3. The Zariski closed algebra $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^p \end{pmatrix} : a \in K \right\}$ is a field, but $KA = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$ is not.

Nevertheless, Zariski closed algebras satisfy many of the structural properties of finite dimensional algebras over algebraically closed fields.

Remark 3.4. Let $I \triangleleft B$ be maximal with respect to $I \cap A = 0$. Passing to B/I, we may assume that every ideal of $B = KA$ intersects A nontrivially.

Remark 3.4 ties the structure of A to the structure of B. The following consequence, parallel to Wedderburn's principal theorem, gives us the basic structure of Zariski closed algebras over arbitrary fields.

Theorem 3.5 (First Representation Theorem, [15, Theorem 3.33]). *Any Zariski closed algebra $A \subset \mathrm{M}_n(K)$ has a Wedderburn decomposition $A = S \oplus J$, where $J = \mathrm{Rad}(A)$ and $S \cong A/J$ is a subalgebra of A that is isomorphic to a direct sum of matrix algebras over fields (which are Zariski closed F-subfields of K).*

3.1. Generic algebras. The algebra that we often study is the Zariski closure of a relatively free algebra. This leads us to the study of generic elements in a relatively free algebra. The construction of a generic algebra over an infinite field is rather classical. One takes a base $b_1, \dots, b_n$ over F, and adjoins indeterminates $\xi_i^{(k)}$ to F ($i = 1, \dots, n$, $k \in \mathbb{N}$), and forms the algebra generated by the "generic" elements $Y_k = \sum_{i=1}^n \xi_i^{(k)} b_i$, $k \in \mathbb{N}$, which is easily seen to be relatively free in the variety defined by $\mathrm{id}(A)$.

The situation is considerably subtler over a finite field, since we encounter the finite component.

Construction 3.6. (General construction of generic algebras, cf. [15, Theorem 7.14]). *Letting $\mathcal{C}_1, \dots, \mathcal{C}_t$ denote the irreducible components of A^{cl} under the Zariski topology, suppose each $\mathcal{C}_i$ is defined over a field with q_i elements. To obtain s "mutually generic" elements $b_{i1}, \dots, b_{is}$ in each component, we take a generic element*

$$b \in \mathcal{C}_1^s \times \cdots \times \mathcal{C}_t^s,$$

where each $\mathcal{C}_i^s$ denotes the direct product of s copies of $\mathcal{C}_i$. Thus b has the form $((b_{11}, \dots, b_{1s}), (b_{21}, \dots, b_{2s}), \dots, (b_{\mu 1}, \dots, b_{\mu s}))$, where each $(b_{i1}, \dots, b_{is}) \in \mathcal{C}_i$; by definition, the b_{ik} are "mutually generic". Next, we define the ***generic coefficient ring***

$$C = F[\xi_{ik} \colon 1 \le i \le s,\ 1 \le k \le \mu] / \langle \xi_{ik}^{q^{d_i}} - \xi_{ik} : 1 \le i \le s,\ 1 \le k \le \mu \rangle,$$

and the generic elements $Y_k = \sum_{i=1}^{s} \bar{\xi}_{ik} b_{ik}$ $(k = 1, \dots, \mu)$, *where* $\bar{\xi}_{ik}$ *is the image of* ξ_{ik} *in* C. *The subalgebra of* B *generated by the* Y_k *serves as our generic algebra for the variety generated by* A.

F is not PI-equivalent to the ring of polynomials $F[\xi]$, since $|F| = q$, so we must pass to $F[\xi]/\langle \xi^q - \xi \rangle$, where the image $\bar{\xi}$ of ξ is a generic element. Since $F[\xi]/\langle \xi^q - \xi \rangle$ is isomorphic to a direct product of q copies of F, we could view our generic element as a q-tuple listing the elements of F. For two generic elements we need to pass to

$$F[\xi_1, \xi_2]/\langle \xi_1^q - \xi_1, \xi_2^q - \xi_2 \rangle,$$

which is isomorphic to a direct product of q^2 copies of F, and so on.

An explicit construction for generic PI-algebras of Zariski closed algebras is given in [15, Theorem 7.19], and requires us to pass from algebras over a field to algebras over a commutative ring, which is the main reason that we do not always assume that the base ring is a field. This sort of construction also works for nonassociative Zariski closed algebras in the framework of universal algebra.

3.2. PI-generic rank over an arbitrary field.

Definition 3.7. *The* ***topological rank*** *of a Zariski closed algebra* A *is defined as the minimal possible number of generators of an* F*-subalgebra* A_0 *of* A *for which the Zariski closure of* A_0 *is* A.

Example 3.8. Let K be an infinite dimensional field extension of a finite field F. The Zariski closed algebra $A = \begin{pmatrix} F & K \\ 0 & F \end{pmatrix}$ has infinite topological rank, since any finite number of elements generates only a finite subspace of K in the $1,2$ position, which is Zariski closed.

Accordingly, we look for an alternative concept which is more closely relevant to PI-theory.

Definition 3.9 *The* ***PI-generic rank*** *of* A *is the minimal number* m *of elements needed to generate a subalgebra satisfying the same PIs as* A*; then the relatively free PI-algebra of* A *could also be generated by* m *elements. In the literature, the PI-generic rank is sometimes called the* ***basic rank***.

Clearly, the PI-generic rank is less than or equal to the topological rank. The PI-generic rank of Example 3.8 is 2.

Corollary 3.10 ([15, Corollary 7.11]). *Any representable algebra* A *(over an arbitrary field has a PI-equivalent algebra with finite PI-generic rank.*

P r o o f. Pass to the Zariski closure, which is easily seen to have finite PI-generic rank by looking at the Wedderburn block form. □

4. Gluing components.

Theorem 4.1 ([15, Theorem 5.14]). *Any Zariski closed F-subalgebra $A = S \oplus J$ of $\mathrm{M}_n(K)$ (where K is an algebraically closed field containing F) can be represented in* ***Wedderburn block form****, in which diagonal blocks comprise the semisimple part S, and the radical J embeds above the diagonal, with certain identifications of the blocks which we call* ***gluing****. Any such identification is obtained via homomorphisms between these blocks, and all identifications among the diagonal blocks are Frobenius gluing.*

The basic question is determining how the radical J interacts with $S \cong A/J$. Gluing occurs separately for the diagonal (semisimple) and the off-diagonal (radical) components. Diagonal gluing is easily described, since any identification of diagonal blocks can be viewed as an isomorphism of matrix algebras, which in turn is described in terms of an isomorphism of their fields of scalars. Any F-automorphism of a finite field is given a power φ_q^ℓ of the Frobenius map $\varphi_q : a \mapsto a^q$, where $|F| = q$, so all diagonal gluing can be described in terms of *Frobenius gluing* of the diagonal blocks. We call q^ℓ the **Frobenius twist** in the gluing. When $\ell = 0$, we call the gluing *identical gluing*. Thus, any diagonal gluing can be expressed (with respect to a suitable choice of base) as identifying the entries $\alpha_{i,j}$ in one component with $\alpha_{i,j}^{q^\ell}$ in the other component. Sometimes idempotents are glued, thereby ruining the effectiveness of the Peirce decomposition, which we refine to the "sub-Peirce decomposition" of [16, Definition 5.20] by looking outside of A.

Example 4.2.

(1) $A = \left\{ \begin{pmatrix} \alpha & x & y & \lambda x \\ 0 & \beta & z & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} : \alpha, \beta, x, y, z \in K \right\}$, where $\lambda \in K$ is fixed. The glued blocks are the sets of indices $T_1 = \{1, 3\}$ and $T_2 = \{2, 4\}$. Accordingly, we have the Peirce decomposition $A = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}$, where $A_{11} = K(e_{11} + e_{33}) + Ke_{13}$, $A_{12} = K(e_{12} + \lambda e_{14})$, $A_{21} = Ke_{23}$ and $A_{22} = K(e_{22} + e_{44})$.

(2) $A = \left\{ \begin{pmatrix} \alpha & x & y & z \\ 0 & \alpha^q & x' & y' \\ 0 & 0 & \alpha & x'' \\ 0 & 0 & 0 & \alpha^q \end{pmatrix} : \alpha, x, x', x'', y, y', z \in K \right\}$. Now there is only one glued component, namely $T_1 = \{1,2,3,4\}$, which decomposes with respect to identical gluing as $T_1 = \{1,3\} \cup \{2,4\}$. A corresponding idempotent decomposition in $\mathrm{M}_4(K)$ (but not in A!) is $\hat{e}_1 = \overline{e}_1^{(1)} + \overline{e}_1^{(2)}$ where $\hat{e}_1 = 1$, $\overline{e}_1^{(1)} = e_{11} + e_{33}$ and $\overline{e}_1^{(2)} = e_{22} + e_{44}$. The sub-Peirce components are $A_{11}^{(11)} = K\overline{e}_1^{(1)} + Ke_{13}$, $A_{11}^{(12)} = Ke_{12} + Ke_{14} + Ke_{34}$, $A_{11}^{(21)} = Ke_{23}$ and $A_{11}^{(22)} = K\overline{e}_1^{(2)} + Ke_{24}$ (similarly to the Peirce components in (1)).

Another kind of gluing (above the diagonal) is called *proportional*, by which we mean that gluing between two radical components is by means of some scalar multiple (perhaps times a Frobenius automorphism). If the Frobenius automorphism is trivial, we define the gluing to be *purely proportional.* Although one could have other possible kinds of gluing, we shall see that they are not needed.

It also is convenient to consider the following kind of gluing:

Definition 4.3. *A relation is called* ***gluing up to infinitesimals*** *if it has the form:*

$$\langle \xi_i^{q^t} - \xi_j \rangle^k = 0 \qquad (2)$$

for suitable k.

Note that when $i = j$ this merely means that we are adjoining an algebraic element to the base field to obtain a commutative algebra which may have nilpotent elements.

Example 4.4. The algebra $A = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in K \right\}$ can also be viewed as the 2×2 matrix representation of the commutative algebra of dual numbers of K, i.e., $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is identified with $a + b\delta$, where $\delta^2 = 0$.

5. The full quiver of a representation. One of the most useful tools in representation theory is the quiver of a finite dimensional algebra. However, we need a more explicit description, which also does not identify Morita equivalent algebras since matrix algebras of different size are not PI-equivalent. In this way, we are led to consider a more elaborate combinatoric object, the **full quiver** of a representation of an algebra, a directed graph Γ, having neither loops, double edges, nor cycles, with the following information attached to the vertices and edges:

The vertices of the full quiver of a representation of A correspond to the diagonal blocks arising from the simple components of the semisimple part S, whereas the arrows come from the radical J. Every vertex likewise corresponds to a central idempotent in the corresponding matrix block.

- The vertices are ordered, say from **1** to **k**, and an edge always takes a vertex to a vertex of higher order. There are identifications of vertices of edges, called **diagonal gluing**, and identification of edges, called **off-diagonal gluing**. Gluing of vertices in full quivers is identical or Frobenius.
- Each vertex is labelled with a roman numeral (I, II etc.); glued vertices are labelled with the same roman numeral. A vertex can be either **filled** or **empty**. When the base field F is finite, superscripts ℓ indicate the **Frobenius twist** between glued vertices.

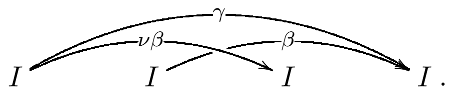
  The first vertex listed in a glued component is also given a pair of subscripts — the **matrix degree** $n_{\mathbf{i}}$ and the **cardinality** of the corresponding field extension of F (which, when finite, is denoted as a power $q^{t_{\mathbf{i}}}$ of $q = |F|$).
- Off-diagonal gluing (i.e., gluing among the edges) includes **Frobenius gluing** and **proportional gluing** with an accompanying **scaling factor** ν.

Our motivating example: The algebra of upper triangular matrices T_n corresponds to the quiver consisting of just one branch of length n, with all vertices corresponding to blocks of dimension 1, and with no gluing. The subtleties arise from gluing, with many examples given in [17]. Here is an example of proportional gluing.

Example 5.1.

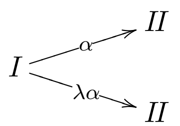
$$A = \left\{ \begin{pmatrix} \alpha & 0 & \nu\beta & \gamma \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix} : \alpha \in F, \beta \in K \right\}$$

has off-diagonal proportional gluing with scaling factor ν. The corresponding quiver is

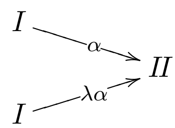
(3) 

A **pseudo-quiver** is the graph obtained by a change of base of the glued diagonal blocks. Such a base change may simplify the arrows in the full quiver in a way that improves the off-diagonal gluing, eliminating linear relations.

Example 5.2. Suppose the quiver contains three vertices with arrows glued in the form



or



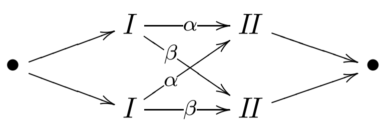
;

for some $\lambda \in K$. Then we can trade two arrows for one, by creating a new vertex, replacing the existing arrows with $I \xrightarrow{(\lambda+1)\alpha} II$.

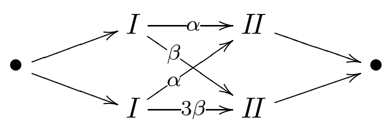
However changing base could result in a representation in which the diagonal blocks corresponding to vertices may have extra linear relations that are not consequences of identical or Frobenius gluing.

The following examples, repeated from [16], illustrate the pseudo-quiver.

Example 5.3. Consider the full quivers

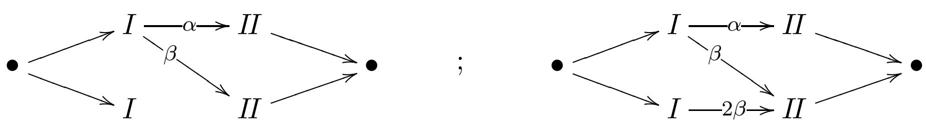
(4) 

and



In the left-hand side of (4), a base change of the components denoted by I, which replaces e_{22}, e_{33} with $e_{22}, e_{33} - e_{22}$ as in Example 5.2, results in the first quiver of (5), which is subdirectly reducible, with no gluing. However the same base change applied to the right-hand side of (4) results in the second quiver of (5), which still has gluing.

