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ON INDEX-EXPONENT RELATIONS OVER HENSELIAN FIELDS WITH LOCAL RESIDUE FIELDS*

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Abstract. Let p be a prime and (K, v) a Henselian valued field with a residue field $\widehat{K}$. This paper determines the Brauer p-dimension of K, in case $p \neq \mathrm{char}(\widehat{K})$ and $\widehat{K}$ is a p-quasilocal field properly included in its maximal p-extension. When $\widehat{K}$ is a local field, it describes index-exponent pairs of central division K-algebras of p-primary degrees. The same goal is achieved, if (K, v) is maximally complete, $\mathrm{char}(K) = p$ and $\widehat{K}$ is local.

1. Introduction. Let E be a field, $\mathbb{P}$ the set of prime numbers, and for each $p \in \mathbb{P}$, let $E(p)$ be the maximal p-extension of E in a separable closure E_{sep}, and $r_p(E)$ the rank of the Galois group $\mathcal{G}(E(p)/E)$ as a pro-p-group (put $r_p(E) = 0$, if $E(p) = E$). Denote by $s(E)$ the class of finite-dimensional associative central simple E-algebras, and by $d(E)$ the subclass of division algebras $D \in s(E)$. For

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each $A \in s(E)$, let $[A]$ be the equivalence class of A in the Brauer group $\mathrm{Br}(E)$, and D_A a representative of $[A]$ lying in $d(E)$. The existence of D_A and its uniqueness, up-to an E-isomorphism, is established by Wedderburn's structure theorem (cf. [27], Sect. 3.5), which implies the dimension $[A\colon E]$ is a square of a positive integer $\deg(A)$ (the degree of A). It is known that $\mathrm{Br}(E)$ is an abelian torsion group, so it decomposes into the direct sum, taken over $\mathbb{P}$, of its p-components $\mathrm{Br}(E)_p$ (see [27], Sects 3.5 and 14.4). The Schur index $\mathrm{ind}(D) = \deg(D_A)$ and the exponent $\exp(A)$, i.e. the order of $[A]$ in $\mathrm{Br}(E)$, are invariants of both A and $[A]$. Their general relations and behaviour under scalar extensions of finite degrees are described as follows (cf. [27], Sects. 13.4, 14.4 and 15.2):

(1.1) (a) $\exp(A) \mid \mathrm{ind}(A)$ and $p \mid \exp(A)$, for each $p \in \mathbb{P}$ dividing $\mathrm{ind}(A)$. For any $B \in s(E)$ with $\mathrm{ind}(B)$ prime to $\mathrm{ind}(A)$, $\mathrm{ind}(A \otimes_E B) = \mathrm{ind}(A).\,\mathrm{ind}(B)$; if A, $B \in d(E)$, then the tensor product $A \otimes_E B$ lies in $d(E)$;
(b) $\mathrm{ind}(A)$ and $\mathrm{ind}(A \otimes_E R)$ divide $\mathrm{ind}(A \otimes_E R)[R\colon E]$ and $\mathrm{ind}(A)$, respectively, for each finite field extension R/E of degree $[R\colon E]$.

As shown by Brauer (see, e.g., [27], Sect. 19.6), (1.1) (a) determines all generally valid index-exponent relations. It is known, however, that, for a number of fields E, the pairs $\mathrm{ind}(A), \exp(A)$, $A \in s(E)$, are subject to much tougher restrictions than those described by (1.1) (a). The Brauer p-dimensions $\mathrm{Brd}_p(E)$, $p \in \mathbb{P}$, contain essential information on these restrictions. We say that $\mathrm{Brd}_p(E) = n$, where $n \in \mathbb{Z}$, if n is the least integer ≥ 0 for which $\mathrm{ind}(D) \leq \exp(D)^n$ whenever $D \in d(E)$ and $[D] \in \mathrm{Br}(E)_p$; if no such n exists, we put $\mathrm{Brd}_p(E) = \infty$. In view of (1.1), $\mathrm{Brd}_p(E) \leq 1$, for a given p, if and only if $\mathrm{ind}(D) = \exp(D)$, for each $D \in d(E)$ with $[D] \in \mathrm{Br}(E)_p$; $\mathrm{Brd}_p(E) = 0$ if and only if $\mathrm{Br}(E)_p = \{0\}$. The absolute Brauer p-dimension $\mathrm{abrd}_p(E)$ of E is defined as the supremum $\mathrm{Brd}_p(R)\colon R \in \mathrm{Fe}(E)$, $\mathrm{Fe}(E)$ being the set of finite extensions of E in E_{sep}. For example, when E is a global or local field, $\mathrm{Brd}_p(E) = \mathrm{abrd}_p(E) = 1$, $p \in \mathbb{P}$, and there exist $Y_n \in d(E)$, $n \in \mathbb{N}$, with $\mathrm{ind}(Y_n) = n$, for any n (see [38], Ch. XII, Sect. 2; Ch. XIII, Sects. 3, 6).

This paper deals with the study of index-exponent K-pairs, for a Henselian (valued) field (K, v), along the lines drawn in [8], Sect. 5. Its purpose is to determine $\mathrm{Brd}_p(K)$ and to describe p-primary index-exponent K-pairs, provided that the residue field $\widehat{K}$ of (K, v) is endowed with a Henselian discrete valuation ω whose residue field is quasifinite, and $p \in \mathbb{P}$ is different from $\mathrm{char}(\widehat{K})$ (for other types of $\widehat{K}$, such as the one of a global field, see [8], Sect. 5). Our main result, presented by the following theorem, concerns the case where $\widehat{K}$ is a local field and the value group $v(K)$ is p-indivisible, i.e. the quotient group $v(K)/pv(K)$

is nontrivial. When K contains a primitive p-th root of unity, it shows that index-exponent p-primary K-pairs are not determined only by $\mathrm{Brd}_p(K)$:

Theorem 1.1. *Let (K, v) be a Henselian field with $\mathrm{Brd}_p(K) < \infty$, for some $p \in \mathbb{P}$, and let $m_p = \min\{\tau(p), r_p(\widehat{K})\}$, where $\tau(p)$ is the dimension of $v(K)/pv(K)$ as a vector space over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Assume that $\tau(p) > 0$, $p \neq \mathrm{char}(\widehat{K})$, $\widehat{K}$ is a local field, and ε_p is a primitive p-th root of unity in $\widehat{K}_{\mathrm{sep}}$, denote by ν the greatest integer for which $\widehat{K}$ contains a primitive p^ν-th root of unity, and in case $\varepsilon_p \in \widehat{K}$, put $r_p'(\widehat{K}) = r_p(\widehat{K}) - 1$ and $m_p' = \min\{\tau(p), r_p'(\widehat{K})\}$. For each $n \in \mathbb{N}$, let $\mu(p, n) = nm_p' + \nu_n(m_p - m_p' + [(\tau(p) - m_p)/2])$, if $\varepsilon_p \in \widehat{K}$, where $\nu_n = \min\{n, \nu\}$, and $\mu(p, n) = nm_p$, if $\varepsilon_p \notin \widehat{K}$. Then $\mathrm{Brd}_p(K) = \mu(p, 1)$; moreover, for a pair $(k, n) \in \mathbb{N}^2$, there exists $D_{k,n} \in d(K)$ with $\mathrm{ind}(D_{k,n}) = p^k$ and $\exp(D_{k,n}) = p^n$ if and only if $n \leq k \leq \mu(p, n)$.*

Assuming that (K, v) is Henselian, $p \in \mathbb{P}$ is not equal to $\mathrm{char}(\widehat{K})$, $\tau(p)$ and ε_p are defined as above, and $(\widehat{K}, \omega)$ is a Henselian discrete valued field (abbr, an HDV-field) with a quasifinite residue field $\tilde{k}$, we obtain the following result:

(1.2)

(a) If $0 < \tau(p) < \infty$, $\mathrm{char}(\widehat{K}) = 0$, and $\tilde{k}$ is infinite with $\mathrm{char}(\tilde{k}) = p$, then $\mathrm{Brd}_p(K) = \tau(p)$ and (p^k, p^n), $k, n \in \mathbb{N}$, $n \leq k \leq n\tau(p)$, are all nontrivial index-exponent p-primary K-pairs;

(b) $\mathrm{Brd}_p(K) = 1$ and (p^n, p^n), $n \in \mathbb{N} \cup \{0\}$, are all index-exponent K-pairs, in case $p \neq \mathrm{char}(\tilde{k})$ and $\varepsilon_p \notin \widehat{K}$; the same holds, if $p \neq \mathrm{char}(\tilde{k})$ and $\tau(p) \leq 1$;

(c) When $p \neq \mathrm{char}(\tilde{k})$, $\varepsilon_p \in \widehat{K}$, and $2 \leq \tau(p) < \infty$, we have $r_p(\widehat{K}) = 2$ and $\mathrm{Brd}_p(K) = 1 + [\tau(p)/2]$;

(d) In the setting of (c), if $\widehat{K}$ contains finitely many roots of unity of p-primary degrees, then index-exponent p-primary K-pairs are determined in accordance with Theorem 1.1; when $\widehat{K}$ contains infinitely many such roots, (p^k, p^n), $k, n \in \mathbb{N}$, $n \leq k \leq n\,\mathrm{Brd}_p(K)$, are index-exponent K-pairs.