(5)



5.1. The monoid grading. We grade paths according to the following rules:

Definition 5.4. *We write $\mathcal{M}_\infty$ for the multiplicative monoid $\{1, q, q^2, \dots, \epsilon\}$, where $\epsilon a = \epsilon$ for every $a \in \mathcal{M}_m$. (In other words, ϵ is the "zero" element adjoined to the multiplicative monoid $\langle q \rangle$.) $\mathcal{M}_m$ denotes the monoid obtained by adjoining a "zero" element ϵ to the subgroup $\langle q \rangle$ of $\mathbb{Z}_{q^m-1}$, namely $\mathcal{M}_m = \{1, q, q^2, \dots, q^{m-1}, \epsilon\}$ where $\epsilon a = \epsilon$ for every $a \in \mathcal{M}_m$. Let $\overline{\mathcal{M}}$ be the semigroup $\mathcal{M}/\sim$, where $\sim$ is the equivalence relation obtained by matching the degrees of glued variables: When two vertices have a Frobenius twist q^ℓ, we identify 1 with q^ℓ in the respective components.*

Example 5.5. The full quiver of

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^q \end{pmatrix} : \alpha \in \mathbb{F}_{q^r}, \beta \in K \right\}, \tag{6}$$

is $I \longrightarrow I^{(1)}$. The grading monoid is $\mathcal{M}_r \times \mathcal{M}_r$ modulo the identification $(1, \epsilon) \equiv (\epsilon, q)$.

The sub-Peirce decomposition has three components, one for each matrix entry, and the generic element $\left\{ \begin{pmatrix} x & y \\ 0 & x^q \end{pmatrix} \right\}$ subdivides into the three components

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x^q \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \right\}.$$

Note that x and x^q have the same grade, by the identification.

5.2. Canonization Theorems. Since arbitrary gluing is difficult to describe, we need some "canonization" theorems to "improve" the gluing. The first theorem shows that we have already specified enough kinds of gluing.

Theorem 5.6 (First Canonization Theorem, cf. [16, Theorem 6.12]). *The Zariski closure of any representable affine PI-algebra A has a representation for whose full quiver all gluing is Frobenius proportional.*

Definition 5.7. *A full quiver (resp. pseudo-quiver) is* ***basic*** *if it has a unique initial vertex r and unique terminal vertex s, and all of its gluing above the diagonal is proportional Frobenius. A basic full quiver (resp. pseudo-quiver) Γ is* ***canonical*** *if any two paths from the vertex r to the vertex s have the same grade.*

(Our notion of basic quiver has nothing to do with the notion of basic algebra in representation theory.)

Theorem 5.8 (Second Canonization Theorem, cf. [17, Theorem 3.7]). *Any relatively free algebra is a subdirect product of algebras having faithful representations whose full quivers are basic.*

Any basic full quiver Γ (resp. pseudo-quiver) of a representable relatively free algebra can be modified (via a change of base) to a canonical full quiver (resp. pseudo-quiver) of an isomorphic algebra (i.e., relatively free algebra of the same variety).

The Third Canonization Theorem describes what happens when one mods out a "nice" T-ideal:

Theorem 5.9 (Third Canonization Theorem, cf. [17, Theorem 3.12]). *Suppose A is a relatively free PI-algebra having a representation with pseudo-quiver Γ, and $\mathcal{I} \neq 0$ is a T-ideal of A closed under multiplication by characteristic coefficients of the elements of A under the given representation. Then $A' = A/\mathcal{I}$ is obtained by means of the following elementary operations:*

(1) *New relations on the base ring and its pseudo-quiver Γ' are obtained by the appropriate new gluing. This means:*

  - *Gluing, perhaps up to infinitesimals, with or without a Frobenius twist. When $i = j$, this means a reduction of the order of the center of the block.*
  - *New quasi-linear relations on arrows, perhaps up to infinitesimals.*
  - *Reducing the matrix degree of a block attached to a vertex. (This is achieved by adjoining every characteristic coefficient for the Hamilton-Cayley relations of smaller degree, and, for Frobenius relations, rela-*

tions of the form

$$c^q f(a_1, \dots, a_m) - c f(a_1, \dots, a_m),$$

for every characteristic coefficient.)

(2) *New linear dependences on vertices (which could include cancelling extraneous vertices) between which any two paths must have the same grade.*

6. Extracting combinatoric information from polynomials.

We get to the crucial point of this paper, which is how to obtain useful information about evaluations of a polynomial $f(x_1, x_2, \dots)$ on a faithful representable algebra A over an integral domain F, in terms of its representation. Our main strategy is to carve out a niche inside a T-ideal which is closed under multiplication of the coefficients of the monic characteristic polynomial of enough elements of a given Zariski closed algebra A to enable us to apply Shirshov's celebrated height theorem [14, Theorem 2.3]. Our procedure, called **hiking**, is described in detail in §6.2.2.

Since the quasi-linearization of f (as defined above) is in the T-ideal generated by f, Lemma 1.7 shows that the evaluations of a quasi-linear polynomial $f(x)$ are spanned by the evaluations obtained by specializations of the indeterminates x_i to $S \cup J$. We work with quasi-linear polynomials and pure substitutions. We want to pinpoint the semisimple substitutions, in order to utilize the well-understood properties of semisimple matrices (especially their characteristic polynomials).

In order to guarantee that the semisimple substitutions (to the matrix components) are indeed semisimple as matrices, we take the Jordan decomposition of the matrix $a = s + r$ where s is semisimple and r is nilpotent with $sr = rs$, and then observe that if $r^k = 0$ and $\overline{q}$ is a p-power greater than k, then

$$a^{\overline{q}} = (s + r)^{\overline{q}} = s^{\overline{q}} + r^{\overline{q}} = s^{\overline{q}} + 0 = s^{\overline{q}},$$

which is semisimple. This leads us to take $\overline{q}$-powers of matrices, and $\overline{q}$-powers of characteristic coefficients, which we call **$\overline{q}$-characteristic coefficients** (for $\overline{q}$ a suitable power of $q = |F|$; if F is infinite we take $q = 1$). In [18, Lemma 2.3] we note that $\sum \alpha_i^{\overline{q}} \lambda^i$ is the characteristic polynomial of $a^{\overline{q}}$.

The difficulty with our strategy is that initially we have little control as to which pure substitutions we have, and to which components they belong. We can increase our control by careful modifications of our polynomial. A polynomial

$f(x_1,\dots,x_t;y_1,\dots,y_t)$ is called $(A;t;\overline{q})$**-quasi-alternating** if f is A-quasi-linear in $x_1,\dots,x_t$ (Definition 1.3) and quasi-homogeneous of degree $\overline{q}$, a q-power, such that f becomes 0 whenever x_i is substituted throughout for x_j for two indices $1 \le i < j \le t$.

Kemer already used quasi-alternating polynomials over infinite fields in [47, Equation (40)]; he uses the terminology **forms** for our characteristic coefficients. If f is $(A;t;\overline{q})$-quasi-alternating, then we still get Kemer's conclusion. As explained in [18], this can also be stated in the language of [14, Theorem J, Equation 1.19, page 27].

Next, we insert an alternating central polynomial h_{n_j} for $\mathrm{M}_{n_j}(C)$, as often as we want, since h_{n_j} takes on scalar values. Since h_{n_j} must vanish on $\mathrm{M}_k(C)$ for all $k < n_j$, this gives us a way of distinguishing the components of matrix degree n_j from smaller degrees, so we focus on the largest matrix degree in the branch.

Next, we utilize the fact that the radical has bounded index of nilpotence.

Remark 6.1. Any nonzero evaluation arises from a string of substitutions $x_i \mapsto \overline{x_i}$ to elements corresponding to some path of the full quiver Γ. (We are permitted to have substitutions repeating in the same matrix block.) Suppose t is the nilpotence index of the radical J. Then any string involving t radical substitutions is 0. If we replace x_i by $h_{n_i}x_{i,1}h_{n_i}x_{i,2}\cdots x_{i,t}h_{n_i}x_i$, then we still get the same evaluation when $x_{i,1},x_{i,2},\cdots,x_{i,t}$ are specialized to the identity matrices in the appropriate blocks, which in particular are semisimple substitutions. On the other hand, the number of radical substitutions must be at most the nilpotence index of A, so at least one of these extra substitutions must be semisimple, if we are still to have a nonzero evaluation. By taking $h_{n_i}x_{i,1}^{\overline{q}}h_{n_i}x_{i,2}\cdots x_{i,t}h_{n_i}x_i$ we force the substitution $\overline{x_{i,1}}^{\,\overline{q}}$ to be semisimple.

Any matrix $a \in \mathrm{M}_n(K)$ can be viewed either as a linear transformation on the n-dimensional space $V = K^{(n)}$, and thus having Hamilton-Cayley polynomial f_a of degree n, or (via left multiplication) as a linear transformation $\tilde{a}$ on the n^2-dimensional space $\tilde{V} = \mathrm{M}_n(K)$ with Hamilton-Cayley polynomial $f_{\tilde{a}}$ of degree n^2. The matrix $\tilde{a}$ can be identified with the matrix

$$a \otimes I \in \mathrm{M}_n(K) \otimes \mathrm{M}_n(K) \cong \mathrm{M}_{n^2}(K),$$

so its eigenvalues have the form $\beta \otimes 1 = \beta$ for each eigenvalue β of a. From this, we conclude:

Proposition 6.2 ([17, Proposition 2.4])**.** *Suppose $a \in \mathrm{M}_n(F)$. Then the characteristic coefficients of a are integral over the F-algebra $\hat{C}$ generated by the characteristic coefficients of $\tilde{a}$.*

P r o o f. The integral closure of $\hat{C}$ contains all the eigenvalues of $\tilde{a}$, which are the eigenvalues of a, so the characteristic coefficients of $\tilde{a}$ also belong to the integral closure. $\square$

6.1. Characteristic coefficient-absorbing polynomials inside T-ideals. Having obtained semisimple substitutions via Remark 6.1, we want to extract their characteristic coefficients, by means of polynomials.

There are two ways of obtaining intrinsically the coefficients of the characteristic polynomial

$$f_a = \lambda^n + \sum_{k=1}^{n-1} (-1)^k \alpha_j(a) \lambda^{n-k}$$

of a matrix a. Fixing k, we write α for α_k. (For example, if $k = 1$ then $\alpha(a) = \mathrm{tr}(a)$.)

Definition 6.3. *In any matrix ring* $\mathrm{M}_n(W)$, *we define*

(7) $$\alpha_{\mathrm{mat}}(a) := \sum_{j=1}^{n} \sum e_{j,i_1} a e_{i_2,i_2} a \cdots a e_{i_k i_k} a e_{i_1,j},$$

the inner sum taken over all vectors of length k.

We can also define the $\overline{q}$-characteristic coefficients via polynomials.

Definition 6.4. *Given a quasi-linear polynomial* $f(x; y)$ *in indeterminates labelled* x_i, y_i, *we say* f *is* ***$\overline{q}$-characteristic coefficient-absorbing*** *with respect to a full quiver* $\Gamma = \Gamma(A)$ *if the following properties hold:*

1. f *specializes to 0 under any substitution in which at least one of the* x_i *is specialized to a radical element of* A. *(In other words, the only nonzero values of* f *are obtained when all substitutions of the* x_i *are semisimple.)*
2. $f(\mathcal{A}(\Gamma))^+$ *absorbs multiplication by any* $\overline{q}$*-characteristic coefficient of any element in a simple (diagonal) matrix block of* $\mathcal{A}(\Gamma)$.

Lemma 6.5. ([18, Lemma 3.6]). *For any* $\mathrm{M}_n(F)$*-quasi-linear polynomial* $f(x_1, x_2, \dots)$ *which is also* $\mathrm{M}_n(F)$*-quasi-homogeneous of degree* $\overline{q}$ *in* x_1, *the polynomial*

$$\hat{f} = f(c_{n^2}(y) x_1 c_{n^2}(z), x_2, \dots)$$

is $\overline{q}$*-characteristic coefficient absorbing in* x_1.

The proof can be formulated in the language of [14, Theorem J, Equation 1.19, page 27] (with the same proof), as follows:

$$\alpha_k^{\overline{q}} f(a_1, \dots, a_t, r_1, \dots, r_m) = \sum f(T^{k_1} a_1, \dots, T^{k_t} a_t, r_1, \dots, r_m), \tag{8}$$

summed over all vectors $(k_1, \dots, k_t)$ with each $k_i \in \{0, 1\}$ and $k_1 + \cdots + k_t = k$, where α_k is the k-th characteristic coefficient of a linear transformation $T : V \to V$, and f is $(A; t; \overline{q})$-quasi-alternating.

Iteration yields:

Proposition 6.6. *For any polynomial $f(x_1, x_2, \dots)$ quasi-linear in x_1 with respect to a matrix algebra $\mathrm{M}_n(F)$, there is a polynomial $\hat{f}$ in the T-ideal generated by f which is $\overline{q}$-characteristic coefficient absorbing.*

Definition 6.7. *Fixing $0 \le k < n$, we denote this implicit definition in Proposition* 6.6 *of $\alpha_k^{\overline{q}}$, the k-th $\overline{q}$-characteristic coefficient of a, as $\alpha_{\mathrm{pol}}^{\overline{q}}(a)$.*

Remark 6.8. If the vertex corresponding to r has matrix degree n_i, taking an $n_i \times n_i$ matrix w, we define $\alpha_{\mathrm{pol}_u}^{\overline{q}}(w)$ as in the action of Definition 6.7 and then the left action

$$a_{u,v} \mapsto \alpha_{\mathrm{pol}_u}^{\overline{q}}(w) a_{u,v}. \tag{9}$$

Likewise, for an $n_j \times n_j$ matrix w we define the right action

$$a_{u,v} \mapsto a_{u,v} \alpha_{\mathrm{pol}_v}^{\overline{q}}(w). \tag{10}$$

(However, we only need the action when the vertex is non-empty; we forego the action for empty vertices.)

Remark 6.9. Notation as in (8), the Cayley-Hamilton identity for $n_i \times n_i$ matrices is

$$0 = \sum_{k=0}^{n_i} \alpha_k^{\overline{q}} f(a_1, \dots, a_t, r_1, \dots, r_m) = \sum_{k_1, \dots, k_t} f(T^{k_1} a_1, \dots, T^{k_t} a_t, r_1, \dots, r_m),$$

which is thus an identity in the T-ideal generated by f.

Definition 6.10. *We call the identity $\sum_{k_1, \dots, k_t} f(T^{k_1} x_1, \dots, T^{k_t} x_t, r_1, \dots, r_m)$ obtained in the above remark, the **Hamilton-Cayley identity induced by** f.*

6.2. Identification of matrix actions. The identification of matrix actions is possible whenever the polynomially-defined characteristic coefficients commute. To achieve this, we treat two cases in turn.

6.2.1. Identification of matrix actions for unmixed substitutions. In the first case, the substitution of an indeterminate is to sums of elements in the same glued Wedderburn component. Here, one replaces f by $c_{n_1^2} f$ and obtains the desired action on the matrix component from Equations (9) and (10).

6.2.2. Hiking. The second case is much subtler. Here, we consider the substitution of an indeterminate to sums of elements in different glued Wedderburn components. We must make sure which substitution is semisimple, and cope with the possibility that our semisimple substitution has been sent to the 'wrong' component, either because its matrix degree is too large or the base field is of the wrong size. To prevent this, we make extra substitutions of indeterminates, called **hiking**, which force the evaluations of f to be 0 in such situations, and also force the matrix characteristic coefficients to commute with each other and with radical substitutions of arrows connecting glued vertices. The hiking procedure is quite subtle, and requires four different stages.

We write $[a, b]_q$ for the **Frobenius commutator** $ab - b^q a$.

Lemma 6.11. *If $f(x_1, \dots, x_n)$ is any polynomial quasi-linear in x_i, then*

$$(11)\quad f(a_1, \dots, [a, a_{i_1} \cdots a_{i_k}], \dots a_n) = \sum_{j=1}^{k} f(a_1, \dots, a_{i_1} \cdots [a, a_{i_j}] \cdots a_{i_k}, \dots a_n),$$

for all substitutions in A.

The proof is by checking first for monomials, and summing.

6.2.3. First stage of hiking. Suppose a quasi-linear nonidentity f of a Zariski closed algebra A has a nonzero value for some radical substitution of some x_i in A, corresponding to an arrow in the full quiver whose initial vertex is labelled by (n_i, t_i) and whose terminal vertex is labelled by (n_i', t_i'). Replacing x_i by $[x_i, h_{\max\{n_i, n_i'\}}]$ (where the h_{n_i} involve new indeterminates) yields a quasi-linear polynomial

$$(12)\qquad \nabla_i f := f(\dots, [x_i, h_{\max\{n_i, n_i'\}}], \dots)$$

in which any substitution of x_i into this diagonal block yields 0, since the evaluations of h_{n_i} in the semisimple part are central; hence, any nonzero value in $\nabla_i f$ forces us into a radical substitution. On the other hand, $\nabla_i f$ does not vanish since there are substitutions in the appropriate diagonal block (the one whose degree is $\max\{n_i, n_i'\}$) for which $h_{\max\{n_i,n_i'\}}$ is a nonzero scalar.

In the case of Frobenius gluing via $x \mapsto x^{q^\ell}$, we need to take instead the substitution

$$x_i \to f_{i+1} := f_i(\dots, [x_i, h_{\max\{n_i,n_i'\}}]_{q^\ell}, \dots).$$

Stage 1 hiking is illustrated via the full quiver given for the Grassmann algebra on two generators:

(13)
$$\begin{array}{ccc} I & \xrightarrow{\alpha} & I \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle -\beta} \\ I & \xrightarrow{\alpha} & I \end{array}$$

Clearly the critical nonidentity for each branch is $f = [x_1, x_2]$, and we get the Grassmann identity $[[x_1, x_3], x_2]$ by hiking x_1.

6.2.4. Second stage of hiking. Next we need a second stage of hiking, to take care of substitutions into the "wrong" component. Consider a path $\mathcal{B}$ in the full quiver, whose vertices are numerated consecutively for convenience. In stage 1 hiking, taking $n_i' = n_{i+1}$ maximal among all the matrix degrees, we have obtained expressions of the form

(14)
$$g_i(x, y, z) = z_{i,1}[h_{\max\{n_i,n_{i+1}\}}(x_{i,1}, x_{i,2}, \dots), y_i]z_{i,2},$$

and then defined

(15)
$$\tilde{f} = f(h_{n_1}, g_1, h_{n_2}, g_2, \cdots, g_\ell, h_{n_\ell+1}),$$

where different indeterminates are used in each polynomial.

Given a nonzero specialization of a given monomial of f under the substitutions $x_i \mapsto \overline{x_i}$, $i \geq 1$, where $\overline{x_i} \in \mathrm{M}_{n_i}(K)$, consider another nonzero specialization into another monomial under the substitutions $x_i \mapsto \overline{x_i}'$, $i \geq 1$, where $\overline{x_i}' \in \mathrm{M}_{n_j}(K)$, for $j \neq i$. Clearly $n_j \geq n_i$. To eliminate the possibility that $n_i < n_j$, we need some procedure to guarantee that the specialization of (14) does not land in the wrong matrix component $\mathrm{M}_{n_j}(K)$ since the polynomial h_{n_i} is not scalar-valued in this component and thus would not eliminate the semisimple substitutions. Letting $H := h_{n_j}^{\overline{q}}$, we consider the polynomials

$$z_{i,1}([h_{n_i}(x_{i,1}, x_{i,2}, \dots), y_i]z_{i,2}g_{i+1}\cdots g_{j-1}H^{q_1} \\ - H^{q_2}[h_{n_i}(x_{i,1}, x_{i,2}, \dots), y_i]z_{i,2}g_{i+1}\cdots g_{j-1}).$$

Consider the product of these operators, taken over all the pairs (q_1, q_2) that occur in Frobenius twists in the branch. Specializing this expression into the j-component (of size n_j) would yield two equal terms which cancel, and thus yield 0. But specializing into the i component n_i (of size n_i) would yield one term nonzero and the other 0, so their difference would be nonzero. In this way, we eliminate the "wrong" specializations while preserving the "correct" one.

Example 6.12. We illustrate this key procedure for the algebra

$$\left\{ \begin{pmatrix} \alpha & * & * & * \\ 0 & \beta & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} : \alpha, \beta \in F \right\},$$

where $*$ denotes an arbitrary element in K. The full quiver $I_{(1,1)} \to II_{(1,1)} \to III_2$, which would normally give us the polynomial

$$z_{1,1}[x_{1,1}, y_1]z_{1,2}z_{2,1}[h_2(x_{2,1}, \dots), y_2]z_{3,1},$$

which could be condensed to $[x_{1,1}, y_1]z[h_2(x_{2,1}, \dots), y_2]$ since various indeterminates can be specialized to 1. But if $f(x_1, x_2, x_3, \dots)$ has both monomials $x_1x_2x_3\cdots$ and $x_3x_2x_1\dots$ then hiking in the second monomial yields the permuted term

$$[h_2(x_{2,1}, \dots), y_2]z[x_{1,1}, y_1]$$

which permits a nonzero evaluation with all substitutions in the lower 2×2 matrix component, and we cannot get a proper hold on the substitutions.

After stage 1 hiking, when we replace x_2 in f by $[y_1, y_2]h_2 - h_2[y_1, y_2]$, stage 2 hiking makes the "wrong" specializations into $\mathcal{C}$ become 0.

6.2.5. Third stage of hiking. When $\mathcal{B}'$ is another branch with the same degree vector, and the corresponding base fields for the i-th vertex of $\mathcal{B}$ and $\mathcal{B}'$ are n_i and n_i' respectively, we take $t_i = q^{n_i'}$ and replace x_i by $(h_{n_i}^{t_i} - h_{n_i})x_i$. This cuts off the specializations to matrices over finite fields of the wrong order.

6.2.6. Fourth stage of hiking. Some of the radical substitutions are **internal** in the sense that they occur in a diagonal block (after "gluing up to infinitesimals"). Hiking absorbs all internal radical substitutions, because of the use of the central polynomial h_{n_i}, so when working with fully hiked polynomials, we need consider only the Peirce decomposition (and not the more complicated sub-Peirce decomposition; cf. [15].)

Remark 6.13. As explained in Proposition 6.6, there is a Capelli polynomial $\tilde{c}_{n_i^2}$ and p-power $\overline{q}$ such that

$$\tilde{c}_{n_i^2}(\alpha_k y)x_i c_{n_i'^2}(y) = \alpha_k^{\overline{q}}(y_1)c_{n_i^2}(y)x_i c_{n_i'^2}(y) \tag{16}$$

on any diagonal block. Since $\overline{q}$-characteristic coefficients commute on any diagonal block, we see from this that

$$\tilde{c}_{n_i^2}(y)x_i c_{n_i'^2(y)}\tilde{c}_{n_i^2}(z)x_i c_{n_i'^2(z)} - \tilde{c}_{n_i^2}(z)x_i c_{n_i'^2(z)}\tilde{c}_{n_i^2}(y)x_i c_{n_i'^2(y)} \tag{17}$$

vanishes identically on any diagonal block, where $z = \alpha_k y$. One concludes from this that substituting (17) for x_i would hike our polynomial one step further. But there are only finitely many ways of performing this hiking procedure. Thus, after a finite number of hikes, we arrive at a polynomial in which we have complete control of the substitutions and the $\overline{q}$-characteristic coefficients commute.

6.3. $\overline{q}$-Characteristic coefficient-absorbing polynomials. Since hiking is applied to the components of maximal matrix degree, the polynomial f is required to have a nonzero evaluation on a maximal vector with respect to the grade. Thus we must consider the following sort of polynomial.

Definition 6.14. *A polynomial f is A-**admissible** on a Zariski-closed algebra A if f takes on some nonzero evaluation on a vector of maximal grade. We denote such a vector as $v_{\mathcal{B}}$, where $\mathcal{B}$ is the branch of the full quiver which gives rise to $v_{\mathcal{B}}$, and call $v_{\mathcal{B}}$ the **matrix vector of** f.*

6.3.1. Symmetrization. Yet another difficulty remains. One could have two strings $I \to II \to III$ and $I \to III \to II$, whereby the substitutions in f go to incompatible components. The following definition, inspired by Drensky [28], enables us to bypass this hazard.

Definition 6.15. *Given matrices $a_1, \dots, a_k$, the **symmetrized** $(t; j)$ characteristic coefficient is the j-elementary symmetric function applied to the t-characteristic coefficients of $a_1, \dots, a_k$.*

For example, taking $t = 1$, the symmetrized $(1, j)$-characteristic coefficients α_t are

$$\sum_{j=1}^{k} \operatorname{trace}(a_j), \quad \sum_{j_1 > j_2} \operatorname{trace}(a_{j_1}) \operatorname{trace}(a_{j_2}), \quad \dots, \quad \prod_{j=1}^{k} \operatorname{trace}(a_j).$$

Lemma 6.16. *Any characteristic coefficient α_t is algebraic over the field with the symmetrized characteristic coefficients adjoined.*

P r o o f. If $\alpha_{t,j}$ denotes the $(t;j)$-characteristic coefficient, then α_t satisfies the usual polynomial $\lambda^k + (-1)^j \sum_{j=1}^{k} \alpha_{t,j}\lambda^{k-j}$. $\square$

Lemma 6.17 (Compatibility Lemma, [18, Lemma 3.26]). *Let A be a representable Zariski-closed algebra. For any A-admissible non-identity f, the T-ideal $\mathcal{I}$ generated by the polynomial f contains a symmetrized $\overline{q}$-characteristic coefficient-absorbing polynomial $\overline{f}$, not an identity of A, in which all substitutions providing nonzero evaluations of f are compatible.*

From this lemma, one readily concludes:

Theorem 6.18 ([18, Theorem 3.25]). *Suppose f is an A-admissible non-identity of a representable, relatively free algebra A. Then the T-ideal $\mathcal{I}$ generated by f contains a $\overline{q}$-characteristic coefficient-absorbing A-admissible non-identity $\tilde{f}$.*

Furthermore, the T-ideal $\mathcal{I}_{\mathcal{B}}$ of all fully hiked A-admissible polynomial obtained from the degree vector $v_{\mathcal{B}}$ is comprised of evaluations of $\overline{q}$-characteristic coefficient-absorbing polynomials, comprised of sums of evaluations on pure specializations in $\mathcal{B}$.

In applying the results of this discussion, we want to avoid situations in which the $\overline{q}$-characteristic coefficient-absorbing polynomials, degenerate into "extraneous" identities because of gluing between branches. In Section 12 below, we give various examples of identities arising from gluing (and also indicate the difficulty in ascertaining id(A) in general). Fortunately, these do not interfere with the proof of Specht's conjecture since they only occur a finite number of times in an ascending chain of T-ideals.

6.4. Application of Shirshov's theorem.

Definition 6.19. *For a Zariski closed algebra $A \subseteq \mathrm{M}_n(K)$ faithful over an integral domain C, we denote by $\hat{C}$ the algebra obtained by adjoining to C the matrix symmetrized $\overline{q}$-characteristic coefficients of products of the sub-Peirce components of the generic generators of A (of length up to the bound of Shirshov's Theorem [14, Chapter 2]).*

Recall in view of Shirshov's theorem that we only need to adjoin a finite number of elements to obtain $\hat{C}$.

Lemma 6.20. *The algebra $\hat{A}$ is a finite module over $\hat{C}$, and in particular is Noetherian.*

P r o o f. Let $\hat{C}'$ be the commutative algebra generated over C, by all the characteristic coefficients of (finitely many) products of the sub-Peirce components of the generic generators of A, as in Definition 6.19. Clearly $\hat{C} \subseteq \hat{C}'$.

Enlarge $\hat{A}$ to $\hat{A}' = \hat{C}'\hat{A}$, which is a finite module over $\hat{C}'$, in view of Shirshov's Theorem. But $\hat{C}'$ is finite over $\hat{C}$, in view of Lemma 6.16, implying $\hat{A}$ is finite over $\hat{C}$. □

Proposition 6.21. *For any $\overline{q}$-characteristic coefficient-absorbing polynomial f with respect the quiver of A, the Hamilton-Cayley identity induced by f is an identity of $\hat{A}$, and thus of A.*

Theorem 6.22 ($\overline{q}$-Characteristic Value Adjunction Theorem)**.** *For any nonidentity f of a representable, relatively free affine algebra A, the T-ideal $\mathcal{I}$ generated by the polynomial f contains a nonzero T-ideal which is also an ideal of the algebra $\hat{A}$.*

Proof. The proof, given in [18, Theorem 3.27], consists of collecting the various pieces. First we quasi-linearize f, and assume it is fully hiked. We may assume that the generators of A are generic elements, say $X_1, \dots, X_t$. Thus we are done by Proposition 6.21. □

7. Solution of Specht's problem for T-ideals over finite fields.

We introduce an induction procedure based on the geometric notion of reducing quivers.

Definition 7.1. *Suppose Γ is a pseudo-quiver. A **reduction** of Γ is a pseudo-quiver Γ' obtained by at least one of the following possible procedures:*

(1) *New relations on the base ring and its pseudo-quiver Γ' are obtained by the appropriate new gluing. This means:*
   - *Gluing, perhaps up to infinitesimals, with or without a Frobenius twist (when the gluing is of a block with itself, with a Frobenius twist, it must become finite);*
   - *New quasi-linear relations on arrows, perhaps up to infinitesimals;*
   - *Reducing the matrix degree of a block attached to a vertex.*

(2) *New linear dependences on vertices (which could include cancelling extraneous vertices) between which any two paths must have the same grade.*

*A **subdirect reduction** $\{\Gamma'_1, \dots, \Gamma'_m\}$ of Γ is a finite set of reductions of Γ.*

Since these elementary operations can only be performed finitely many times on any given full quiver, we will have a solution of Specht's problem once we can modify arbitrary T-ideals to the "nice" T-ideals corresponding to these reductions. This is our next objective.