When (K, v) is a maximally complete field with $\mathrm{char}(K) = p$ and $\widehat{K}$ a local field, $\mathrm{Brd}_p(K)$ and index-exponent p-primary K-pairs are determined as follows:

Proposition 1.2. *Assume that (K, v) is a maximally complete field, $\mathrm{char}(K) = p > 0$, and $\widehat{K}$ is a local field, and define $\tau(p)$ as in Theorem 1.1. Then:*

(a) $\mathrm{Brd}_p(K) = \infty$ *if and only if* $\tau(p) = \infty$*; when this holds,* (p^k, p^n) *is an index-exponent pair over* K*, for any* $k, n \in \mathbb{N}$ *with* $k \geq n$*;*

(b) $\mathrm{Brd}_p(K) = \tau(p)$*, provided that* $\tau(p) < \infty$*; in this case,* (p^k, p^n) *is an index-exponent* K*-pair, where* $k, n \in \mathbb{N}$*, if and only if* $n \leq k \leq n\tau(p)$*.*

Proposition 1.2 is deduced in Section 3 from our description of index-exponent p-primary pairs over maximally complete fields of characteristic p with perfect residue fields (see Corollary 3.6 and Proposition 3.5). The proofs of (1.2) and Theorem 1.1 rely on the fact that HDV-fields with quasifinite residue fields are quasilocal, i.e. their finite extensions are p-quasilocal fields with respect to every $p \in \mathbb{P}$ (see [31], Ch. XIII, Sect. 3). As in [5], a field E with $r_p(E) > 0$, for some p, is called p-quasilocal, if the relative Brauer group $\mathrm{Br}(E'/E)$ equals the group ${}_p\mathrm{Br}(E) = \{b \in \mathrm{Br}(E)\colon pb = 0\}$, for every degree p extension E' of E in $E(p)$; when $r_p(E) = 0$, we say that E is p-quasilocal, if $\mathrm{Br}(E)_p = \{0\}$. The part of Theorem 1.1 concerning $\mathrm{Brd}_p(K)$ is a special case of a formula for $\mathrm{Brd}_p(K)$, deduced when $\widehat{K}$ is any p-quasilocal field with $\mathrm{char}(\widehat{K}) \neq p$ and $r_p(\widehat{K}) > 0$ (see Section 4). To prove this formula we use the inequality $\mathrm{Brd}_p(\widehat{K}) \leq 1$, the surjectivity of the scalar extension map $\mathrm{Br}(\widehat{K})_p \to \mathrm{Br}(\widehat{K}')_p$, for every extension $\widehat{K}'$ of $\widehat{K}$ in $\widehat{K}(p)$, and the following relations between finite extensions of $\widehat{K}$ in $\widehat{K}(p)$ and algebras $\Delta_p \in d(\widehat{K})$ of p-primary degrees (see [5], I, Sects. 3 and 4):

(1.3) $\mathrm{ind}(\Delta_p) = \mathrm{g.c.d.}\{[L_p\colon \widehat{K}], \mathrm{ind}(\Delta_p)\}\,\mathrm{ind}(\Delta_p \otimes_{\widehat{K}} L_p)$ whenever L_p is a finite extension of $\widehat{K}$ in $\widehat{K}(p)$. Specifically, L_p embeds in Δ_p as a $\widehat{K}$-subalgebra if and only if $[L_p\colon \widehat{K}] \mid \mathrm{ind}(\Delta_p)$; $[\Delta_p] \in \mathrm{Br}(L_p/\widehat{K})$ if and only if $\mathrm{ind}(\Delta_p) \mid [L_p\colon \widehat{K}]$.

Statements (1.2) and the concluding assertion of Theorem 1.1 are proved in Section 5. Their proofs are based on Morandi's theorem [26], the theory of division algebras over Henselian fields developed in [17], and the structure of the (continuous) character group $C(\widehat{K}(p)/\widehat{K})$ of $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ as an abstract abelian group (see (5.2), (5.3) and Remark 5.3). Our proofs also rely on the fact that if $\widehat{K}$ is a local field or $p \neq \mathrm{char}(\tilde{k})$, then $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ is a Demushkin group if $\widehat{K}$ contains a primitive p-th root of unity, and $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ is a free pro-p-group, otherwise (cf. [32], Ch. II, 2.2 and 5.6). By a Demushkin group, we mean a pro-p-group G_p whose continuous cohomology groups $H^i(G_p, \mathbb{F}_p)$ with coefficients in $\mathbb{F}_p$, for $i = 1, 2$, satisfy the following conditions: $H^2(G_p, \mathbb{F}_p)$ is of order p, $H^1(G_p, \mathbb{F}_p)$ is finite and abelian of period p, and for any nonzero $a \in H^1(G_p, \mathbb{F}_p)$, the homomorphism $\varphi_a\colon H^1(G_p, \mathbb{F}_p) \to H^2(G_p, \mathbb{F}_p)$, mapping each $b \in H^1(G_p, \mathbb{F}_p)$ into the cup-product $a \cup b$, is surjective. We also use the well-known fact that local fields contain finitely many roots of unity, and take into account that Brauer groups of HDV-fields with quasifinite residue fields are isomorphic to the quotient group $\mathbb{Q}/\mathbb{Z}$ of the additive group of rational numbers by the subgroup of integers (cf. [31], Ch. XIII, Sect. 3).

The basic notation and terminology used and conventions kept in this paper are standard, like in [5], I, and [7, 8]. We write $Z(B)$ for the centre of an associative ring B. Given a Henselian field (K, v), $K_{\rm ur}$ denotes the compositum of inertial extensions of K in $K_{\rm sep}$; the notions of an inertial, a nicely semi-ramified (abbr, NSR), and a totally ramified (division) K-algebra, are defined in [17]. Section 2 includes valuation-theoretic preliminaries used in the sequel. By a Pythagorean field, we mean a formally real field whose set of squares is additively closed. As usual, $[r]$ stands for the integral part of a real number $r \geq 0$, and for any $p \in \mathbb{P}$, a $\mathbb{Z}_p$-extension means a Galois extension whose Galois group is isomorphic to the additive group $\mathbb{Z}_p$ of p-adic integers. The set of intermediate fields of a field extension Λ/Ψ is denoted by $I(\Lambda/\Psi)$. Symbol algebras are defined, e.g., in [17] and [27], Sect. 15.4. Galois groups are viewed as profinite with respect to the Krull topology, and by a profinite group homomorphism, we mean a continuous one. The reader is referred to [23], [14], [17], [16], [27] and [32], for missing definitions concerning field extensions, orderings and valuations, m-dimensional local fields, simple algebras, Brauer groups and Galois cohomology.

2. Preliminaries. Let (K, v) be a Krull valued field with a residue field $\widehat{K}$ and a (totally ordered) value group $v(K)$. We say that (K, v) is Henselian, if v extends uniquely, up-to an equivalence, to a valuation v_L on each algebraic extension L/K. This occurs, for example, if (K, v) is maximally complete, i.e. it has no immediate proper extension (a valued extension (K', v'), such that $K' \neq K$, $\widehat{K}' = \widehat{K}$ and $v'(K') = v(K)$). When (K, v) is Henselian, we denote by $\widehat{L}$ the residue field of (L, v_L) and put $v(L) = v_L(L)$, for any algebraic extension L/K. Clearly, $\widehat{L}/\widehat{K}$ is an algebraic extension and $v(K)$ is an ordered subgroup of $v(L)$; $e(L/K)$ denotes the index of $v(K)$ in $v(L)$. By Ostrowski's theorem (cf. [14], Theorem 17.2.1), when L/K is finite, $[L\colon K]$, $[\widehat{L}\colon \widehat{K}]$ and $e(L/K)$ are related as follows:

(2.1) $[\widehat{L}\colon \widehat{K}]e(L/K)$ divides $[L\colon K]$ and $[L\colon K][\widehat{L}\colon \widehat{K}]^{-1}e(L/K)^{-1}$ is not divisible by any $p \in \mathbb{P}$, $p \neq \mathrm{char}(\widehat{K})$; $[L\colon K] = [\widehat{L}\colon \widehat{K}]e(L/K)$, if $\mathrm{char}(\widehat{K}) \nmid [L\colon K]$.

The Henselity of (K, v) ensures that each $\Delta \in d(K)$ has a unique, up-to an equivalence, valuation v_Δ extending v and possessing an abelian value group $v(\Delta)$ (cf. [30], Ch. 2, Sect. 7). This group is totally ordered and includes

$v(K)$ as an ordered subgroup of index $e(\Delta/K) \leq [\Delta\colon K]$. Also, the residue division ring $\widehat{\Delta}$ of (Δ, v_Δ) is a $\widehat{K}$-algebra, and by Ostrowski-Draxl's theorem [12], $e(\Delta/K)[\widehat{\Delta}\colon \widehat{K}] \mid [\Delta\colon K]$ and if $\mathrm{char}(\widehat{K}) \nmid \mathrm{ind}(\Delta)$, then $[\Delta\colon K] = e(\Delta/K)[\widehat{\Delta}\colon \widehat{K}]$. Statement (2.1) and the Henselity of (K, v) imply the following:

(2.2) The quotient groups $v(K)/pv(K)$ and $v(L)/pv(L)$ are isomorphic, if $p \in \mathbb{P}$ and $[L\colon K] < \infty$. When $\mathrm{char}(\widehat{K}) \nmid [L\colon K]$, the natural embedding of K into L induces canonically an isomorphism $v(K)/pv(K) \cong v(L)/pv(L)$.