Lemma 7.2. *Suppose A is a relatively free affine algebra in the variety of a Zariski closed algebra B.*

Consider a maximal path in the full quiver of B with the corresponding degree vector v_A. Let $\mathcal{J}$ be the ideal generated by the homogeneous elements of the degree vector v_A. $A/\mathcal{J}$ is the relatively free algebra of a Zariski closed algebra, and hence representable, and its full quiver has fewer maximal paths of degree v_A than A.

Proof. The proof is similar to that of the Second Canonization Theorem, Theorem 5.8. Consider a maximal graded component in A. Add characteristic coefficients of the generators of the generic algebra constructed from B, and note that they agree with the grading of the paths. Factoring out the product corresponding to the maximal degree vector we obtain a representable algebra, B'. Construct the full quiver of B' as in the proof of the Second Canonization Theorem. Then all paths in B' have fewer maximal paths of degree v_A, and $A/\mathcal{J}$ is the relatively free algebra of B'. □

Theorem 7.3 (Positive solution of Specht's Problem over fields)**.** *Let F be an arbitrary field (possibly finite). Any chain of T-ideals in the free algebra of $F\{x\}$ ascending from the ideal of identities of an affine PI-algebra A, stabilizes.*

Proof. We repeat the proof from [18]. We want to show that any ascending chain of T-ideals

$$\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_3 \subseteq \cdots \tag{18}$$

in the relatively free algebra A, stabilizes.

By Lemma 2.2 (for fields) we may assume that the algebra A is representable. Let Γ be the full quiver of A, so $\mathrm{id}(A) = \mathrm{id}(\Gamma)$. Let v_A denote the maximal degree vector for a non-zero evaluation of some polynomial in A. For each j, let $\mathcal{I}_j^{(1)}$ be the maximal subideal of $\mathcal{I}_j$ closed under multiplication by $\hat{C}$, where $C = F$. For each j, let $\mathcal{I}_j^{(0)} \subseteq \mathcal{I}_j$ denote the T-ideal of A generated by symmetrized $\overline{q}$-characteristic coefficient-absorbing polynomials of $\mathcal{I}_j$ having a non-zero specialization with maximal degree vector.

Then we get the chain

$$\mathcal{I}_1{}^{(1)} \subseteq \mathcal{I}_2{}^{(1)} \subseteq \mathcal{I}_3{}^{(1)} \subseteq \cdots \tag{19}$$

of ideals of A, which are also ideals in the Noetherian algebra $\hat{A}$ of Definition 6.19, and thus stabilize at some $\mathcal{I}_{j_0}^{(1)}$.

Passing to $A/\mathcal{I}_{j_0}^{(1)}$, we may assume that $\mathcal{I}_j^{(1)} = 0$ for each $j > j_0$. Note that $A/\mathcal{I}_{j_0}^{(1)}$ is representable as a subalgebra of $\hat{A}/\mathcal{I}_{j_0}^{(1)}$. Now, if $\mathcal{I}_j^{(0)} \neq 0$ then the fully hiked polynomial of some $0 \neq f \in \mathcal{I}_j^{(0)}$ is in $\mathcal{I}_j^{(1)}$, which is thus non-zero as well, so we conclude that $\mathcal{I}_j^{(0)} = 0$ for each $j > j_0$. It follows that $\mathcal{I}_j$ has only zero evaluations in degree v_A.

Finally, let $\mathcal{J}$ be the T-ideal defined in Lemma 7.2. Thus $\mathcal{J} \cap \mathcal{I}_j = \mathcal{I}_j^{(0)}$, so passing to $A/\mathcal{J}$, which is relatively free and representable by Lemma 7.2, we lower the maximum degree, and are done by induction. □

8. Various aspects of torsion. The proof for algebras over Noetherian rings is wrapped up after a few remarks about torsion in Noetherian modules over a commutative ring C. To avoid confusion in our notation, we write z for elements of C since c_k denotes the Capelli polynomial.

Definition 8.1. *An element $z \in C$ **makes** a **torsion** if there is $k > 0$ such $z^k a = 0$. For given $z \in C$, we say the torsion made by z is **bounded** (**by** ℓ) if $z^\ell a = 0$ whenever z makes $a \in M$ torsion. (In other words, if $z^k a = 0$ then $z^\ell a = 0$ for every $a \in M$.)*

We define $\overline{\mathrm{tor}}(C; a) = \{z \in C\colon\ z$ makes the element a torsion$\}$. For a subset $S \subseteq M$, we define $\overline{\mathrm{tor}}(C; S) = \cap_{a\in S}\overline{\mathrm{tor}}(C; a)$.

*The p-**torsion index** for $a \in M$ is the minimal k such that $z^{p^k} a = 0$ whenever z makes a torsion. Similarly the p-torsion index of a subset $S \subseteq M$ is the minimal k such that $z^{p^k} S = 0$ whenever $z \in \overline{\mathrm{tor}}(C; S)$.*

Notice that $\overline{\mathrm{tor}}(C; a)$ is an ideal of C, in fact the radical ideal of $\mathrm{Ann}_C(a)$.

Definition 8.2. *Given a module M over a commutative integral domain C and $I \subseteq C$ (permitting $I = C$), define $\mathrm{tor}_I(M)$ to be the set of elements of M annihilated by all elements of I. Likewise, given $\mathcal{I} \subseteq M$ we define $\mathrm{tor}(C)_{\mathcal{I}}$ to be the set of elements of C annihilated by all elements of $\mathcal{I}$.*

Remark 8.3. We have inverse correspondences {Ideals of C} → {T-Ideals of A} and {T-Ideals of A} → {Ideals of C} given respectively by

$$I \mapsto \mathrm{tor}_I(A), \qquad \mathcal{I} \mapsto \mathrm{tor}(C)_{\mathcal{I}}.$$

Clearly $\mathrm{tor}_{\mathrm{tor}(C)_{\mathcal{I}}}(A) \supseteq \mathcal{I}$, defining a "closure" operation on T-ideals.

We also consider torsion with respect to a single element $I = \{z\}$ of C.

Lemma 8.4. *If C is a principal ideal domain, then* $\mathrm{tor}_{z_1}(A)\cap\mathrm{tor}_{z_2}(A)=0$ *for any relatively prime elements z_1, z_2 of C.*

8.1. Torsion in commutative Noetherian algebras over a field of positive characteristic. For the remainder of this section, we consider the special situation in which F is a field of characteristic $p>0$, and C is a commutative Noetherian F-algebra.

Lemma 8.5 ([18, Lemma 4.2]). 1. *Every element of a C-module has finite p-torsion index.*

2. *Every Noetherian C-module has finite p-torsion index.*

Proof. 1. Define $\overline{\mathrm{tor}}(C;a)_k = \left\{z\in C:\ z^{p^k}a=0\right\}$; these are ideals of C because of the Frobenius endomorphism in characteristic p. The series $\overline{\mathrm{tor}}(C;a)_1\subseteq\overline{\mathrm{tor}}(C;a)_2\subseteq\cdots$ stabilizes since C is Noetherian, which proves a has a finite p-torsion index.

2. Let M' be a submodule of the Noetherian module M maximal with respect to having finite p-torsion index, and assume $a\notin M'$. Let k be larger than the p-torsion indices of M' and of a. For every $z\in\overline{\mathrm{tor}}(C;M'+Ca)=\overline{\mathrm{tor}}(C;M')\cap\overline{\mathrm{tor}}(C;a)$, we have that $z^{p^k}(M'+Ca)=z^{p^k}M'+z^{p^k}Ca=0$, proving that $M'+Ca$ has p-torsion index $\le k$, contrary to the maximality of M'. □

Proposition 8.6 ([18, Proposition 4.3]). *Suppose $\hat{A}=\hat{C}\{a_1,\dots,a_t\}$ is a relatively free, affine algebra over a commutative Noetherian F-algebra $\hat{C}$. Then $\hat{A}$ is a finite subdirect product of an algebra $\hat{A}'$ defined over the* $C/\overline{\mathrm{tor}}(C;a_i)$, *$1\le i\le t$, together with the $\left\{\hat{A}/z^j\hat{A}: z\in\hat{C},\ j<k\right\}$ where k is the maximum of the p-torsion indices of $a_1,\dots,a_t$.*

9. Solution of Specht's problem for proper T-ideals over arbitrary commutative Noetherian rings.

Using the same ideas, we now prove Specht's problem for affine PI-algebras over a commutative Noetherian ring C, in the case that each T-ideal is *PI-proper*, in the sense that the ideal of the base ring generated by the coefficients of the PIs is all of C.

Lemma 9.1. *If A is a relatively free algebra then every homomorphic image of A with respect to a T-ideal is relatively free. Furthermore, for every $z\in C$,* $\mathrm{Ann_A z}$ *is a T-ideal, with $zA\cong A/$*$\mathrm{Ann_A z}$ *as C-modules (but not necessarily as C-algebras).*

Proof. $\mathrm{Ann_A z}$ is clearly a T-ideal, and the rest of the assertion is standard. □

(Note that associativity of A was not used in the proof of Lemma 9.1, and is not used in Lemma 9.3.)

Definition 9.2. *We say that a Noetherian ring C is **Specht** if Specht's conjecture holds for PI-proper T-ideals defined over C, i.e., any PI-proper T-ideal generated by polynomials $f_1, f_2, \dots$ is finitely based; we say C is **almost Specht** if C/I is Specht for every nonzero ideal I of C.*

By Noetherian induction, we may assume that C is almost Specht.

Lemma 9.3 (A general reduction of Specht's problem to the case where C is an integral domain)**.** *If C is almost Specht but not Specht, then C is an integral domain.*

Proof. Suppose $z_1z_2 = 0$ for $0 \neq z_1, z_2 \in C$. The system of images $\{f_j + z_2C\{x\}\}$ in $C\{x\}/z_2C\{x\}$ is defined over C/z_1C, and thus by hypothesis is finitely based, say by the cosets of $f_1, \dots, f_t$, so taking $\mathcal{I}$ to be the T-ideal generated by $f_1, \dots, f_t$,, we have $f_j = g_j + z_2h_j$ where $g_j \in \mathcal{I}$, for each $j \geq t$. The polynomials f_j can be replaced by z_2h_j for all $j > t$. But the T-ideal generated by $\{z_2h_j : j > t\}$ in A/z_1A is also finitely based, by hypothesis. □

In view of this lemma, coupled with the fact that any Zariski closed algebra can be defined over a field, one might expect the proof of our next theorem (Theorem 9.9) to be an easy generalization of Theorem 7.3. This is not the case, since modding out a T-ideal from a relatively free faithful C-algebra might yield a non-faithful C-algebra, so we may encounter torsion, and furthermore there are difficulties in finding a representable algebra in which to get started. Lewin's theorem is no longer applicable directly over an arbitrary commutative base ring, so we need a more intricate argument to supplement Lemma 2.2. We can handle torsion in Noetherian modules, so the main effort in the proof is to find the correct Noetherian environment in which to eliminate the torsion.

The following fact is well known.

Lemma 9.4 (Baby Artin-Rees Lemma)**.** *Let M be a C-module, with $z \in C$, and take any $k \in \mathbf{N}$. Suppose $\mathrm{Ann}_M(z^{k+1}) \subseteq \mathrm{Ann}_M(z^k)$. Then $z^kM \cap \mathrm{Ann}_M(z) = 0$.*

Proof. If $z^ka \in \mathrm{Ann}_M(z)$, then $z^{k+1}a = 0$, implying $z^ka = 0$ by assumption. □

We start with a key special case.

Proposition 9.5. *Let C be an almost Specht, commutative Noetherian ring, and A an affine PI-algebra containing a representable T-ideal $\mathcal{I}$. Then any chain of T-ideals in the free algebra of $C\{x\}$ ascending from $\mathrm{id}(A)$ stabilizes.*

Proof. By Lemma 9.3, C is an integral domain. We need to show that any ascending chain of PI-proper T-ideals

$$\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_3 \subseteq \cdots \tag{20}$$

of A, stabilizes. Since $\mathcal{I} \subseteq \mathcal{I}_1$, we may replace A by $A/\mathcal{I}$, and assume that A is representable. We view $A \subseteq M_n(K)$, where K is an algebraically closed field containing C. If C is finite, then it is a field, and we are done by Theorem 7.3. So we may assume that C is an infinite integral domain. Denoting AK as A_K, we work with respect to a quiver Γ of A_K as a K-algebra.

As in Theorem 7.3, let $\mathcal{I}_j^{(1)}$ be the maximal subideal of $\mathcal{I}_j$ closed under multiplication by $\hat{C}$ of Definition 6.19. Thus

$$\mathcal{I}_1^{(1)} \subseteq \mathcal{I}_2^{(1)} \subseteq \mathcal{I}_3^{(1)} \subseteq \cdots \tag{21}$$

are ideals in the Noetherian algebra $\hat{A} = \hat{C}A$, so this chain stabilizes, and we may assume $\mathcal{I}_j^{(1)} = \mathcal{I}_{j_0}^{(1)}$ for $j > j_0$.

For a T-ideal $\mathcal{I}$ of A, let $\overline{\mathcal{I}} = K\mathcal{I}$, taken in A_K. Define $\widetilde{\mathcal{I}} = K\mathcal{I} \cap A \supseteq \mathcal{I}$. Let $A' = A/\mathcal{I}_{j_0}^{(1)}$. Passing down to A', we shall pass further to $A/\widetilde{\mathcal{I}_{j_0}^{(1)}}$.