A finite extension R of K is said to be inertial, if $[R\colon K] = [\widehat{R}\colon \widehat{K}]$ and $\widehat{R}/\widehat{K}$ is separable. We say that R/K is totally ramified, if $[R\colon K] = e(R/K)$; R/K is called tamely ramified, if $\widehat{R}/\widehat{K}$ is separable and $\mathrm{char}(\widehat{K}) \nmid e(R/K)$. The properties of K_{ur}/K used in the sequel are essentially those presented in [17], page 135, and restated in [6], (3.3) (see also [33], Theorem A.24). Here we recall some results on central division K-algebras (most of which can be found in [17]):

(2.3) (a) If $D \in d(K)$ and $\mathrm{char}(\widehat{K}) \nmid \mathrm{ind}(D)$, then $[D] = [S \otimes_K V \otimes_K T]$, for some S, V, $T \in d(K)$, such that S/K is inertial, V/K is NSR, T/K is totally ramified, $T \otimes_K K_{\mathrm{ur}} \in d(K_{\mathrm{ur}})$, $\exp(T \otimes_K K_{\mathrm{ur}}) = \exp(T)$, and T is a tensor product of totally ramified cyclic K-algebras (see also [12], Theorem 1);

(b) The set $\mathrm{IBr}(K) = \{[S'] \in \mathrm{Br}(K)\colon S' \in d(K), S'/K \text{ inertial}\}$ is a subgroup of $\mathrm{Br}(K)$ canonically isomorphic to $\mathrm{Br}(\widehat{K})$; $\mathrm{Brd}_p(\widehat{K}) \leq \mathrm{Brd}_p(K)$, $p \in \mathbb{P}$, and equality holds when $p \neq \mathrm{char}(\widehat{K})$ and $v(K) = pv(K)$;

(c) With assumptions and notation being as in (a), if $T \neq K$, then K contains a primitive root of unity of degree $\exp(T)$; in addition, if $T_n \in d(K)$ and $[T_n] = n[T] \neq 0$, for some $n \in \mathbb{N}$, then T_n/K is totally ramified;

Statement (2.3) can be supplemented as follows (see, e.g., [8], Sect. 4):

(2.4) If D, S, V and T are related as in (2.3) (a), then:

(a) $n[D] \in \mathrm{IBr}(K)$, for a given $n \in \mathbb{N}$, if and only if $\exp(V) \mid n$ and $\exp(T) \mid n$;

(b) D/K is inertial if and only if $V = T = K$; D/K is inertially split, i.e. $[D] \in \mathrm{Br}(K_{\mathrm{ur}}/K)$, if and only if $T = K$;

(c) $\exp(D) = \mathrm{lcm}(\exp(S), \exp(V), \exp(T))$.

The following result of [8] gives a formula for $\mathrm{Brd}_p(K)$ whenever $p \neq \mathrm{char}(\widehat{K})$ and $\mathrm{Brd}_p(\widehat{K}) = 0$:

Theorem 2.1. *Assume that* (K, v) *is a Henselian field with* $\mathrm{Brd}_p(\widehat{K}) < \infty$, *for some* $p \in \mathbb{P}$, $p \neq \mathrm{char}(\widehat{K})$, *and let* $\tau(p)$, ε_p *and* m_p *be as in Theorem 1.1. Then:*

(a) $\mathrm{Brd}_p(K) = \infty$ *if and only if* $m_p = \infty$ *or* $\tau(p) = \infty$ *and* $\varepsilon_p \in \widehat{K}$;

(b) $[(\tau(p)+m_p)/2] \leq \mathrm{Brd}_p(K) \leq \mathrm{Brd}_p(\widehat{K}) + [(\tau(p)+m_p)/2]$, *if* $\tau(p) < \infty$ *and* $\varepsilon_p \in \widehat{K}$; *when* $m_p < \infty$ *and* $\varepsilon_p \notin \widehat{K}$, $m_p \leq \mathrm{Brd}_p(K) \leq \mathrm{Brd}_p(\widehat{K}) + m_p$.

As shown in [8], Sect. 4, Theorem 2.1 leads to the following description of index-exponent p-primary K-pairs, in the case where $\mathrm{Brd}_p(K) = \infty$:

Corollary 2.2. *Let* (K, v) *be a Henselian field with* $\mathrm{Brd}_p(\widehat{K}) < \infty = \mathrm{Brd}_p(K)$, *for some* $p \neq \mathrm{char}(\widehat{K})$. *Then the following alternative holds:*

(a) (p^k, p^n): $k, n \in \mathbb{N}, n \leq k$, *are index-exponent* K*-pairs;*

(b) $p = 2$ *and* $\widehat{K}$ *is a Pythagorean field; such being the case, the group* $\mathrm{Br}(K)_2$ *has period* 2, *and there are* $D_m \in d(K)$, $m \in \mathbb{N}$, *with* $\mathrm{ind}(D_m) = 2^m$.

This Section ends with a lemma that is implicitly used in the proofs of the main results of the following Section.

Lemma 2.3. *Let* (K, v) *be a valued field with* $\mathrm{char}(K) = p > 0$ *and* $v(K) \neq pv(K)$, *and let* $\pi \in K^*$ *be an element of value* $v(\pi) \notin pv(K)$. *Assume that* G *is a finite* p*-group of order* p^t. *Then there exists a Galois extension* M *of* K *in* $K(p)$, *such that* $\mathcal{G}(M/K) \cong G$, v *is uniquely, up-to an equivalence, extendable to a valuation* v_M *of* M, *and* $v(\pi) \in p^t v_M(M)$; *in particular,* $v_M(M)/v(K)$ *is cyclic and* $(M, v_M)/(K, v)$ *is totally ramified.*

P r o o f. One may assume, for the proof, that $v(\pi) < 0$. Let $\mathbb{F}$ be the prime subfield of K, $(K_v, \bar{v})$ a Henselization of (K, v), $\rho(K_v) = \{u^p - u\colon\ u \in K_v\}$, ω the valuation of the field $\Phi = \mathbb{F}(\pi)$ induced by v and for each $m \in \mathbb{N}$, let L_m and Λ_m be the root fields in K_{sep} over K and Φ, respectively, of the polynomial $f_m(X) = X^p - X - \pi_m$, where $\pi_m = \pi^{1+qm}$. Identifying K_v with its K-isomorphic copy in K_{sep}, take a Henselization $(\Phi_\omega, \bar{\omega})$ of (Φ, ω) among the valued subfields of $(K_v, \bar{v})$ (this is possible, by [14], Theorem 15.3.5), and put $\Psi_m = \Lambda_1 \dots \Lambda_m$ and $M_m = L_1 \dots L_m$, for each m. It is well-known that $(K_v, \bar{v})/(K, v)$ and $(\Phi_\omega, \bar{\omega})/(\Phi, \omega)$ are immediate extensions, i.e. $\widehat{K}_v = \widehat{K}$, $\bar{v}(K_v) = v(K)$ and $\widehat{\Phi}_\omega = \widehat{\Phi}$, $\bar{\omega}(\Phi_\omega) = \omega(\Phi)$. Also, it is easily verified that $\rho(K_v)$ is an $\mathbb{F}$-subspace of K_v, and $\bar{v}(u') \in pv(K)$ whenever $u' \in \rho(K_v)$ and $\bar{v}(u') < 0$. This implies the cosets $\pi_m + \rho(K_v)$, $m \in \mathbb{N}$, are linearly independent over $\mathbb{F}$, so the Artin-Schreier theorem (cf. [23], Ch. VIII, Sect. 6) enables one to prove the following statement, for each $m \in \mathbb{N}$:

(2.5) L_m/K, L_mK_v/K_v, Λ_m/Φ and $\Lambda_m\Phi_\omega/\Phi_\omega$ are degree p cyclic extensions; M_m/K, M_mK_v/K_v, Ψ_m/Φ and $\Psi_m\Phi_\omega/\Phi_\omega$ are abelian of degree p^m.

Let now G_r be a finite p-group of rank $r > 0$ and order $p^{\mu(r)}$. Since $\mathrm{char}(\Phi) = p$, and therefore, $\mathcal{G}(\Phi(p)/\Phi)$ is a free pro-p-group (cf. [32], I, 1.5, 4.2; II, 2.2), there exists a Galois extension Γ_r of Φ in K_{sep}, such that $\mathcal{G}(\Gamma_r/\Phi) \cong G_r$ and $\Psi_r \in I(\Gamma_r/\Phi)$. Hence, by Galois theory, the field Γ_rK is a Galois extension of K with $\mathcal{G}(\Gamma_rK/K) \cong \mathcal{G}(\Gamma_r/\Phi) \cong G_r$. We prove that Γ_rK/K, G_r and π are related in agreement with Lemma 2.3. Firstly, it is easy to see that Ψ_r equals the fixed field of the Frattini subgroup of $\mathcal{G}(\Gamma_r/\Phi)$. Secondly, it follows from the Artin-Schreier theorem and the definition of Ψ_r that every degree p extension of Φ_ω in $\Psi_r\Phi_\omega$ is totally ramified (relative to $\bar{\omega}$). Note also that $\widehat{\Phi}$ is finite, so the Henselity of $\bar{\omega}$ ensures that each finite extension Φ' of Φ_ω contains as a subfield an inertial lift of $\widehat{\Phi}'$ over Φ_ω. At the same time, $\bar{\omega}$ is discrete, which shows that Φ'/Φ_ω is defectless if it is separable (see [23], Ch. XII, Sect. 6, Corollary 2). These facts make it easy to deduce from (2.5) and Galois theory that $\Gamma_r\Phi_\omega/\Phi_\omega$ is totally ramified and $[\Gamma_rK\colon K] = [\Gamma_r\Phi_\omega\colon \Phi_\omega] = [\Gamma_r\colon \Phi] = p^{\mu(r)}$. Therefore, Γ_r/Φ is totally ramified, i.e. it possesses a primitive element θ whose minimal polynomial $f_\theta(X)$ over Φ is Eisensteinian relative to ω (cf. [16], Ch. 2, (3.6), and [23], Ch. XII, Sects 2, 3 and 6). Let θ_0 be the free term of $f_\theta(X)$. As $\pi \in \Phi$, $v(\pi) \notin pv(K)$ and Γ_r/Φ is a Galois extension, the conditions on θ guarantee that it is a primitive element of Γ_rK/K (and Γ_rK_v/K_v), $p^{\mu(r)}w(\theta) = v(\theta_0) = \omega(\theta_0)$ and $v(\pi) \in p^{\mu(r)}w(\Gamma_rK)$, for any valuation w of Γ_rK extending v. This implies w is unique, up-to an equivalence, and so completes the proof of Lemma 2.3. □

The conclusion of Lemma 2.3 need not be true in the mixed-characteristic setting. It has been established by Kurihara (cf. [20], Corollary 2) that there exists an HDV-field (K, v) with $\mathrm{char}(K) = 0$, $\widehat{K}$ imperfect and $\mathrm{char}(\widehat{K}) = p > 0$, which does not admit a totally ramified cyclic extension of degree p^t, for any sufficiently large $t \in \mathbb{N}$ depending on K.

3. Brauer p-dimensions in characteristic p. In this Section we consider index-exponent relations of p-algebras over Henselian fields of characteristic p. First we supplement Lemma 2.3 as follows:

Lemma 3.1. *Let (K, v) be a valued field with* $\mathrm{char}(K) = p > 0$ *and $v(K) \neq pv(K)$, and let $\tau(p)$ be defined as in Theorem 1.1. Suppose that L is a finite abelian extension of K in $K(p)$ satisfying the following conditions:*

(a) $[L\colon K] = p^m$ *and* $\mathcal{G}(L/K)$ *has period* $p^{m'}$ *and rank* t*;*

(b) L *has a unique, up-to an equivalence, valuation* v_L *extending* v*, and the group* $v_L(L)/v(K)$ *is cyclic of order* p^m*.*

Then there is $T \in d(K)$ *with* $\exp(T) = p^{m'}$*, possessing a maximal subfield* K*-isomorphic to* L*, except, possibly, in case* $\tau(p) < \infty$ *and* $p^{t-\tau(p)} \geq [\widehat{K}\colon \widehat{K}^p]$.

P r o o f. It is clear from Galois theory and the structure of finite abelian groups that $L = L_1 \dots L_t$ and $[L\colon K] = \prod_{j=1}^{t} [L_j\colon K]$, for some cyclic extensions L_j/K, $j = 1, \dots, t$. Take an element $\pi \in K$ so that $v(\pi) \in p^m v_L(L)$, put $\pi_0 = \pi$, and suppose that there exist $\pi_j \in K^*$, $j = 1, \dots, t$, and $\mu \in \mathbb{Z}$ with $0 \leq \mu \leq t$, such that the cosets $v(\pi_i) + pv(K)$, $i = 0, \dots, \mu$, are linearly independent over $\mathbb{F}_p$, and in case $\mu < t$, $v(\pi_u) = 0$ and the residue classes $\hat{\pi}_u$, $u = \mu+1, \dots, t$, generate an extension of $\widehat{K}^p$ of degree $p^{t-\mu}$ (this assumption is admissible unless $\tau(p) \leq t$ and $p^{t-\tau(p)} \geq [\widehat{K}\colon \widehat{K}^p]$). Fix a generator λ_j of $\mathcal{G}(L_j/K)$, for $j = 1, \dots, t$, denote by T the K-algebra $\otimes_{j=1}^{t} (L_j/K, \lambda_j, \pi_j)$, where $\otimes = \otimes_K$, and put $T' = T \otimes_K K_v$. We show that $T \in d(K)$ (whence $\mathrm{ind}(T) = p^m$) and $\exp(T) = p^{m'}$. Clearly, $T' \cong \otimes_{j=1}^{t} (L'_j/K_v, \lambda'_j, \pi_j)$ over K_v, where $\otimes = \otimes_{K_v}$, $L'_j = L_j K_v$ and λ'_j is the unique K_v-automorphism of L'_j extending λ_j, for each j (as in the proof of Lemma 2.3, we identify K_v with its K-isomorphic copy in K_{sep}). Therefore, it suffices for the proof of Lemma 3.1 to show that $T' \in d(K_v)$. Since, by the proof of Lemma 2.3, K_v and $L' = LK_v$ are related as in our lemma, this amounts to proving that $T \in d(K)$, for (K, v) Henselian. Note that if $m = 1$, then our assertion is a special case of [6], Lemma 4.2. Henceforth, we assume that $m \geq 2$ and view all value groups considered in the rest of the proof as (ordered) subgroups of a fixed divisible hull of $v(K)$. Let L_0 be the degree p extension of K in L_t, and $R_j = L_0 L_j$, $j = 1, \dots, t$. Put $\rho_t = \lambda_t^p$, and when $t \geq 2$, denote by ρ_j the unique L_0-automorphism of R_j extending λ_j, for $j = 1, \dots, t-1$. Then the centralizer C of L_0 in T is L_0-isomorphic to $\otimes_{j=1}^{t} (R_j/L_0, \rho_j, \pi_j)$, where $\otimes = \otimes_{L_0}$; in particular, $\deg(C) = p^{m-1}$. Using (2.1), Lemma 2.3 and this result, one easily obtains that it is sufficient to prove that $T \in d(K)$, under the extra hypothesis that $C \in d(L_0)$.

Let w be the valuation of C extending v_{L_0}, $\widehat{C}$ its residue division ring, and for each $\xi \in C$ with $w(\xi) = 0$, let $\widehat{\xi} \in \widehat{C}$ be the residue class of ξ. It follows from the Ostrowski-Draxl theorem that $w(C)$ equals the sum of $v(L)$ and the group generated by $[R_{i'}\colon L_0]^{-1} v(\pi_{i'})$, $i' = 1, \dots, \mu$. Similarly, it is proved that $\widehat{C}/\widehat{K}$ is a purely inseparable field extension. Moreover, one sees that $\widehat{C} \neq \widehat{K}$ if and only if $\mu < t$, and when this is the case, $[\widehat{C}\colon \widehat{K}] = \prod_{u=\mu+1}^{t} [R_u\colon L_0]$ and

$\widehat{C} = \widehat{K}(\hat{\eta}_{\mu+1}, \dots, \hat{\eta}_t)$, where $\eta_u \in (R_u/L_0, \rho_u, \pi_u)$ is a root of π_u of degree $[R_u : L_0]$ acting on R_u by conjugation as the automorphism ρ_u, for each index u. In view of (2.1) and well-known general properties of purely inseparable finite extensions (cf. [23], Ch. VII, Sect. 7), these results show that $w(\eta_t) \notin pw(C)$, if $\mu = t$, and $w(\eta_t) = 0$ and $\hat{\eta}_t \notin \widehat{C}^p$, otherwise. Observe now that there is a K-isomorphism $\bar{\rho}_t$ of C extending λ_t, such that $\bar{\rho}_t^p(\bar{c}) = \eta_t \bar{c} \eta_t^{-1} \colon \bar{c} \in C$, and $\bar{\rho}_t(\eta_t) = \eta_t$. This implies $w(c) = w(\bar{\rho}_t(c))$, for each $c \in C$, the products $c' = \prod_{\kappa=0}^{p-1} \bar{\rho}_t^{\kappa}(c)$, $c \in C$, have values $w(c') \in pw(C)$, and $\hat{c}' \in \widehat{C}^p$, if $w(c) = 0$. Therefore, $c' \neq \eta_t$, for any $c \in C$, so it follows from [1], Ch. XI, Theorems 11 and 12, that $T \in d(K)$. Let now Λ be the fixed field of the maximal subgroup of $\mathcal{G}(L/K)$ of period p. Then [27], Sect. 15.1, Corollary b, implies the class $p[D] \in \mathrm{Br}(K)$ is represented by a crossed product of Λ/K, defined similarly to D. As Λ/K and π are related like L/K and π, and $\mathcal{G}(\Lambda/K)$ is of period $p^{m'-1}$, this enables one to prove inductively that $\exp(D) = p^{m'}$, as claimed. □

Corollary 3.2. *Let E be a field with $\mathrm{char}(E) = p > 0$ and $[E \colon E^p] = p^\nu < \infty$, and F/E a finitely-generated extension of transcendency degree $n > 0$. Then $n + \nu - 1 \leq \mathrm{Brd}_p(F) \leq \mathrm{abrd}_p(F) \leq n + \nu$, and when $n + \nu \geq 2$, $(p^t, p^s) \colon t, s \in \mathbb{N}, s \leq t \leq (n + \nu - 1)s$, are index-exponent pairs over F.*

Proof. We have $n + \nu - 1 \leq \mathrm{Brd}_p(F) \leq \mathrm{abrd}_p(F) \leq n + \nu$, by [6], Theorem 2.1 (c). Note also that F has a valuation v trivial on E, such that $v(F) = \mathbb{Z}^n$ and $\widehat{F}$ is a finite extension of E (see, e.g. [6], (4.1)). Therefore, $[\widehat{F} \colon \widehat{F}^p] = p^\nu$ (cf. [23], Ch. VII, Sect. 7) and $v(F)/pv(F)$ is of order p^n, which makes it easy to deduce the concluding assertion of Corollary 3.2 from Lemma 3.1. □

Remark 3.3. It is known [28], (3.19) (see also [17], Corollary 6.10) that if (K, v) is a Henselian field and $T \in d(K)$ is a tame K-algebra, in the sense of [28] or [17], then the period $\mathrm{per}(T/K)$ of the group $v(T)/v(K)$ divides $\exp(T)$. At the same time, by Lemma 3.1 with its proof, if $\mathrm{char}(K) = p > 0$ and $v(K)/pv(K)$ is infinite, then there are $T_n \in d(K)$, $n \in \mathbb{N}$, such that $\mathrm{ind}(T_n) = per(T_n/K) = p^n$, $\exp(T_n) = p$ and T_n/K is defectless, for each n.

Next we describe index-exponent p-primary pairs over some maximally complete fields of characteristic p, including those with perfect residue fields.