The quotient $\widetilde{\mathcal{I}_{j_0}^{(1)}}/\mathcal{I}_{j_0}^{(1)}$ is torsion, so there is $0 \neq z \in \hat{C}$ such that $z\widetilde{\mathcal{I}_j^{(1)}} = z\widetilde{\mathcal{I}_{j_0}^{(1)}} \subseteq \mathcal{I}_{j_0}^{(1)}$. The chain $\operatorname{Ann}_{\hat{A}} z \subseteq \operatorname{Ann}_{\hat{A}} z^2 \subseteq \cdots \subseteq \operatorname{Ann}_{\hat{A}} z^k \subseteq \cdots$ stabilizes at some k, by the Noetherianity of $\hat{A}$. Now, applying the baby Artin-Rees lemma to $\hat{A}/\mathcal{I}_{j_0}^{(1)}$, we see that

$$z^k \hat{A} \cap \widetilde{\mathcal{I}_j^{(1)}} \subseteq \mathcal{I}_{j_0}^{(1)}.$$

In particular the natural map

$$A' \Longrightarrow (A'/z^k A') \oplus (A/\widetilde{\mathcal{I}_{j_0}^{(1)}})$$

is an injection. The image of the chain (20) of the first summand on the right stabilizes by applying Noetherian induction. Thus, we pass to the second summand of the right, which has no C-torsion. Letting $\mathcal{J}$ be the ideal constructed in Lemma 7/2, we have for every $j > j_0$ that $\mathcal{I}_j \cap \mathcal{J} = 0$ in $A_K/A_K\widetilde{\mathcal{I}_{j_0}^{(1)}}$ as in the last paragraph of the proof of Theorem 7.3. Hence, a fortiori, $\mathcal{I}_j \cap \mathcal{J} = 0$ in $A/\widetilde{\mathcal{I}_{j_0}}$. We are done by induction on the degree vector. $\square$

In order to conclude our proof in the general case, we need a way of reducing to the hypothesis of this proposition, by means of a careful analysis of torsion. For reference in future work, we formulate the next step in greater generality, since it does not require associativity and enables one to reduce Specht's problem (over any commutative associative Noetherian base ring) to the case of algebras over a field, even for nonassociative varieties. If the reader feels more comfortable, one can continue in the class of associative PI-algebras.

Lemma 9.6. *Suppose A is a relatively free (not necessarily associative) C-algebra, viewed as a C-module, and $z \in C$ such that C/zC is Specht. Then the torsion of A made by z is bounded.*

Proof. Take $\mathcal{I}_0 = 0$ and for each j define inductively $\mathcal{I}_j = \{a \in A : za \in \mathcal{I}_{j-1}\}$. This defines an ascending chain of T-ideals of A over C/zC, which thus must stabilize, and this stage bounds the torsion of A made by z. $\square$

Proposition 9.7. *Suppose C is almost Specht, and $z \in C$ is such that C/zC is a field. Suppose A is a relatively free (not necessarily associative) C-algebra, viewed as a C-module. Consider the commutative diagram*

$$
\begin{array}{ccccccccc}
A/\mathcal{I}_1 & \longrightarrow & zA/z\mathcal{I}_1 & \longrightarrow & z^2A/z^2\mathcal{I}_1 & \longrightarrow & z^3A/z^3\mathcal{I}_1 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A/\mathcal{I}_2 & \longrightarrow & zA/z\mathcal{I}_2 & \longrightarrow & z^2A/z^2\mathcal{I}_2 & \longrightarrow & z^3A/z^3\mathcal{I}_2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A/\mathcal{I}_3 & \longrightarrow & zA/z\mathcal{I}_3 & \longrightarrow & z^2A/z^2\mathcal{I}_3 & \longrightarrow & z^3A/z^3\mathcal{I}_3 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
\tag{22}
$$

whose rows are the maps $z^iA/z^i\mathcal{I}_m \to z^{i+1}A/z^{i+1}\mathcal{I}_m$ (given by multiplication by z) and whose columns are given by the natural projections $z^iA/z^i\mathcal{I}_m \to z^iA/z^i\mathcal{I}_{m+1}$.

Then there is some ℓ such that all entries in the diagram past the $\ell \times \ell$ square are isomorphic.

Proof. These maps are all defined over the field C/zC, so Lewin's theorem yields us a representable T-ideal contained in $\mathcal{I}_1$ and we can apply Proposition 9.5 to each column to see that it stabilizes. On the other hand, the chain of maps $A \to zA \to z^2A \to \dots$ stabilizes at some z^k, by Lemma 9.6, so each row stabilizes at the k position. Taking ℓ to be the maximal length of these k columns until they stabilize yields the desired result. □

Theorem 9.8. *Suppose $\mathcal{C}$ is a variety of (not necessarily associate) algebras defined over commutative, associative base rings. If every field is Specht with respect to T-ideals in $\mathcal{C}$, then every Noetherian base ring C is Specht.*

Proof. By Noetherian induction, we may assume that the base ring C is almost Specht.

CASE I. C is a principal ideal domain. If A has torsion made by some element z of C, then writing z as a product of prime elements, we may assume that z is prime. By Proposition 9.7, any chain of T-ideals of A can be continued to a chain defined over C/pC, which by Noetherian induction must stabilize.

Hence A is torsion-free over C, and we pass to $A \otimes_C K$ where K is the field of fractions of C, where we are done by hypothesis.

Having proved CASE I, we do the general case. We can replace C by the center of A, which is affine by the version of the Artin-Tate lemma given in [83, Proposition 6.25]. (Note that the proof there does not require associativity.) Thus, it is enough to prove the theorem for C affine.

Next, by Noetherian induction, we may assume that C is almost Specht. Write $C = C_0[z_1, \dots, z_u]$, where C_0 is the subring of C generated by 1, and let $C_j = C_0[z_1, \dots, z_j]$ for $0 < j \leq u$.

We proceed by induction on j. Indeed, we are done by Lemma 9.6 for $j = 0$. Furthermore, that result enables us to break up our chain into ℓ chains of T-ideals defined over the field C_0/zC_0, (where the prime element z is taken as in CASE I), so we may assume that C_0 is a field.

For arbitrary j, we assume that A is torsion-free over C_{j-1}. As before, adjoining a commutative indeterminate to C and A, we may assume that C_{j-1} is an infinite domain. Replacing A by $A \otimes_{C_{j-1}} K_{j-1}$, where K_{j-1} is the field of fractions of C_{j-1}, we may replace C_0 by K_{j-1}, and assume that $C_j = F[z_j]$, which is a principal ideal ring. Now we conclude by CASE I. □

We are finally ready to prove Specht's conjecture for associative PI-algebras over arbitrary Noetherian rings.

Theorem 9.9. *Every commutative Noetherian ring C is Specht (for the class of associative algebras).*

Proof. The hypothesis of 9.8 is satisfied, in view of Theorem 7.3. □

Here is an alternate argument for the proof of CASE I of Theorem 9.8, based instead on finiteness principles. We want to reduce to the case that A is representable, and apply Proposition 9.5. A well-known theorem of Amitsur [82, Theorem 1.6.46] says that every affine PI-algebra over an arbitrary commutative ring satisfies an identity all of whose nonzero coefficients are ± 1, in fact a power s_m^k of the standard polynomial. Thus, $\mathcal{I}_1$ contains some polynomial s_m^k, and we can replace $\mathcal{I}_1$ by the T-ideal $\mathcal{I}_0$ generated by s_m^k and still have the chain (20). Replacing A by $C\{X\}/\mathcal{I}_0$, we may assume that A is the relatively free affine algebra with respect to the identity s_m^k. Furthermore, adjoining a commuting indeterminate to A and to C, we may shrink $\mathcal{I}_1$ still further and assume that C is an infinite integral domain.

Let $\overline{K}$ be the algebraic closure of the field of fractions K of C, and let N be the radical of $A_{\overline{K}} := A \otimes_C \overline{K}$. Then the matrix units of the semisimple algebra $A_{\overline{K}}/N$ can be written in terms of a finite number of elements of A, and the characteristic coefficients in the largest dimension (say n_i^2) matrix component of $A_{\overline{K}}/N$ can be described in terms of evaluations of the Capelli polynomial $c_{n_i^2}$, by Remark 6.9. Furthermore, in view of Shirshov's theorem, when passing to $\hat{C}$, since we only used a finite number of elements, we see that $m\mathcal{I} \subseteq \mathcal{I} \otimes_C \hat{C}$, for some natural number m. Proposition 9.5 yields the ACC for T-ideals of the relatively free algebra $A \otimes_C \hat{C}$, so we may pass to A/mA. Then A/mA is naturally a C/mC-algebra, so we are done unless $mC = 0$. By Lemma 9.3, m can be taken to be some prime number p. Thus C is an F-algebra, where $F = \mathbb{Z}/p\mathbb{Z}$.

We conclude the proof of Theorem 9.9 as before, reducing to the case that C is affine using the Artin-Tate lemma, and proceeding by induction on the number of generators of C.

10. Solution of Specht's problem in general, where the T-ideals are not necessarily PI-proper. Using the same ideas, we can extend Theorem 9.9 still further, considering the general case where the T-ideals are not PI-proper; in other words, the ideals of C generated by the coefficients of the polynomials in the T-ideals of $C\{X\}$ do not contain the element 1. Towards this end, given a set S of polynomials in $C\{X\}$, define its **coefficient ideal** to be the ideal of C generated by the coefficients of the polynomials in S. We need a few observations about the multilinearization procedure.

Lemma 10.1. *If a T-ideal $\mathcal{I}$ contains a $f \notin id(A)$ with coefficient c, then $\mathcal{I}$ also contains an A-quasi-linear polynomial with coefficient c.*

Proof. First we note that one of the blended components of f has coefficient c, and then we quasi-linearize it. □

10.1. L'vov's question. It is easy to see [18, Corollary 6.3] that a T-ideal is PI-proper iff its coefficient ideal contains 1, but the following question of L'vov remains open:

Question 10.2. *If the ideal generated by the coefficients of an identity f of degree m contains 1, does the T-ideal generated by f contain some PI of the same degree m having some coefficient equal to 1?*

L'vov showed that the T-ideal generated by f contains such an identity of degree m^2.

Definition 10.3. *An algebra A is* ***irreducible*** *if the intersection of two nonzero ideals is nonzero.*

The following observation reduces the question to the case when C is a finite, irreducible, local ring whose order is a prime power.

Lemma 10.4 *Let u be a multilinear word of degree m, N the C-submodule of the polynomial algebra generated by words of length m that are lexicographically smaller than the word u, and $\overline{M} = M/N$.*

(1) *Suppose $\sum z_i = 1$.*
If $(C/Cz_i) \otimes \overline{M} = 0$ for every z_i, then $\overline{M} = 0$.
(2) *If $(\mathbb{Q} \otimes M)/(\mathbb{Q} \otimes N) = 0$ and $(\mathbb{Z}_{p^k} \otimes M)/(\mathbb{Z}_{p^k} \otimes N) = 0$ for each prime number p and $k \in N$, Then $\overline{M} = 0$.*

It seems reasonable to consider the case for which $C = \mathbb{Z}_{p^k}$, in particular for $k = 2$. The identity f connects its homogeneous components, for example, for $C = \mathbb{Z}_{p^2}$ and $f = f_1 + pf_2$, where f_1 is multilinear and f_2 quasilinear over $\mathbb{Z}_p$ of order p, and $\deg(f_2) = p\deg(f_1)$. Such polynomials establish relations between multilinear and quasilinear representations of the symmetric group. In this way, the question of L'vov raises the interesting connection between different quasilinear representations, and can be generalized to the case when the ideal I generated by the coefficients of f is a proper ideal of C.

L'vov's question can also be generalized for improper case when the ideal I generated by the coefficients of f does not contain 1. Does f imply some multilinear identity g, $\deg(g) = \deg(f)$, with coefficients from I? This question becomes easier when we only require $\deg g \leq \deg(f)^2$.

Proposition 10.5. *Suppose $\mathcal{I}$ is a T-ideal with coefficient ideal I. Then there is a polynomial $f \in \mathbb{Z}\{x\}$ for which $cf \in \mathcal{I}$ for all $c \in I$.*

Proof. Since C is Noetherian, we can write $I = \sum_{i=1}^{t} Cz_i$, and then it is enough to prove the assertion for $c = z_i$, $1 \leq i \leq t$.

Let M_m denote the space of multilinear words of degree m in $\{y_1, \dots, y_m\}$ in the countably generated algebra with generators $\{y_1, y_2, \dots, \}$. In view of Shirshov's Height Theorem [14, Theorem 2.3], the space $\sum_i z_i M_m$ has bounded rank as a $\mathbb{Z}$-module. On the other hand, there is a well-known action of the symmetric group S_m on the indices of $y_1, \dots, y_m$ described in [14, Chapter 5]. In particular, [14, Theorem 5.51] gives us a rectangle such that any multilinear polynomial f whose Young diagram contains this rectangle satisfies $z_i f \in \mathcal{I}$. □

Corollary 10.6. *If $\mathcal{I}$ is a T-ideal with coefficient ideal I, there is a PI-proper T-ideal of $C\{x\}$ whose intersection with $I\{x\}$ is contained in $\mathcal{I}$.*

The proof of this corollary relies on some deep work of Zubkov [99], as explained in [18, Corollary 6.5].

Remark 10.7. Define $\mathcal{M}(n, C) := \mathrm{M}_n(\mathbb{Z}[\xi_1, \xi_2, \dots]) \otimes_{\mathbb{Z}} C$. The algebra $\mathcal{M}(n, C)$ is rather tricky. The natural map $\mathcal{M}(n, C) \to \mathrm{M}_n(C[\xi_1, \xi_2, \dots])$ is onto but need not be 1:1, as noted in the discussion following Question 11.11 below. Nevertheless, we do see from [99] that $\mathcal{M}(n, C)$ is a free C-module, whose base is $\{we_{ij} \otimes 1\}$ where w are the monomials in $\xi_1, \xi_2, \dots$, and all identities of $\mathcal{M}(n, C)$ are of the form cf where $f \in \mathrm{id}(\mathrm{M}_n(\mathbb{Z}[\xi_1, \xi_2, \dots])) = \mathrm{id}(\mathrm{M}_n(\mathbb{Z})) = \mathrm{id}(\mathrm{M}_n(\mathbf{Q}))$ and $c \in C$.

Remark 10.8. In view of Remark 10.7, the T-ideal of f of Proposition 10.5 contains all identities of $\mathcal{M}_n(C)$, for n large enough, by the argument of [18, Proposition 4.4]. Alternatively, when reducing to characteristic p, we can apply a theorem of Kemer [49] to the proof of Proposition 10.5. Kemer has obtained a sharp bound in [51, Theorem 3].

We are ready for the final version of our main theorem.

Theorem 10.9. *Any T-ideal in the free algebra $C\{x\}$ is finitely based, for any commutative Noetherian ring C.*

Proof. By Noetherian induction, we may assume that the theorem holds over C/I for every nonzero ideal I of C. Thus, by Lemma 9.3, C is an integral domain. If C is finite, then it is a field, and we are done by Theorem 7.3. So we may assume that C is an infinite integral domain. We need to show that any T-ideal generated by a given set of polynomials $\{g_1, g_2, \dots\}$ is finitely based. The coefficient ideals of $\{g_1, g_2, \dots g_j\}$ stabilize to some ideal I of C at some j_0, since C is Noetherian. We let A_0 denote the relatively free algebra with respect to the T-ideal generated by $g_1, \dots, g_{j_0}$. Inductively, we let A_i denote the relatively free algebra with respect to the T-ideal generated by $f_{j_0+1}, \dots, f_{j_0+i}$, and take a PI-proper polynomial f_{i+1}, not in $\mathrm{id}(A_i)$ such that cf_{i+1} is in the T-ideal generated by g_{i+1} in A_i for all c in the coefficient ideal of g_{i+1}. (Such a polynomial exists in view of Proposition 10.5.) This gives us an ascending chain of PI-proper T-ideals of A_0, which must terminate in view of Theorem 9.9, a contradiction. □

12. The T-ideal structure of relatively free algebras over commutative Noetherian rings. Theorem 10.9 enables us to generalize previously known results about relatively free PI-algebras to a variety which is not necessarily PI-proper. For example, as noted in the introduction, the ACC on T-ideals formally gives us the following Noetherian-type results.

Definition 11.1. *An algebra A is* ***T-prime*** *if the product of two nonzero T-ideals is nonzero. A T-ideal $\mathcal{I}$ of A is* ***T-prime*** *if $A/\mathcal{I}$ is a T-prime algebra.*

Proposition 11.2. *Any relatively free algebra A over a commutative Noetherian ring has a unique maximal nilpotent T-ideal $N(A)$, which we call the* ***T-radical****. The T-radical is the intersection of a finite number of T-prime T-ideals.*

Proof. Just copy the standard arguments, using Noetherian induction. □

Proposition 11.3 ([18, Proposition 6.5]). *Each T-prime, relatively free algebra A with 1 over a commutative Noetherian ring C is either the free algebra over C/J for some prime ideal $J \lhd C$, or is PI-equivalent to a relatively free algebra over a field. In particular, either A is free or PI.*

Proof. The center Z of A is an integral domain over which A is torsion free; indeed, if $c \in C$ has torsion, then $0 = (cA)\operatorname{Ann}_A(c)$ implies $cA = 0$ so $c = 0$. If Z is finite then it is a field. If Z is infinite then A is PI-equivalent to $A \otimes_Z K$ where K is the field of fractions of Z. □

The celebrated theorem of Razmyslov-Kemer-Braun [22] becomes:

Theorem 11.4 ([18, Theorem 6.13]). *The Jacobson radical of any relatively free affine algebra (not necessarily PI-proper) is nilpotent.*

Of course there is no hope to generalize this result to non-relatively free algebras, since the nilradical of an affine algebra need not be nilpotent. Here is a cute consequence. Define the **algebraic radical** of an algebra to be the largest ideal of algebraic elements. (This exists by Zorn's lemma, since the sum of algebraic ideals is algebraic.)

Corollary 11.5. *The algebraic radical of any relatively free algebra A satisfies the identity $x^m - x^n = 0$ for suitable $m > n$.*

P r o o f. Adjoining a unit element if necessary, we may assume that A has a unit element. The upper nilradical N is a nilpotent T-ideal, so one can pass to A/N and assume that A has T-radical 0. In view of Proposition 11.2, we may assume that A is T-prime. If not free, A is PI. If the center K of A is infinite, then a generic element cannot be algebraic, so we assume that K is a finite field. This case is Exercise 1.4 of [14]. Namely, each element has a bounded power that is semisimple, so we conclude with Fermat's little theorem. □

11.1. Consequences of torsion for relatively free algebras. Torsion has been so useful in this theory that we apply a few more elementary properties from [18] to relatively free algebras. The underlying idea is to use Remark 8.3 to pass from T-ideals to more familiar ideals of the commutative Noetherian base ring C. For $k \in \mathbf{N}$ recall that $\mathrm{Ann}_A(k) = \{a \in A : ka = 0\}$.

Lemma 11.6. *Suppose A is a relatively free affine C-algebra.*

(1) *A has p-torsion for only finitely many prime numbers p.*

(2) *There is some k_0 such that $\mathrm{Ann}_A(p^k) = \mathrm{Ann}_A(p^{k+1})$ for all $k > k_0$ and all prime numbers p.*

(3) *Let $\phi_k : A \to A \otimes \mathbb{Z}/p^k\mathbb{Z}$ denote the natural homomorphism. If $p^kA \neq p^{k+1}A$, then $\ker \phi_k \neq \ker \phi_{k+1}$.*

P r o o f. $\mathrm{Ann}_A(p^k)$ is a T-ideal for each p^k. Let $\mathcal{I}_k$ be the T-ideal generated by p^k-torsion elements. The $\mathcal{I}_k$ stabilize for some k_0, yielding (2), and (3) follows since once the chain stabilizes we have $p^kA = p^{k+1}A$. Likewise, the direct sum of these T-ideals taken over all primes stabilizes, yielding (1). □

Lemma 11.7. *Suppose A is a relatively free affine C-algebra. For any prime ideal P of C, there is a single element $z_0 \in C \setminus P$ annihilating*

$$\mathcal{I} = \{a \in A : ca = 0 \text{ for some } c \in C \setminus P\}.$$

In particular, if C is an integral domain, then there is a single element $z_0 \in C$ annihilating all torsion elements of A.

P r o o f. We call an annihilator ideal of the form $\mathrm{Ann}_A\, c$ **principal**. The sum of two principal annihilator ideals is contained in a principal annihilator ideal, since

$$\mathrm{Ann}(z_1z_2) \supseteq \mathrm{Ann}(z_1) + \mathrm{Ann}(z_2).$$

Thus any maximal principal annihilator ideal annihilates all torsion elements of A. □

Definition 11.8. *A **Vandermonde matrix** of d elements $z_1, \dots, z_d \in C$ is the matrix*

$$(z_i^{j-1}) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_d \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{d-1} & z_2^{d-1} & \dots & z_d^{d-1} \end{pmatrix}.$$

*Its determinant is called the **Vandermonde determinant**.*

Proposition 11.9.

(1) *Each homogeneous component of an identity f in the variable x_1 vanishes under multiplication by the (Vandermonde) determinant of any Vandermonde matrix formed by any $d+1$ elements of C, where $d = \deg f$ in x_1.*
(2) *Suppose that the ring $\overline{C} = C/P$ is infinite. Then for any homogeneous component h of an identity of A, there is $c \in C \setminus P$ such that $ch \in \mathrm{id}(A)$.*
(3) *Suppose that for each d the ring C contains the inverse of a $d \times d$ Vandermonde determinant. Then the homogeneous components of any identity of A are also identities of A.*

Proof.

(1) This is a standard Vandermonde argument. If f_j are the homogeneous components of degree j in x_1 then $f(z_i a_1, a_2, \dots, a_m) = \sum_{j=0}^{d} z_j f_j(a_1, a_2, \dots, a_m)$. Hence the vector $f_1(a_1, a_2, \dots, a_m), \dots, f_d(a_1, a_2, \dots, a_m)$ is annihilated by the Vandermonde matrix, and thus by its determinant (which is the matrix times its adjoint).
(2) Take the Vandermonde determinant of elements in $C \setminus P$.
(3) Take this Vandermonde determinant in the argument of (1). □

Thus, we see that every homogeneous component of an identity has torsion with respect to the complement of a given prime ideal P.

Question 11.10. *Is every PI-algebra (not necessarily affine) a homomorphic image of a torsion-free PI-algebra?*

Torsion also relates to the following question of Procesi.

Question 11.11. *Is the kernel of the canonical homomorphism $\mathrm{M}_n(\mathbb{Z}) \to \mathrm{M}_n(\mathbb{Z}/p\mathbb{Z})$ equal to $p\mathrm{M}_n(\mathbb{Z})$?*

Procesi's question relates to the important matter of the classification of T-prime ideals. (Their classification in characteristic 0 is due to Kemer, in terms of the Grassman envelope of simple superalgebras.) There are many other examples in positive characteristic, obtained by Razmyslov [81] and Kemer, and the main hope would be to classify those T-prime ideals generated by multilinear polynomials.

Schelter [87] found a counterexample to Procesi's question for $n = p = 2$ (whose sharpness is demonstrated in [4]). Kemer [52] showed that for each prime $p > 2$ there exists $n \leq p$ such that the conjecture of Procesi fails for the pair p, n. More precisely, if ℓ is the matrix type of a "non-regular" prime variety, then Procesi's question has a counterexample for some $n < \ell$.

On the other hand, Kemer and Averyanov [53] obtained a positive answer for 2-generated algebras, for $n = 3$ and $p > 3$. For any n there is a positive answer for a fixed number of generators and sufficiently large primes p, in view of Lemma 11.6. A further discussion can be found in Kemer and Averyanov [54]. Kemer [52] had proved that any counterexample to Procesi's question would yield a non-regular T-prime variety. But Kemer's student Samoĭlov [86] showed that there are no non-regular T-prime varieties of matrix type n if $\mathrm{char}(p) > n$. Consequently, Procesi's question has a positive solution whenever $\mathrm{char}(p) > n$.

Zubkov [99] verifed Procesi's question affirmatively for algebras with characteristic coefficients adjoined.

12. Polynomial identities arising from gluing in full quivers.

In this section, we give various examples of identities arising from gluing (and also indicate the difficulty in ascertaining $\mathrm{id}(A)$ in general). As already noted, these do not interfere with the proof of Specht's conjecture, so in this sense this section is a digression, but the examples given here do throw considerable light on the identities of Zariski closed algebras and the effect of gluing.

Example 12.1. Suppose A is the algebra of the full quiver which is the path

$$I_{n_1} \longrightarrow II_{n_2} \longrightarrow III_{n_3} \longrightarrow \cdots$$

of length $\ell - 1$. In this case, $\mathrm{id}(A) = \mathrm{id}(\mathrm{M}_{n_1}(F))\,\mathrm{id}(\mathrm{M}_{n_2}(F)) \cdots$ by a theorem of Lewin [65].

(i) Let $n = n_1 + n_2 + \cdots + n_\ell$. It follows from the previous paragraph that the product of standard polynomials $s_{2n_1} s_{2n_2} \cdots s_{2n_\ell}$ is the minimal identity of A, where we substitute distinct indeterminates in each standard polynomial.

For example, the full quiver

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet,$$

a single path of length $n-1$ where each $n_i = 1$, corresponds to the algebra of upper triangular $n \times n$ matrices under the natural representation, and satisfies the identity

$$[x_1, x_2] \cdots [x_{2n-1}, x_{2n}].$$

(ii) One could use any identity of $n_i \times n_i$ matrices in (i), instead of the standard polynomial. In particular, using the Capelli polynomial c_n, we have the identity

$$c_{n_1^2+1} c_{n_2^2+1} \cdots c_{n_\ell^2+1},$$

where again we use distinct indeterminates for each occurrence of the Capelli polynomial. However, $c_{n_1^2} c_{n_2^2+1} \cdots c_{n_\ell^2+1} \notin \mathrm{id}(A)$, and is a critical non-identity for F infinite.

Example 12.2. Notation as in (15), for the first two consecutive glued vertices of a glued component $\mathcal{C}$, we start with

$$f_i = y_i \tag{23}$$

(since the diagonal substitutions in $\mathcal{C}$ are scalar, and thus central). If the portion of $\mathcal{B}$ corresponding to $\mathcal{C}$contains another glued vertex, we hike by replacing y_i by the Lie commutator $[y_i, z_{i,k}]$, for a new indeterminate $z_{i,k}$, in order to force two radical substitutions of the y_i as long as we encounter glued vertices in $\mathcal{C}$. Thus, if $\mathcal{C}$ has t_i glued vertices, we replace y_i by the higher Lie commutator

$$[[\dots[[y_i, z_{i,t_i}], z_{i,t_i-1}], \dots], z_{i,1}].$$

Example 12.3. When nonconsecutive vertices are glued, we write $\{a, b, c\}$ for $abc - cba$.

(i) $I \longrightarrow II \longrightarrow I$, with identical gluing. The corresponding algebra is

$$\left\{ \begin{pmatrix} \alpha & * & * \\ 0 & \beta & * \\ 0 & 0 & \alpha \end{pmatrix} : \alpha, \beta \in F \right\} \subset \mathrm{M}_3(F).$$

We consider the nonidentity $x_1[y_{1,1}, y_{1,2}]x_2[y_{2,1}, y_{2,2}]x_3$. In order not to produce 0 at the outset, x_1 and x_3 must both be specialized to the glued block corresponding to I, so

$$\{x_1,\ [y_{1,1}, y_{1,2}]x_2[y_{2,1}, y_{2,2}],\ x_3\} \\ = x_1[y_{1,1}, y_{1,2}]x_2[y_{2,1}, y_{2,2}]x_3 - x_3[y_{1,1}, y_{1,2}]x_2[y_{2,1}, y_{2,2}]x_1$$

is also an identity, which is critical over an infinite field. (Note that x_4 was superfluous in this example, since there was no gluing involved in the second component.) We can lower the degree of our identities as follows: For the full quiver $I \longrightarrow II \longrightarrow III$ without gluing, we would have the trace-absorbing non-identities

$$x_1[y_{1,1}, y_{1,2}][y_{2,1}, y_{2,2}], \quad [y_{1,1}, y_{1,2}][y_{2,1}, y_{2,2}]x_1$$

(and x_1 is superfluous, since it could be specialized to 1.)

These nonidentities are not so useful in detecting the presence of gluing, since we do not know which substitution (of $y_{2,1}$ or $y_{2,2}$) is semisimple, to the diagonal block corresponding to I, and which substitution is radical. However, their *difference*

$$[x_1, [y_{1,1}, y_{1,2}][y_{2,1}, y_{2,2}]]$$

is an identity.

(ii) $I \longrightarrow II \longrightarrow III \longrightarrow II \longrightarrow I$, where each gluing is identical.

Without gluing we would have the critical non-identity

$$x_1[y_{1,1}, y_{1,2}]x_2[y_{2,1}, y_{2,2}]x_3[y_{3,1}, y_{3,2}]x_4[y_{4,1}, y_{4,2}]x_5.$$

With identical gluing, we have various identities, including

$$\{x_1,\ [y_{1,1}, y_{1,2}]x_2[y_{2,1}, y_{2,2}]x_3[y_{3,1}, y_{3,2}]x_4[y_{4,1}, y_{4,2}],\ x_5\}$$

and

$$x_1[y_{1,1}, y_{1,2}]\{x_2, [y_{2,1}, y_{2,2}], x_3\}[y_{3,1}, y_{3,2}]x_4[y_{4,1}, y_{4,2}]x_5.$$

12.1. Identities arising from infinitesimal gluing.

Example 12.4. Although the algebras corresponding to the paths

$$II \longrightarrow I \longrightarrow I, \qquad II \longrightarrow I$$

are PI-equivalent (since the first ends in an isolated glued triangle), the algebras corresponding to the paths

$$I \longrightarrow II \longrightarrow I \longrightarrow I, \qquad I \longrightarrow II \longrightarrow I$$

are not PI-equivalent. Indeed, $[[x_1, [y_{1,1}, y_{1,2}][y_{2,1}, y_{2,2}]], x_2]$ is an identity for the second, but is a nonidentity for the first since x_2 could be specialized to an infinitesimal.)