Proposition 3.4. *Let (K, v) be a valued field of characteristic $p > 0$. Suppose that $v(K)/pv(K)$ is infinite or $[\widehat{K} \colon \widehat{K}^p] = \infty$, where $\widehat{K}^p = \{\hat{a}^p \colon \hat{a} \in \widehat{K}\}$. Then $(p^k, p^n) \colon k, n \in \mathbb{N}, n \leq k$, are index-exponent K-pairs.*

P r o o f. Lemma 3.1, [8], Remark 4.3, and our assumptions show that there are tensor products $D_n \in d(K)$, $n \in \mathbb{N}$, of degree p cyclic K-algebras with $\exp(D_n) = p$ and $\mathrm{ind}(D_n) = p^n$, for each n. Hence, by [7], Lemma 5.2, it suffices to prove that (p^n, p^n), $n \in \mathbb{N}$, are index-exponent K-pairs. By Witt's lemma (cf. [11], Sect. 15, Lemma 2), each cyclic extension L of K in $K(p)$ lies in $I(L'/K)$, for some $\mathbb{Z}_p$-extension L' of K in $K(p)$. Fix a topological generator σ of $\mathcal{G}(L'/K)$, and for any $n \in \mathbb{N}$, let L_n be the extension of K in L' of degree p^n, and σ_n the automorphism of L_n induced by σ. Clearly, L_n/K is cyclic and σ_n generates $\mathcal{G}(L_n/K)$. Choosing L' so that $(L_1/K, \sigma_1, c) \cong D_1$, for some $c \in K^*$, one gets $\mathrm{ind}(\Delta_n) = \exp(\Delta_n) = p^n$ from [27], Sect. 15.1, Corollary b, for the cyclic K-algebras $\Delta_n = (L_n/K, \sigma_n, c)$, $n \in \mathbb{N}$, which completes our proof. □

Proposition 3.5. *Let (K, v) be a maximally complete field with $\mathrm{char}(K) = p > 0$, $v(K) \neq pv(K)$ and $[K\colon K^p] = p^n$, for some $n \in \mathbb{N}$, and let G_p be a Sylow pro-p-subgroup of $\mathcal{G}(\widehat{K}_{\mathrm{sep}}/\widehat{K})$. Then $n - 1 \leq \mathrm{Brd}_p(K) \leq n$. Moreover, the following holds when $\widehat{K}$ is perfect:*

(a) $\mathrm{Brd}_p(K) = n - 1$ *if and only if* $n > r_p(\widehat{K})$;

(b) $(p^k, p^s)\colon k, s \in \mathbb{N}$, $s \leq k \leq \mathrm{Brd}_p(K)s$, *are index-exponent K-pairs.*

(c) $\mathrm{abrd}_p(K) = n-1$ *if and only if either* $G_p = \{1\}$ *or* $n \geq 2$ *and* $G_p \cong \mathbb{Z}_p$.

P r o o f. Our assumptions show that $[K\colon K^p] = [\widehat{K}\colon \widehat{K}^p]e(K/K^p)$ (cf. [37], Theorem 31.21), so it follows from Lemma 3.1 and Albert's theory of p-algebras [1], Ch. VII, Theorem 28, that $n - 1 \leq \mathrm{Brd}_p(K) \leq n$, as claimed. In the rest of the proof, we suppose that $\widehat{K}$ is perfect. First we consider the case of $r_p(\widehat{K}) \geq n$. Then one gets from Galois theory and Witt's lemma that $\mathbb{Z}_p^n$ is realizable as a Galois group over $\widehat{K}$. Hence, by [33], Theorem A.24, there is a Galois extension U_n of K in K_{ur} with $\mathcal{G}(U_n/K) \cong \mathbb{Z}_p^n$. This implies each finite abelian p-group H of rank $\leq n$ is isomorphic to $\mathcal{G}(U_H/K)$, for some Galois extension U_H of K in U_n. Observing also that $v(K)/pv(K)$ has order p^n, and using [17], Example 4.3, one proves the existence of an NSR-algebra $N_H \in d(K)$ with a maximal subfield $U'_H \cong U_H$ over K. Therefore, $\exp(N_H) = \mathrm{per}(H)$ and $\mathrm{ind}(N_H) = [U_H\colon K]$, so $\mathrm{Brd}_p(K) = n$, which reduces the rest of our proof to the case of $n > r_p(\widehat{K})$. Note that $(L', v_{L'})$ is maximally complete and $[L'\colon L'^p] = p^n$ whenever L'/K is a finite extension (cf. [37], Theorem 31.22, and [23], Ch. VII, Sect. 7). This enables one to deduce from [2], Theorem 3.3, by the method of proving [8], (5.5), that for each $D_e \in d(K)$ with $\exp(D_e) = p^e$, where $e \in \mathbb{N}$, $[D_e] \in \mathrm{Br}(K_e/K)$, for some purely inseparable extension K_e/K such that $[K_e\colon K] \mid p^{(n-1)e}$. In view of (1.1) (b), the obtained result yields $\mathrm{ind}(D_e) \mid p^{(n-1)e}$ and $\mathrm{Brd}_p(K) = n - 1$, so Proposition 3.5 (a) is proved. Applying Lemmas 2.3 and 3.1, one concludes that (p^t, p^m), $t, m \in \mathbb{N}$, $0 < m \leq t \leq (n-1)m$, are index-exponent K-pairs, which reduces Proposition 3.5 (b) to a consequence of Proposition 3.5 (a). It remains for us to prove Proposition 3.5 (c). Clearly, if $G_p = \{1\}$, then $r_p(\widehat{L}) = 0$, for every $L \in \mathrm{Fe}(K)$. At the same time, it follows from Galois cohomology and Nielsen-Schreier's formula for open subgroups of free pro-p-groups (cf. [32], Ch. I, 3.3, 4.2; Ch. II, 2.2) that if G_p is not procyclic, then $r_p(K_1) \geq n$, for some finite extension K_1 of K in K_{ur}. Note finally that if G_p has rank 1 as a pro-p-group, then its open subgroups are isomorphic to $\mathbb{Z}_p$, which implies $r_p(L) \leq 1$, $L \in \mathrm{Fe}(K)$. As (L, v_L) is maximally complete and $[L\colon K] = p^n$, these facts give us the possibility to deduce Proposition 3.5 (c) from Proposition 3.5 (a). □

We are now prepared to generalize Proposition 1.2 as follows.

Corollary 3.6. *Let (K, v) be a maximally complete field with $\mathrm{char}(K) = p > 0$ and $\tau(p)$ defined as in Theorem 2.1. Suppose further that $\widehat{K}$ is complete with respect to a discrete valuation ω with a quasifinite residue field $\tilde{k}$. Then:*

(a) $\mathrm{Brd}_p(K) = \infty$ *if and only if* $\tau(p) = \infty$*; when this holds, (p^k, p^n) is an index-exponent pair over K, for any $k, n \in \mathbb{N}$ with $k \geq n$;*

(b) $\mathrm{Brd}_p(K) = \tau(p)$*, provided that $\tau(p) < \infty$; in this case, (p^k, p^n) is an index-exponent K-pair, where $k, n \in \mathbb{N}$, if and only if $n \leq k \leq n\tau(p)$.*

Proof. It is known (cf. [14], Sect. 5.2) that K has a valuation φ (a refinement of v), such that $\varphi(K) = v(K) \oplus \omega(\widehat{K})$, $\omega(\widehat{K})$ is an isolated subgroup of $\varphi(K)$, v and ω are canonically induced by φ and $\omega(\widehat{K})$ on K and $\widehat{K}$, respectively, and $\widehat{K}_\varphi \cong \tilde{k}$, where $\widehat{K}_\varphi$ is the residue field of (K, φ). Observing that, by theorems of Krull and Hasse-Schmidt-MacLane (cf. [14], Theorems 12.2.3, 18.4.1, and [37], Theorem 31.24 and page 483), $(\widehat{K}, \omega)$ is maximally complete and (K, φ) possesses an immediate extension (K', φ') which is a maximally complete field, one obtains that $(K', \varphi') = (K, \varphi)$. As $r_p(\tilde{k}) = 1$ and $\tilde{k}$ is perfect, Corollary 3.6 can now be deduced from Propositions 3.4 and 3.5. □

When (K, v) is a Henselian field, such that $\mathrm{char}(K) = p > 0$, $v(K)$ is a non-Archimedean group, $v(K)/pv(K)$ is finite and $[\widehat{K}\colon \widehat{K}^p] = p^\nu < \infty$, there is, generally, no formula for $\mathrm{Brd}_p(K)$ involving only invariants of $\widehat{K}$ and $v(K)$. This is illustrated below in the case of $v(K) = \mathbb{Z}^t$, for any integer $t \geq 2$.

Example 3.7. Let Y_0 be a field with $\mathrm{char}(Y_0)=p$ and $[Y_0 : Y_0^p]=p^\nu<\infty$, and let $Y_t = Y_0((T_1))\dots((T_t))$ be the iterated formal Laurent power series field in t variables over Y_0. Denote by w_t the natural $\mathbb{Z}^t$-valued valuation of Y_t trivial on Y_0. It is known (see [3], page 181 and further references there) that there exist elements $X_n \in Y_{t-1}$, $n \in \mathbb{N}$, algebraically independent over the field $Y_{t-2}(T_{t-1})$, where $Y_{t-2} = Y_0((T_1))\dots((T_{t-2}))$ in the case of $t \geq 3$. Put $F_n = Y_{t-2}(T_{t-1}, X_1, \dots X_n)$, for each $n \in \mathbb{N}$, $F_\infty = \cup_{n=1}^\infty F_n$, and $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. For any $n \in \mathbb{N}_\infty$, denote by F'_n the separable closure of F_n in Y_{t-1}, and by v_n the valuation of the field $K_n = F'_n((T_t))$ induced by w_t. It is easily verified that (K_n, v_n) is Henselian, $v_n(K_n) = \mathbb{Z}^t$ and $\widehat{K}_n = Y_0$, for each index n. Note also that $[F'_\infty \colon F'^p_\infty] = \infty$, so Proposition 3.4, applied to the valuation of K_n induced by the natural discrete valuation of Y_t trivial on Y_{t-1}, yields $\mathrm{Brd}_p(K_\infty) = \infty$. When $n \in \mathbb{N}$, we have $[K_n \colon K_n^p] = p^{\nu+t+n} = p[F'_n \colon F'^p_n]$, which enables one to deduce from Lemma 3.1, [6], Lemma 4.1, and [1], Ch. VII, Theorem 28 (see also [23], Ch. VII, Sect. 7) that $\nu + t + n - 1 \leq \mathrm{Brd}_p(K_n) \leq \nu + n + t$.