We also should consider the issue of empty vertices.

Example 12.5. Let us consider the algebra without 1 corresponding to the path

$$I \longrightarrow \circ \longrightarrow I \longrightarrow I.$$

Explicitly, first we consider the polynomial $[y_{1,1}, y_{1,2}]x_1[y_{2,1}, y_{2,2}]x_2$. Any nonzero substitutions of $[y_{1,1}, y_{1,2}]$ and $[y_{2,1}, y_{2,2}]$ must be to the radical, above the diagonal, which means that x_1 must go to a diagonal element. In order for the polynomial $z[y_{1,1}, y_{1,2}]x_1[y_{2,1}, y_{2,2}]x_2$ not to vanish, the positioning of the empty block forces a radical substitution of z. Consequently, the polynomial

$$[x_3, z[y_{1,1}, y_{1,2}]x_1[y_{2,1}, y_{2,2}]x_2]$$

vanishes on this algebra because the radical components have all been exhausted in the substitutions.

There is a strange effect when some, but not all, of the edges are glued.

Example 12.6. The identities of the path

$$I \xrightarrow{\alpha} I \xrightarrow{\alpha} I \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} I \,; \tag{24}$$

are a consequence of the Lie nilpotence identity of degree $k+1$ and the metabelian identity $[[x, y], [z, t]]$. In this case we can represent the algebra by 2×2 matrices over the ring $F[t]/\langle t^{k+1}\rangle$.

Example 12.7. (i) $I^{(1)} \longrightarrow II \longrightarrow I$, with $q = 2$; the Frobenius gluing is by squaring. Now we have the identity $[x_1, [y_{1,1}, y_{1,2}]x_2[y_{2,1}, y_{2,2}], x_3]_2$.

Example 12.8. (i) Suppose A is the algebra of a path of length ℓ with nonidentical Frobenius gluing, with each $n_i = 1$. Then the algebra $A \otimes \overline{F}$ is isomorphic to the algebra T of upper triangular $\ell \times \ell$ matrices over $\overline{F}$, which corresponds to the path $I \to II \to III \to \cdots$.

(ii) On the other hand, the conclusion of (i) fails if A has identical gluing or Frobenius gluing up to infinitesimals, and in fact $\mathrm{id}(A \otimes \overline{F}) \supset \mathrm{id}(A)$. In this case, we obtain two different varieties which satisfy the same multilinear identities and thus have the same codimension growth.

12.2. The effect of gluing between branches on identities. We turn briefly to the trickiest aspect of this section, the interesting question as to how gluing between branches can produce identities for the full quiver Γ. This is a very difficult problem, and we cannot give a full description of $\mathrm{id}(\Gamma)$; rather we show how the full quivers naturally yield identities via gluing between branches.

Gluing between branches is called **degenerate** if each edge of one branch is glued to the corresponding edge of the other branch, in the same order.

Gluing between branches is **permuted** if the edges (including labelling) are permuted from the first branch to the second branch.

A **permuted branch** of a given branch $\mathcal{B}$ is a branch with permuted gluing to $\mathcal{B}$.

When we take the (weighted) difference in the products, then the two branches may cancel, as illustrated in the following full quiver:

Example 12.9.

(25)
$$\begin{array}{ccc} I & \xrightarrow{\ \alpha\ } & II \\ \downarrow{\scriptstyle -\alpha} & & \downarrow{\scriptstyle \beta} \\ II & \xrightarrow{\ \beta\ } & III \end{array}$$

Since all evaluations along the whole branches cancel out, we are left only with evaluations for partial branches. Consequently, the algebra of this full quiver is PI-equivalent to the algebra of the full quiver with two disconnected branches:

(26)
$$I \longrightarrow II; \qquad II \longrightarrow III$$

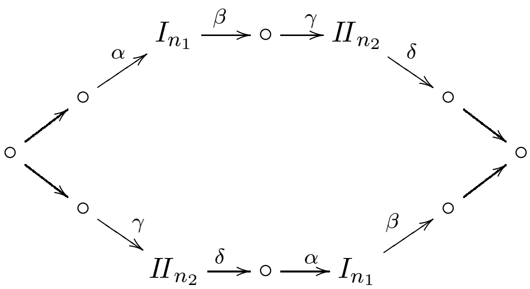
Different permuted branches $\mathcal{B}_\pi$ are obtained by first creating another branch with degenerate gluing and then permuting its edges. Thus, applying these permutations to the factors of the critical nonidentities of $\mathcal{B}$ yields evaluations that satisfy a given linear relation, and thus give rise to an identity.

For example, two permuted branches $\mathcal{B}_1$ and $\mathcal{B}_2$ will have different polynomials f_1 and f_2 whose evaluations on A are proportional, and thus for suitable μ, $f_1 - \mu f_2$ is an identity.

These considerations give rise to the following example that counters intuition.

Example 12.10 Suppose the full quiver Γ of the algebra A is of the form

(27)



Here we take $n_1 \neq n_2$, so that the corresponding central polynomials distinguish the blocks. There is one initial vertex and one terminal vertex. In the upper branch a block of type I precedes a block of type II, and the reverse is true in the lower branch.

Consider the polynomial

$$P = P_1 - P_2,$$

where $P_1 = f_1 h_{n_2} f_2 f_3 h_{n_4} f_4$ and $P_2 = f_3 h_{n_4} f_4 f_1 h_{n_2} f_2$. The evaluations corresponding to each branch cancel, so $P = [f_1 h_{n_2} f_2, f_3 h_{n_4} f_4] \in \mathrm{id}(A)$. Note that A is a factor algebra of $A_1 \oplus A_2$, where A_1 and A_2 are the algebras corresponding to the two branches. Thus, letting $\mathcal{F}$ denote the free algebra, we see that $\mathrm{id}(A) = \mathrm{id}(A_1) \cap \mathrm{id}(A_2)$, and thus the relatively free algebra $\hat{A} = \mathcal{F}/\mathrm{id}(A)$ of A is a subdirect product of $\mathcal{F}/\mathrm{id}(A_1)$ and $\mathcal{F}/\mathrm{id}(A_2)$.

On the other hand, the natural morphism

$$\hat{A} \quad \to \quad \mathcal{F}/(\langle P\rangle + \mathrm{id}(A_1)) \;\oplus\; \mathcal{F}/(\langle P\rangle + \mathrm{id}(A_2))$$

is not 1:1 since $\langle P\rangle + \mathrm{id}(A_1)$ contains P_1 and thus contains $P_2 = P - P_1$ (and likewise $\langle P\rangle + \mathrm{id}(A_2)$ contains P_2 and $P_1 = P - P_2$, whereas $\mathrm{id}(A)$ contains neither P_1 nor P_2.

A **parametric identity** of an algebra A is a polynomial $f(x_1, \dots, x_d) \in K[\Lambda]$, for some purely transcendental field extension $K = F(\xi_1, \dots, \xi_m)$ of the base field F, whose homogeneous components (in the x_i) are not identities, but which becomes an identity for any specialization of the ξ_i to F.

In [17, Example 7.2] an example is given of an algebra A without unit element (over a base field F of arbitrary characteristic), whose full quiver is a path, and with a parametric PI defined over the field $K = F(\xi_1, \dots, \xi_{2n})$, where ξ_i are purely transcendental over F.

13. Specht's problem for nonassociative algebras. Since the notion of algebraic variety is appropriate in any class of universal algebras, one can pose Specht's problem for arbitrary classes of algebraic varieties, including classes of nonassociative algebras. Some of the steps of the program of Remark 2.3 can be extended to this situation.

(1) In order to get started, one can work with "linear" algebras, i.e., those that are representable (in the appropriate context) inside matrices.
(2) The Zariski closure can be defined for any representable algebra, not necessarily associative (since the proof of [15, Theorem 3.11] does not use associativity), and belongs to the same variety, by [15, Lemma 3.18]. The Zariski closure could be useful in any class of algebras for which there are appropriate structure theorems for f.d. algebras. For example, one could utilize Levi's decomposition theorem for Lie algebras.
(3) Gluing can be described in terms of the appropriate decomposition.
(4) The big challenge with passing to nonassociative algebras is to find a workable alternative to full quivers and pseudo-quivers of representations, which essentially are associative in nature.
(5) The principle of hiking does not rely on associativity.
(6) Shirshov proved his theorem for arbitrary classes of nonassociative algebras.
(7) Step (6) enables us to find representable T-ideals inside arbitrary T-ideals, thereby providing a procedure to reduce quivers. Characteristic coefficient-absorbing polynomials can be defined in general using **Capelli systems** of k-alternating polynomials (written with all possible placement of indeterminates) in place of the Capelli polynomials; in fact, for this step Kemer used a polynomial-oriented approach developed by Zubrilin, expounded in [14], which generalizes [14, Theorem J, p. 25]. (Of course, one extends the theory to quasi-linear polynomials.)

(8) Quiver reduction is a formal geometric process. So this notion would likely apply to the nonassociative replacement for quivers.

(9) Noetherian induction on the base ring also is formal, and works in generality.

(10) The facts about torsion can be formulated module-theoretically over the commutative associative base ring, and do not rely on the properties of the given multiplication of the algebra, so the remaining steps of the proof are applicable in general.

REFERENCES

[1] S. I. Adian. Infinite irreducible systems of group identities. *Izv. Akad. Nauk SSSR Ser. Mat.* **34** (1970), 715–734 (in Russian); English translation in: *Math. USSR, Izv.* **4** (1970), 721–739.

[2] E. Aljadeff, A. Kanel-Belov. Representability and Specht problem for G-graded algebras. *Adv. in Mat.* **225**, *5* (2010), 2391–2428.

[3] A. Z. Anan'in. The representability of fiitely generated algebras with chain condition. *Arch. Math.* **59** (1992), 275–277.

[4] T. Asparouhov, V. Drensky, P. Koev, D. Tsiganchev. Generic 2×2 matrices in positive characteristic. *J. Algebra* **225** (2000), 451–486.

[5] A. V. Badeev. On the Specht property of varieties of commutative alternative algebras over a field of characteristic 3 and commutative Moufang loops. *Sibirsk. Mat. Zh.* **41**, *6* (2000), 1252–1268 (in Russian); English translation in: *Siberian Math. J.* **41**, *6* (2000), 1027–1041.

[6] A. V. Badeev. On the Specht property of soluble varieties of commutative alternative algebras over a field of characteristic 3. Ph.D. Thesis, Moscow State Pedagogial Univ., Moscow, 1999 (in Russian).

[7] Yu. A. Bakhturin. Identitical relations in Lie algebras. Transl. from the Russian. Itrecht, VNU Science Press, 1987.

[8] Yu. A. Bakhturin, A. Yu. Olshanskii. Identical relations in finite Lie rings. *Mat. Sb.* **96 (138)**, *4* (1975), 543–559 (in Russian); English translation in: *Math. USSR-Sb.* **25**, *4* (1975), 507–523.

[9] Zbl 0673.16001 Yu. A. Bakhturin, A. Yu. Olshanskii. Identities, Algebra – 2. *Itogi Nauki Tehn. Ser. Sovrem. Probl. Mat. Fundam. Napravleniya*, **18**, VINITI, Moscow, 1988, 117–240 (in Russian); English translation in: Algebra II: Noncommutative rings, Encyclopaedia Math. Sci., vol. **18**, Berlin, Springer, 1991, 107–221.

[10] A. Belov. On non-Specht varieties. *Fundam. Prikl. Mat.* **5**, *1* (1999), 47–66 (in Russian. English summary).

[11] A. Belov. Counterexamples to the Specht problem. *Mat. Sb.* **191**, *3*, (2000), 13–24 (in Russian); English translation in: *Sb. Math.* **191**, *3* (2000), 329–340.

[12] A. Belov. Local finite basis property and local representability of varieties of associative rings. *Izv. Ross. Akad. Nauk Ser. Mat. 1* (2010), 3–134 (in Russian); English translation in: *Izv. Math.* **74**, *1* (2010), 1–126.

[13] A. Belov, V. Borisenko, V. Latyshev. Monomial algebras. *J. Math. Sci., New York* **87**, *3* (1997), 3463–3575.

[14] A. Belov, L. H. Rowen. Computational Aspects of Polynomial Identities. Research Notes in Mathematics vol. **9**, AK Peters, Wellesley, MA, 2005.

[15] A. Belov, L. H. Rowen, U. Vishne. Structure of Zariski closed algebras. *Trans. Amer. Math. Soc.* **362** (2010), 4695–4734.

[16] A. Belov, L. H. Rowen, U. Vishne. Full quivers of representations of algebras. *Trans. Amer. Math. Soc.* **364** (2012), 5525–5569.

[17] A. Belov, L. H. Rowen, U. Vishne. PI-varieties associated to full quivers of representations of algebras. *Trans. Amer. Math. Soc.*, to appear.

[18] A. Belov, L. H. Rowen, U. Vishne. Specht's problem for associative affine algebras over commutative Noetherian rings. Preprint.

[19] A. Belov, L. H. Rowen, U. Vishne. Application of full quivers of representations of algebras, to polynomial identities. *Commun. Algebra* **39**, *12*, (2011), 4536–4551.

[20] G. Bergman, W. Dicks. On universal derivations, *J. Algebra* **36** (1975), 193–211.

[21] L. A. Bokut, I. P. Shestakov. Some results by A. I. Shirshov and his school. In: Proceedings of the Second International Conference on Algebra

dedicated to the memory of A. I. Shirshov (Eds L. A. Bokut et al.) August 20-25, 1991, Barnaul, Russia. Contemp. Math. vol. **184**, Providence, RI, Amer. Math. Soc., 1995, 1–12.

[22] A. Braun. The nilpotence of the radical in aa finitely generated PI-ring. *J. Algebra* **89** (1984), 375–396.

[23] R. M. Bryant, M. R. Vaughan-Lee. Soluble varieties of Lie algebras. *Q. J. Math., Oxf. II. Ser.* **23** (1972), 107–112.

[24] V. S. Drensky. Identities in Lie algebras. *Algebra Logika* **13** (1974), 265–290 (Russian).

[25] V. S. Drensky. A minimal basis for the identities of a second-order matrix algebra over a field of characteristic 0. *Algebra Logika* **20** (1981), 282–290 (in Russian); English translation in: *Algebra Logic* **20** (1981), 188–194.

[26] V. S. Drensky. Representations of the symmetric group and varieties of linear algebras. *Mat. Sb.* **115(157)**, *1(5)* (1981), 98–115 (in Russian); English translation in: *Math. USSR-Sb.* **43**, *1* (1982), 85–101.