4. Brauer *p*-dimensions of Henselian fields with *p*-quasilocal residue fields. Let (K, v) be a Henselian field with $\widehat{K}$ p-quasilocal and $r_p(\widehat{K}) > 0$. Then $\mathrm{Brd}_p(\widehat{K}) \leq 1$, so Theorem 2.1 yields $\mathrm{Brd}_p(K) = \infty$ if and only if $m_p = \infty$ or $\tau(p) = \infty$ and $\varepsilon_p \in \widehat{K}$. When $\mathrm{Brd}_p(K) = \infty$, index-exponent p-primary K-pairs are described by Corollary 2.2 (and the Pythagorean property of formally real 2-quasilocal fields, see [5], I, Lemma 3.5). The main result of this Section concerns the case of $\mathrm{Brd}_p(K) < \infty$ and can be stated as follows:

Theorem 4.1. *Let (K, v) be a Henselian field with* $\mathrm{Brd}_p(K) < \infty$, *for some $p \in \mathbb{P}$, and set ε_p, $\tau(p)$ and m_p as in Theorem 2.1. Suppose that $\widehat{K}$ is p-quasilocal, $p \neq \mathrm{char}(\widehat{K})$ and $m_p > 0$. Then:*

(a) $\mathrm{Brd}_p(K) = u_p$, *where $u_p = [(\tau(p) + m_p)/2]$, if $\varepsilon_p \in \widehat{K}$ and $\widehat{K}$ is a nonreal field; $u_p = m_p$, if $\varepsilon_p \notin \widehat{K}$;*

(b) $\mathrm{Br}(K)_2$ *is a group of period* 2 *and* $\mathrm{Brd}_2(K) = 1 + [\tau(2)/2]$, *provided that $\widehat{K}$ is formally real and $p = 2$.*

Before proving Theorem 4.1, note that it yields $\mathrm{Brd}_p(K) = \tau(p)$ whenever $r_p(\widehat{K}) = \infty$. This holds in all presently known cases where $\widehat{K}$ is p-quasilocal and $\mathrm{Br}(\widehat{K})_p$ does not embed in $\mathbb{Q}/\mathbb{Z}$ or, equivalently, in the quasicyclic p-group $\mathbb{Z}(p^\infty)$ (see [35], the end of Sect. 3, [9], Theorem 1.2, and e.g., [25], [34]).

Proof of Theorem 4.1. Suppose first that $\widehat{K}$ is formally real and $p = 2$. Then, by [5], I, Lemma 3.5, $\widehat{K}$ is Pythagorean, $\widehat{K}(2) = \widehat{K}(\sqrt{-1})$ and $\mathrm{Br}(\widehat{K})_2$ is of order 2. Therefore, $r_2(\widehat{K}) = 1$ and $r_2(\widehat{K}(\sqrt{-1})) = 0$, so it follows from the Merkur'ev-Suslin theorem [24], (16.1), that $\mathrm{Br}(\widehat{K}(\sqrt{-1}))_2 = \{0\}$. Note further that K is Pythagorean, which implies $2\,\mathrm{Br}(K) = \{0\}$ (cf. [22], Theorem 3.16, and [13], Theorem 3.1). These observations and [8], Corollary 5.5, prove Theorem 4.1 (b). We turn to the proof of Theorem 4.1 (a), so we assume that $p > 2$ or $\widehat{K}$ is a nonreal field. Then $\mathrm{Br}(\widehat{K})_p$ is a divisible group, by [5], I, Theorem 3.1. Our argument also relies on the following results concerning inertial algebras $I \in d(K)$ with $[I] \in \mathrm{Br}(K)_p$, and inertial extensions U of K in $K(p)$:

(4.1) (a) $\mathrm{ind}(I) = \exp(I)$ and I is a cyclic K-algebra;
(b) $[I] \in \mathrm{Br}(U/K)$ if and only if $\mathrm{ind}(I) \mid [U\colon K]$; U is embeddable in I as a K-subalgebra if and only if $[U\colon K] \mid \mathrm{ind}(I)$;
(c) $\mathrm{ind}(I \otimes_K I')$ equals $\mathrm{ind}(I)$ or $\mathrm{ind}(I')$, if $I' \in d(K)$, I'/K is NSR, and $[I'] \in \mathrm{Br}(K)_p$.

Statements (4.1) can be deduced from (1.3), (2.3) (b) and [17], Theorems 3.1 and 5.15. They imply in conjunction with [8], Lemma 4.1, that $\mathrm{ind}(W) \mid \exp(W)^{m_p}$, for each $W \in d(K)$ inertially split over K. At the same time, it follows from [6], (3.3) and (3.6), and [26], Theorem 1 (see also [17], Example 4.3), that there is an NSR-algebra $W' \in d(K)$ with $\mathrm{ind}(W') = p^{m_p}$ and $\exp(W') = p$. Observe now that, by (2.3) (c), $\mathrm{Br}(K)_p \subseteq \mathrm{Br}(K_{\mathrm{ur}}/K)$ in case $\varepsilon_p \notin \widehat{K}$ or $\tau(p) = 1$. In view of (4.1) and [17], Theorem 4.4 and Lemma 5.14, this yields $\mathrm{Brd}_p(K) = m_p$, so it remains for us to prove Theorem 4.1, under the extra hypothesis that $\varepsilon_p \in \widehat{K}$ and $\tau(p) \geq 2$. It is easily obtained from [26], Theorem 1, and [8], Lemmas 4.1 and 4.2, that there exists $\Delta \in d(K)$ with $\exp(\Delta) = p$ and $\mathrm{ind}(\Delta) = p^{\mu(p)}$, where $\mu(p) = [(m_p + \tau(p))/2]$. This means that $\mathrm{Brd}_p(K) \geq \mu(p)$, so we have to prove that $\mathrm{Brd}_p(K) \leq \mu(p)$. Note first that $2 \leq m_p$, provided $\mathrm{Br}(\widehat{K})_p \neq \{0\}$. Assuming the opposite and taking into account that $\varepsilon_p \in \widehat{K}$, one obtains from the other conditions on $\widehat{K}$ that it is a nonreal field with $r_p(\widehat{K}) = 1$. Hence, by [39], Theorem 2, $\widehat{K}(p)/\widehat{K}$ is a $\mathbb{Z}_p$-extension. In view of [24], (11.5) and (16.1), and Galois cohomology (cf. [32], Ch. I, 4.2), this requires that $\mathrm{Br}(\widehat{K})_p = \{0\}$. As $\tau(p) \geq 2$, the obtained contradiction proves that $r_p(\widehat{K}) \geq m_p \geq 2$, as claimed. Now take an algebra $D \in d(K)$ so that $\exp(D) = p^n$, for some $n \in \mathbb{N}$, attach S, V and $T \in d(K)$ to D as in (2.3) (a), and fix $\Theta \in d(K)$ so that $[\Theta] = [V \otimes_K T]$. To prove that $\mathrm{ind}(D) \mid p^{n\mu(p)}$ we need the following statements:

(4.2) (a) If $n = 1$, then S, V and T can be chosen so that $V \otimes_K T = \Theta$, and $S = K$ or $V = K$.
(b) If $n \geq 2$, then there is a totally ramified extension Y of K in $K(p)$, such that $[Y\colon K] \mid p^{\mu(p)}$ and either $\exp(D_Y) \mid p^{n-1}$, or $\exp(D_Y) = \exp(S_Y) = p^n$, $[Y\colon K] \mid p^{[\tau(p)/2]}$ and $\exp(V_Y \otimes_Y T_Y) \mid p^{n-1}$, where $S_Y, V_Y, T_Y \in d(Y)$ are attached in accordance with (2.3) (a) to a representative $D_Y \in d(Y)$ of $[D \otimes_K Y]$.

Statement (4.2) (a) can be deduced from (4.1), [8], (4.7), and well-known properties of cyclic algebras (cf. [27], Sect. 15.1, Proposition b). Since $m_p \geq 2$, (4.2)

(a) implies the assertion of Theorem 4.1 (a) in the case of $n = 1$, so we assume further that $n \geq 2$. The conclusion of (4.2) (b) is obvious, if $\exp(\Theta) \mid p^{n-1}$ (one may put $Y = K$). Therefore, by (2.4) (c), it suffices to prove (4.2) (b) under the hypothesis that $\exp(\Theta) = p^n$. Take $D_{n-1} \in d(K)$ so that $[D_{n-1}] = p^{n-1}[D]$ and attach to it a triple S_{n-1}, V_{n-1}, $T_{n-1} \in d(K)$ in agreement with (4.2) (a). Then $V_{n-1} \otimes_K T_{n-1}$ contains as a maximal subfield an abelian and totally ramified extension Y of K. Observing that $[V_{n-1} \otimes_K T_{n-1}] \in \mathrm{Br}(Y/K)$, identifying Y with its K-isomorphic copy in $K(p)$, and using (2.4) (a) and (1.1) (a), one sees that it has the properties required by (4.2) (b).