[27] V. S. Drensky. Infinitely based varieties of Lie algebras. *Serdica Bulg. Math. Publ.* **9** (1983), 79–82 (in Russian).

[28] V. S. Drensky. On the Hilbert series of relatively free algebras. *Commun. Algebra* **12** (1984), 2335–2347.

[29] V. S. Drensky. Identities in matrix Lie algebras. Trudy Seminara Imeni I. G. Petrovskogo **6** (1981), 47–55 (in Russian); English translation in: *J. Sov. Math.* **33** (1986), 987–994.

[30] V. S. Drensky. Free Algebras and PI-Algebras. Graduate Course in Algebra. Singapore, Springer, 2000.

[31] G. K. Genov. Some Specht varieties of associative algebras. *Pliska Stud. Math. Bulgar.* **2** (1981), 30–40 (in Russian).

[32] G. K. Genov. A basis of identities of the algebra of third-order matrices over a finite field. *Algebra Logika* **20**, *4* (1981), 365–388; English translation in: *Algebra Logic* **20**, *4* (1981), 241–257.

[33] G. K. Genov, P. N. Siderov. A basis of identities of the algebra of fourth-order matrices over a finite field. I, II. *Serdica Bulg. Math. Publ.* **8**, *3* (1982), 313–323; **8**, *4* (1982), 351–366. (in Russian)

[34] A. Giambruno, M. Zaicev. Minimal varieties of algebras of exponential growth. *Adv. Math.* **174** (2003), 310–323.

[35] A. Giambruno, M. Zaicev. Polynomial Identities and Asymptotic Methods. Mathematical Surveys and Monographs vol. **122**. Providence, RI, American Mathematical Society, 2005.

[36] A. V. Grishin. Examples of T-spaces and T-ideals in Characteristic 2 without the Finite Basis Property. *Fundam. Prikl. Mat.* **5**, *1* (1999), 101–118 (in Russian. English summary).

[37] A. V. Grishin. A commutative Algebraic Approach to the Study of Polynomial Identities and T-spaces. Dr. Sci. Thesis, Moscow State Univ., Moscow, 2000 (in Russian).

[38] H. Helling. Eine Kennzeichnung von Charakteren auf Gruppen und assoziativen Algebren. *Commun. Algebra* **1** (1974), 491–501.

[39] G. Higman. Ordering by divisibility in abstract algebras. *Proc. Lond. Math. Soc., III. Ser.* **2** (1952), 326–336.

[40] A. V. Iltyakov. Representability of Algebras and Finite Bases of Identities. Dr. Sci. Thesis, IM SB RAS, Novosibirsk, 1990 (in Russian).

[41] A. V. Iltyakov. Finiteness of basis identities of a finitely generated alternative PI-algebra, Sibir. Mat. Zh. **31**, *6* (1991), 87–99 (in Russian); English translation in: *Sib. Math. J.* **31** (1991), 948–961.

[42] A. V. Iltyakov. On finite basis identities of identities of Lie algebra representations. *Nova J. Algebra Geom.* **1**, *3* (1992), 207–259.

[43] A. V. Iltyakov. Polynomial identities of Finite Dimensional Lie Algebras, 2003. `http://www.maths.usyd.edu.au/res/Algebra/Ilt/1998-6.html`

[44] I. M. Isaev. Finite right alternative algebras with infinite bases of identities. Abstracts of XVII All-union Algebraic Conf., Part I, Novosibirsk, 1985 (in Russian).

[45] N. Jacobson. Basic Algebra II. San Francisco, W. H. Freeman and Company, 1980.

[46] A. R. Kemer. The representability of reduced-free algebras. *Algebra Logika* **27**, *3* (1988), 274–294 (in Russian); English translation in: *Algebra Logic* **27**, *3* (1988), 167–184.

[47] A. R. Kemer. Identities of finitely generated algebras over an infinite field. *Izv. Akad. Nauk SSSR, Ser. Mat.* **54**, *4* (1990), 726–753 (in Russian); English translation in: *Math. USSR Izv.* **37**, *1*(1991), 69–97.

[48] A. R. Kemer. Identities of associative algebras. In: Transl., Ser. 2, Am. Math. Soc. vol. **148** (1991), 65–71. Transl. Math. Monogr., **87**, Amer. Math. Soc. (1991).

[49] A. R. Kemer. Multilinear identities of the algebras over a field of characteristic p. *Int. J. Algebra Comput.* **5**, *2* (1995), 189–197.

[50] A. R. Kemer. On the multilinear components of the regular prime varieties. In: Methods in ring theory, Proceedings of the Trento conference, Trento, Italy, (Eds V. Drensky et al.). Lect. Notes Pure Appl. Math. **198**, New York, Marcel Dekker, 1998, 171–183.

[51] A. R. Kemer. Matrix type of some algebras over a field of characteristic p. *J. Algebra* **251**, *2* (2002), 849–863.

[52] A. R. Kemer. On some problems in PI-theory in characteristic p connected with dividing by p. In: Proceedings of the 3rd International Algebra Conference, Tainan, Taiwan, June 16–July 1, 2002 (Eds Yu. Fong et al). Dordrecht, Kluwer Academic Publishers, 2003, 53–66.

[53] A. R. Kemer, I. Averyanov. Conjecture of Procesi for 2-generated algebra of generic 3×3 matrices. *J. Algebra* **299**, *1* (2006), 151–170.

[54] A. R. Kemer, I. Averyanov. Some problems in PI-theory. In: Advances in Algebra and Combinatorics. Proceedings of the 2nd International Congress in Algebra And Combinatorics, Guangzhou, China, July 2–4, 2007, Beijing, China, July 6–11, 2007 and Xian, China, July 12–15, 2007 (Eds K. P. Shum et al.), Hackensack, NJ, World Scientific, 2008, 189–204.

[55] Yu. G. Kleiman. The basis of a product variety of groups. I, II . *Izv. Akad. Nauk SSSR, Ser. Mat.* **37** (1973), 95–97; **38** (1974), 475–483 (in Russian); English translation in: *Math. USSR, Izv.* **7** (1973), 91–94; **8** (1974), 481–489.

[56] D. Krakowski, A. Regev. The polynomial identities of the Grassmann algebra. *Trans. Amer. Math. Soc.* **181** (1973), 429–438.

[57] A. N. Krasilnikov. On the property of having a finite basis of some varieties of Lie algebras. *Vestnik Moskov. Univ. Ser. I, Mat. Mekh. 2* (1982),

34–38 (in Russian); English translation in : *Moscow State Univ. Bull.* **37**, *2* (1982), 44–48.

[58] R. L. Kruse. Identities satisfied by a finite ring. *J. Algebra* **26** (1973), 298–318.

[59] V. N. Latyshev. On algebras with identity relations. Dokl. Akad. Nauk SSSR **146** (1962), 1003–1006 (in Russian). English translation in: *Sov. Math., Dokl.* **3** (1962), 1423–1427.

[60] V. N. Latyshev. Generalization of the Hilbert theorem on the finiteness of bases. *Sibirsk. Mat. Zh.* **7**, *6* (1966), 1422–1424 (in Russian); English translation in: *Siberian Math. J.* **7**, *6* (1966), 1112–1113.

[61] V. N. Latyshev. On the Specht property of certain varieties of associated algebras. *Algebra Logika* **8**, *6* (1969), 660–673 (in Russian); English translation in: *Algebra Logic* **8**, *6* (1969), 374–382.

[62] V. N. Latyshev. Partially ordered sets and nonmatrix identities of associative algebras. *Algebra Logika* **15** (1976), 53–70. (in Russian); English translation in: *Algebra Logic* **15** (1976), 34–45.

[63] V. N. Latyshev. Finite-basedness of identities in some rings. *Uspekhi Mat. Nauk* **32**, *4(196)* (1977), 259–260 (in Russian).

[64] V. N. Latyshev. Non-matrix Varieties of Associative Algebras. Dr. Sci. Thesis, Moscow State Univ., Moscow, 1977 (in Russian).

[65] J. Lewin. A matrix representation for associative algebras, I, II. *Trans. Amer. Math. Soc.* **188**, *2* (1974), 293–317.

[66] I. V. L'vov. On varieties of associative rings. I, II *Algebra Logika* **12** (1973), 269–297, 667–688 (in Russian); English translation in: *Algebra Logic* **12** (1973), 150–167, 381–393.

[67] A. I. Maltsev. On algebras with identity relations. *Mat. Sb.* **26(68)**, *1* (1950), 19–33 (in Russian).

[68] Yu. N. Maltsev, E. N. Kuzmin. A basis for identities of the algebra of second order matrices over a finite field. *Algebra Logika* **17** (1978), 28–32 (in Russian); English translation in: *Algebra Logic* **17** (1978), 17–21.

[69] YU. A. MEDVEDEV. An example of a variety of solvable alternative algebras over a field of characteristic 2 having no infinite basis of identities. *Algebra Logika* **19**, *3* (1980), 300–313; English translation in: *Algebra Logic* **19**, *3* (1980), 191–201.

[70] B. H. NEUMANN. Identical relations in groups I. *Math. Ann.* **114** (1937), 506–525.

[71] H. NEUMANN. Varieties of groups, (Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. **37**.) Berlin-Heidelberg-New York, Springer-Verlag, 1967.

[72] S. OATES, M. B. POWELL. Identical relations in finite groups. *J. Algebra* **1**, (1964) 11–39.

[73] A. YU. OLSHANSKII. On the problem of a finite basis of identities in groups. *Izv. Akad. Nauk SSSR, Ser. Mat.* **34** (1970), 376–384.(in Russian); English translation in: *Math. USSR, Izv.* **4** (1970), 381–389.

[74] S. V. PCHELINTSEV. The structure of weak identities on Grassmannian envelopes of central-methaabelian alternative superalgebras of superrank 1 over a field of characteristic 3. *Fundam. Prikl. Mat.* **7**, *3* (2001), 849–871 (in Russian. English summary)

[75] A. P. POPOV. The Specht property of some varieties of associative algebras. *Pliska Stud. Math. Bulgar.* **2** (1981), 41–53 (in Russian).

[76] C. PROCESI. Rings with Polynomial Identities. Pure and Applied Mathematics vol. **17**, New York, Marcel Dekker, 1973.

[77] C. PROCESI. The invariant theory of $n \times n$ matrices, *Adv. Math.* **19** (1976), 306–381.

[78] YU. P. RAZMYSLOV. Trace identities of full matrix algebras over a field of characteristic zero. *Izv. Akad. Nauk SSSR, Ser. Mat.* **38** (1974), 723–756 (in Russian); English translation in: *Math. USSR Izv.* **8** (1974), 727–760.

[79] YU. P. RAZMYSLOV. Finite basing of the identities of a matrix algebra of second order over a field of characteristic zero. *Algebra Logika* **12** (1973), 83–113 (in Russian); English translation in: *Algebra Logic* **12** (1973), 47–63.

[80] YU. P. RAZMYSLOV. Existence of a finite basis for certain varieties of algebras. *Algebra Logika* **13** (1974), 685–693 (in Russian); English translation in: *Algebra Logic* **13** (1974), 394–399.

[81] Yu. P. Razmyslov. Identities of Algebras and their Representations. Sovremennaya Algebra vol. **14**, Moscow, Nauka, 1989.

[82] L. H. Rowen. Polynomial Identities in Ring Theory. Pure and Applied Mathematics vol. **84**, New York etc., Academic Press, 1980.

[83] L. H. Rowen. Ring Theory II. Pure and Applied Mathematics vol. **128**, Boston, MA etc., Academic Press, 1988.

[84] L. H. Rowen. Graduate algebra: Noncommutative View. Graduate Studies in Mathematics vol.**91**, Providence, RI, American Mathematical Society, 2008.

[85] L. H. Rowen, L. W. Small. Representability of algebras finite over their center. Preprint, 2012.

[86] L. M. Samoĭlov. Prime Varieties of Associative Algebras and Related Nill-problems. Dr. Sci. Thesis, Moscow, 2010, 161 pp.

[87] W. Schelter. On a question concerning generic matrices over the integers. *J. Algebra* **96** (1985), 48–53.

[88] V. V. Shchigolev. Examples of T-spaces with an infinite basis. *Mat. Sb.* **191**, *3* (2000), 143–160 (in Russian); English translation in: *Sb. Math.* **191** *3* (2000), 459–476.

[89] I.P. Shestakov. Superalgebras and counterexamples. *Sibirsk. Mat. Zh.* **32**, *6* (1991), 187–196(in Russian); English translation in: *Siberian Math. J.* **32**, *6* (1991), 1052–1060.

[90] A. I. Shirshov. Selected works of A. I. Shirshov. Translated from the Russian by Murray Bremner and Mikhail V. Kotchetov. Edited by Leonid A. Bokut, Victor Latyshev, Ivan Shestakov and Efim Zelmanov. Contemporary Mathematicians. Basel, Birkhauser Verlag, 2009.

[91] W. Specht. Gesetze in Ringen. I. *Math. Z.* **52** (1950), 557–589.

[92] I. Sviridova. Identities of PI-algebras graded by a finite Abelian group. *Commun. Algebra* **39**, *9* (2011), 3462–3490.

[93] A. Ja. Vais, E. I. Zelmanov. Kemer's theorem for finitely generated Jordan algebras. *Izv. Vyssh. Uchebn. Zved. Mat.*, *6(325)* (1989), 42–51 (in Russian); English translation in: *Sov. Math.* **33**, *6* (1990), 38–47.

[94] M. R. Vaughan-Lee. Varieties of Lie algebras. *Quart. J. of Math., Oxf. II. Ser.* **21** (1970), 297–308.

[95] M. R. Vaughan-Lee. Uncountably many varieties of groups. *Bull. London Math. Soc.* **2**, *3* (1970), 280–286.

[96] M. R. Vaughan-Lee. Abelian-by-nilpotent varieties of Lie algebras. *J. London Math. Soc., II. Ser.* **11** (1975), 263–266.

[97] I. B. Volichenko. On varieties of centre-by-metabelian Lie algebras. Institute of Math. Belarussian Academy of Sciences, preprint No. 16, 1980.

[98] I. B. Volichenko. Varieties of Lie algebras with identity $[[x_1, x_2, x_3], [x_4, x_5, x_6]] = 0$ over a field of characteristic 0. *Sibirsk. Mat. Zh.* **25**, *3* (1984), 40–54 (in Russian); English translation in: *Siberian Math. J.* **25** (1984), 370–382.

[99] A. N. Zubkov. Matrix invariants over an infinite field of finite characteristic. *Sibirsk. Mat. Zh.* **34**, *6* (1993), 68–74 (in Russian); English translation in: *Siberian Math. J.* **34**, *6* (1993), 1059–1065.

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