We continue with the proof of Theorem 4.1 (a). In view of (2.2) and (4.2) (a), a standard inductive argument allows us to proceed under the extra hypothesis that $\mathrm{ind}(D') \mid \exp(D')^{\mu(p)}$, for each $D' \in d(K')$ with $\exp(D') \mid p^{n-1}$, where K'/K is an arbitrary totally ramified finite extension. It is known (cf. [17], Corollary 6.8) that if $J, J' \in d(K)$, J/K is inertial and $[J'] = [J \otimes_K \Theta]$, then $v(J') = v(\Theta)$, $Z(\widehat{J'}) = Z(\widehat{\Theta})$ and $[\widehat{J'}] = [\widehat{J} \otimes_{\widehat{K}} \widehat{\Theta}] \in \mathrm{Br}(Z(\widehat{\Theta}))$. Note also that the period of the group $v(J')/v(K)$ divides $\exp(J')$ (see Remark 3.3). At the same time, by [5], I, Theorem 4.1, the scalar extension map $\mathrm{Br}(\widehat{K}) \to \mathrm{Br}(Z(\widehat{\Theta}))$ induces a surjective homomorphism $\mathrm{Br}(\widehat{K})_p \to \mathrm{Br}(Z(\widehat{\Theta}))_p$. As $\mathrm{Brd}_p(\widehat{K}) \leq 1$ and $m_p \geq 2$, these results, combined with (1.3), (4.1) (a), (b), the Ostrowski-Draxl theorem, and the inductive hypothesis, prove the following:

(4.3)
(a) If $\exp(\Theta) \mid p^{n-1}$, then $\mathrm{ind}(D) \mid p.\mathrm{ind}(S_0 \otimes_K V \otimes_K T)$, for some $S_0 \in d(K)$ inertial over K with $\exp(S_0) \mid p^{n-1}$;
(b) If $\exp(\Theta) \mid p^{n-1}$ and $\mathrm{ind}(D) > \mathrm{ind}(I \otimes_K V \otimes_K T)$ whenever $I \in d(K)$, $[I] \in \mathrm{IBr}(K)$ and $\exp(I) \mid p^{n-1}$, then $[Z(\widehat{D})\colon \widehat{K}] = p^k$ and $[\widehat{D}\colon Z(\widehat{D})] = p^{2n-2k}$, for some $k \in \mathbb{Z}$ with $0 \leq k < n$; hence, $\mathrm{ind}(D)^2 \mid p^{2n}e(\Theta/K) \mid p^{2n}\exp(\Theta)^{\tau(p)}$, which yields $\mathrm{ind}(D)^2 \mid p^{2n+(n-1)\tau(p)} \mid p^{m_p n+(n-1)\tau(p)}$.

Now fix an extension Y/K and Y-algebras D_Y, S_Y, V_Y, T_Y as in (4.2) (b), and take $\Theta_Y \in d(Y)$ so that $[\Theta_Y] = [V_Y \otimes_Y T_Y]$. Observing that, by (1.1) (b), $\mathrm{ind}(D) \mid \mathrm{ind}(D_Y)[Y\colon K]$, and applying (4.3) in case $\exp(D_Y) = p^n$ to D_Y, V_Y, T_Y and Θ_Y, instead of D, V, T and Θ, respectively, one concludes that $\mathrm{ind}(D)^2 \mid p^{n(m_p+\tau(p))}$. Theorem 4.1 is proved. □

Theorem 4.1 (a) retains its validity, if (K, v) is a Henselian field, such that $\tau(p) < \infty$, $r_p(\widehat{K}) = 0$ and $\mu_p(\widehat{K}) \neq \{1\}$. Then it follows from [24], (16.1), that $\mathrm{Brd}_p(\widehat{K}) = 0$, so Theorem 2.1 (a) implies $\mathrm{Brd}_p(K) = [\tau(p)/2]$.

Remark 4.2. Let (K, v) be a Henselian field with $\widehat{K}$ formally real and 2-quasilocal. Then the symbol K-algebra $D' = A_{-1}(-1, -1; K)$ lies in $d(K)$, and it follows from [8], Lemma 4.2, that if $\tau(2) \geq 2$, then there exist $D_n \in d(K)$, $n = 1, \dots, [\tau(2)/2]$, totally ramified over K with $\exp(D_n) = 2$ and $\mathrm{ind}(D_n) = 2^n$, for each n. As D'/K is inertial, this implies together with [26], Theorem 1, that $D' \otimes_K D_n \in d(K)$ (and $\mathrm{ind}(D' \otimes_K D_n) = 2^{n+1}$), $n = 1, \dots, [\tau(2)/2]$. In view of (2.3) (b) and Theorem 4.1 (b), these facts prove that if $0 \leq \tau(2) < \infty$, then $(1, 1)$ and $(2^n, 2)$, $n = 1, \dots, 1 + [\tau(2)/2]$, are all index-exponent 2-primary K-pairs.

Corollary 4.3. *Let K_m be an m-dimensional local field with a quasifinite m-th residue field K_0, for some $m \in \mathbb{N}$. Suppose that $p \in \mathbb{P}$ is different from* $\mathrm{char}(K_0)$, *and ε_p is a primitive p-th root of unity in $K_{0,\mathrm{sep}}$. Then* $\mathrm{Brd}_p(K_m) = [(1+m)/2]$, *if $\varepsilon_p \in K_0$;* $\mathrm{Brd}_p(K_m) = 1$, *otherwise.*

P r o o f. This is in fact a special case of Theorem 4.1, since our assumptions imply the existence of a Henselian $\mathbb{Z}^m$-valued valuation on K_m with $\widehat{K}_m = K_0$. □

When $\varepsilon_p \in K_0$, the equality $\mathrm{Brd}_p(K_m) = [(1+m)/2]$ can also be obtained from [6], Lemma 4.1, and Khalin's formula for the number of isomorphism classes of K_m-algebras $D_{p,k} \in d(K_m)$ with $\exp(D_{p,k}) = p$ and $\mathrm{ind}(D_{p,k}) = p^k$, for a fixed $k \in \mathbb{N}$ (Khalin's formula has been deduced in [18], under the hypothesis that K_0 is finite, but it clearly holds in the setting of Corollary 4.3 as well).

Proposition 4.4. *Let K_m be an m-dimensional local field with* $\mathrm{char}(K_m) = 0$, *K_0 finite and* $\mathrm{char}(K_0) = p$. *Then $m - 1 \leq \mathrm{abrd}_p(K_m) \leq m$. Moreover,* $\mathrm{Brd}_p(K_m) \geq m - 1$ *unless $m \geq 4$,* $\mathrm{char}(K_1) = 0$ *and $r_p(K_1) < m - 1$, where K_1 is the last but one residue field of K_m.*

P r o o f. Note that if $m = 1$, then $\mathrm{Brd}_p(K_m) = \mathrm{abrd}_p(K_m) = 1$ (cf. [31], Ch. XIII, Sect. 3), which proves our assertions. We assume further that $m \geq 2$. It is well-known that finite extensions of K_m are m-dimensional local fields, so the equality $\mathrm{abrd}_p(K_m) \leq m$ reduces to a consequence of [7], Lemma 4.1, and the Corollary to [19], Theorem 2. To prove the other inequalities stated in Proposition 4.4, we consider the i-th residue field K_{m-i} of K_m, where $i \geq 0$ is the maximal integer for which $\mathrm{char}(K_{m-i}) = 0$. Clearly, if $i > 0$, then K_m has a $\mathbb{Z}^i$-valued Henselian valuation v_i with a residue field K_{m-i}. When $i = m - 1$, Theorem 4.1, applied to (K_m, v_i), gives a formula for $\mathrm{Brd}_p(K_m)$, which indicates that $\mathrm{Brd}_p(K_m) \leq m - 1$ and equality holds if and only if $r_p(K_1) \geq m - 1$. This, combined with [32], Ch. II, Theorems 3 and 4 (applied to finite extensions of K_1), proves that $\mathrm{abrd}_p(K_m) = m - 1$. Thus it follows that $\mathrm{Brd}_p(K_m) = m - 1$ in case $m \leq 3$. It remains to be seen that $\mathrm{Brd}_p(K_m) \geq m - 1$, provided that $i < m - 1$. Then $K_{m-i'}$, $i' = i, i + 1$, is an $(m - i')$-dimensional local field with last residue field K_0; in particular, $K_{m-i'}$ is complete with respect to a discrete valuation $\omega_{m-i'}$ whose residue field is $K_{m-i'-1}$. In view of Lemma 2.3 and Proposition 3.5, this means that $r_p(K_{m-i-1}) = \infty$, and in the case where $i < m-2$, $\mathrm{Brd}_p(K_{m-i-1}) = m-i-2$. More precisely, there exist $D_0 \in d(K_{m-i-1})$, defined as in the proof of Lemma 3.1 when $i < m-2$ (and equal to K, if $i = m-2$), and totally ramified Galois extensions M'_n/K_{m-i-1}, $n \in \mathbb{N}$, relative to ω_{m-i-1}, such that $\deg(D_0) = e(D_0/K_{m-i-1}) = p^{m-i-2}$, $[D_0] \in {}_p\mathrm{Br}(K_{m-i-1})$, $\widehat{D}_0$ is a field with $\widehat{D}_0^p \subseteq \widehat{K}$, and for each index n, $D_0 \otimes_{K_{m-i-1}} M'_n \in d(M'_n)$ and $\mathcal{G}(M'_n/K_{m-i-1})$ is elementary abelian of order p^n. Let D and M_n be inertial lifts over K_{m-i} (relative to ω_{m-i}) of D_0 and M'_n, respectively. Then M_n/K_{m-i} are inertial Galois extensions, $\mathcal{G}(M_n/K_{m-i}) \cong \mathcal{G}(M'_n/K_{m-i-1})$ and $D \otimes_{K_{m-i}} M_n$ lies in $d(M_n)$, for every $n \in \mathbb{N}$. This enables one to deduce (in the spirit of the proof of [8], Proposition 6.3) from [17], Example 4.3 (or [7], (3.6) (a)), and [26], Theorem 1, that there exists $T \in d(K_{m-i})$ with $\deg(T) = p$, T/K_{m-i} NSR relative to ω_{m-i}, and $\Sigma \in d(K_{m-i})$, where $\Sigma = D \otimes_{K_{m-i}} T$. Clearly, $\exp(\Sigma) = p$ and $\deg(\Sigma) = p^{m-i-1}$, so $\mathrm{Brd}_p(K_{m-i}) \geq m-i-1$, proving Proposition 4.4 in case $i = 0$. Let finally $i > 0$. Considering inertial lifts over K_m relative to v_i of Σ and any $L_i \in I(M_{i+1}/K_{m-i})$ with $\Sigma \otimes_{K_{m-i}} L_i \in d(L_i)$ and $[L_i \colon K_{m-i}] = p^i$, one obtains similarly that $\mathrm{Brd}_p(K_m) \geq m-1$. □

The inequalities $m-1 \leq \mathrm{Brd}_p(K) \leq m$ also hold under the assumption that (K, v) is an HDV-field, $\mathrm{char}(K) = 0$ and $\mathrm{char}(\widehat{K}) = p > 0$, where $\widehat{K}$ is an $(m-1)$-dimensional local field with a finite last residue field, for some $m \geq 2$. The lower bound $\mathrm{Brd}_p(K) \geq m-1$ is obtained as in the proof of Proposition 4.4, and the inequality $\mathrm{Brd}_p(K) \leq m$ is implied by Proposition 4.4 and the injectivity of the scalar extension map $\mathrm{Br}(K) \to \mathrm{Br}(\widetilde{K})$, $\widetilde{K}$ being the completion of K with respect to v [10], Theorem 1.

5. Proof of Theorem 1.1. Let (K, v) be a Henselian field, $p \in \mathbb{P}$, $\widehat{K}(p)_{\mathrm{ab}}$ the maximal abelian extension of $\widehat{K}$ in $\widehat{K}(p)$, and $\mu_p(\widehat{K})$, $\mu_p(K)$ the groups of roots of unity of p-primary degrees lying in $\widehat{K}$ and K, respectively. First, we describe index-exponent p-primary K-pairs, assuming that $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ is a Demushkin group and $\mu_p(\widehat{K})$ is a nontrivial finite group.

Lemma 5.1. *Let (K, v) be a Henselian field containing a primitive p-th root of unity, for some $p \in \mathbb{P}$, $p \neq \mathrm{char}(\widehat{K})$. Suppose that $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ is a Demushkin pro-p-group, $\mu_p(\widehat{K})$ is a finite group of order p^ν, and $r_p(\widehat{K}) = r < \infty$. Put $r' = r-1$, $m' = \min\{\tau(p), r'\}$, and for each $n \in \mathbb{N}$, let $\nu_n = \min\{n, \nu\}$ and $\mu(p, n) = nm' + \nu_n(m_p - m' + [(\tau(p) - m_p)/2])$. Then (p^k, p^n), where $k, n \in \mathbb{N}$, is an index-exponent pair over K, if and only if $n \leq k \leq \mu(p, n)$.*

P r o o f. First we prove the following assertions:

(5.1) (a) $C(\widehat{K}(p)/\widehat{K})$ is isomorphic to the direct sum $\mathbb{Z}(p^{\infty})^{r'} \oplus \mathbb{Z}/p^{\nu}\mathbb{Z}$ and $\mathcal{G}(\widehat{K}(p)_{\rm ab}/\widehat{K}) \cong \mathbb{Z}_p^{r'} \oplus \mathbb{Z}/p^{\nu}\mathbb{Z}$;

(b) A cyclic extension M of $\widehat{K}$ in $\widehat{K}(p)$ lies in $I(M_{\infty}/\widehat{K})$, for some $\mathbb{Z}_p$-extension M_{∞} of $\widehat{K}$ in $\widehat{K}(p)$ if and only if there is $M' \in I(\widehat{K}(p)/M)$, such that $M'/\widehat{K}$ is cyclic and $[M'\colon M] = p^{\nu}$; this is the case if and only if $\mu_p(\widehat{K}) \subset N(M/\widehat{K})$.

The nontriviality of $\mu_p(\widehat{K})$ and the Demushkin property of $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ ensure that $r \geq 2$, $\widehat{K}$ is a p-quasilocal nonreal field (see [5], I, Lemma 3.8). Hence, by [5], I, Theorem 3.1, $\mathrm{Br}(\widehat{K})_p$ is divisible, which enables one to deduce from [24], (11.5), and the condition on the order of $H^2(\mathcal{G}(\widehat{K}(p)/\widehat{K}), \mathbb{F}_p)$ that $\mathrm{Br}(\widehat{K})_p \cong \mathbb{Z}(p^{\infty})$. The rest of the proof of (5.1) (a) relies on our assumption on $\mu_p(\widehat{K})$, which shows that $\widehat{K}$ contains a primitive p^{ν}-th root of unity δ not lying in $\widehat{K}^{*p}$. Consider an extension $\widehat{K}_{\delta}$ of $\widehat{K}$ obtained by adjunction of a p-th root of δ. It is easily verified that $\widehat{K}_{\delta}/\widehat{K}$ is a cyclic extension of degree p. As $\widehat{K}$ is p-quasilocal and $\mathrm{Br}(\widehat{K})_p \cong \mathbb{Z}(p^{\infty})$, this means that $\mathrm{Br}(\widehat{K}_{\delta}/\widehat{K})$ has order p. In view of Kummer theory, cyclic $\widehat{K}$-algebras of degree p are symbol algebras, so the noted fact indicates that there is a cyclic degree p extension $\widehat{K}'/\widehat{K}$, such that the cyclic $\widehat{K}$-algebra $(\widehat{K}'/\widehat{K}, \sigma', \delta)$ lies in $d(\widehat{K})$ (σ' is a generator of $\mathcal{G}(\widehat{K}'/\widehat{K})$). Therefore, by [27], Sect. 15.1, Proposition b, δ does not lie in the norm group $N(\widehat{K}'/\widehat{K})$. Applying Albert's height theorem to $\widehat{K}'/\widehat{K}$ (cf. [15], Sect. 2), one proves the nonexistence of a cyclic extension $\widehat{K}_1'/\widehat{K}$, such that $[\widehat{K}_1'\colon \widehat{K}] = p^{1+\nu}$ and $\widehat{K}' \in I(\widehat{K}_1'/\widehat{K})$. This result allows us to obtain from Galois theory that the complement $C(\widehat{K}(p)/\widehat{K}) \setminus p^{\nu}C(\widehat{K}(p)/\widehat{K})$ contains an element of order p. Similarly, it can be deduced from Kummer theory that $p^{\nu-1}C(\widehat{K}(p)/\widehat{K})$ contains all elements of $C(\widehat{K}(p)/\widehat{K})$ of order p. Observe now that the Demushkin condition on $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ ensures that $C(\widehat{K}(p)/\widehat{K}) \cong \mathbb{Z}(p^{\infty})^{r'} \oplus C$, for some cyclic p-group C (cf. [21], page 106). Summing-up the noted properties of $C(\widehat{K}(p)/\widehat{K})$, one concludes that $C \cong \mathbb{Z}/p^{\nu}\mathbb{Z}$ and so proves (5.1) (a). As to (5.1) (b), it is implied by (5.1) (a) and Albert's height theorem.

We continue with the proof of Lemma 5.1. Statement (2.3) (b), the isomorphism $\mathrm{Br}(\widehat{K})_p \cong \mathbb{Z}(p^{\infty})$, and the equality $\mathrm{Brd}_p(\widehat{K}) = 1$ imply that (p^m, p^m), $m \in \mathbb{N}$, are index-exponent pairs over both $\widehat{K}$ and K. In view of Theorem 4.1, this proves Lemma 5.1 in the case where $\tau(p) = 1$, so we assume that $\tau(p) \geq 2$. Suppose first that $n \in \mathbb{N}$ and $n \leq \nu$. Then, by Theorem 4.1, $\mathrm{ind}(\Delta_n) \mid p^{\mu(p,n)}$, for each $\Delta_n \in d(K)$ with $\exp(\Delta_n) = p^n$. Using [26], Theorem 1, and the natural bijection between $I(Y/K)$ and the set of subgroups of $v(Y)/v(K)$, for any finite abelian tamely and totally ramified extension Y/K (cf. [30], Ch. 3, Sect. 2), one obtains that, for each $k \in \mathbb{N}$ with $n \leq k \leq \mu(p,n)$, there exist an NSR-algebra $V_{n,k} \in d(K)$ and a totally ramified $T_{n,k} \in d(K)$, such that $V_{n,k} \otimes_K T_{n,k} \in d(K)$, $\exp(V_{n,k} \otimes_K T_{n,k}) = p^n$ and $\mathrm{ind}(V_{n,k} \otimes_K T_{n,k}) = p^k$. These observations and the former part of (1.1) (a) prove Lemma 5.1 when $n \leq \nu$. The rest of the proof is carried out by induction on $n \geq \nu$. The basis of the induction is provided by the preceding argument, which allows us to assume that $n > \nu$ and $\mathrm{ind}(X) \mid p^{\mu(p,(n-1))}$ whenever $X \in d(K)$ and $\exp(X) \mid p^{n-1}$. Fix an algebra $D \in d(K)$ so that $\exp(D) = p^n$ and attach to D a triple S, V, $T \in d(K)$ as in (2.3) (a). Clearly, if $\exp(V) \mid p^{n-1}$, then $\exp(V \otimes_K T) \mid p^{n-1}$, so (4.3) and the inductive hypothesis imply $\mathrm{ind}(D) \mid p^{1+\mu(p,(n-1))} \mid p^{\mu(p,n)}$, as claimed. In view of (2.4), it remains to consider the case where $\exp(V) = p^n$. Let Σ, $D_\nu \in d(K)$ satisfy $[\Sigma] = [S \otimes_K V]$ and $[D_\nu] = p^\nu[D]$ $(= p^\nu[\Sigma])$. Then, by (2.4) (c), $\exp(\Sigma) = p^n$, and it follows from (4.1) and [27], Sect. 15.1, Corollary b and Proposition b, that Σ/K is NSR. Note also that $\exp(D_\nu) \mid p^{n-\nu}$, and (2.3) (c) and [27], Sect. 15.1, Corollary b, imply D_ν/K is NSR; in particular, D_ν contains as a maximal subfield an inertial extension U_ν of K. By [17], Theorem 4.4, U_ν/K is abelian with $\mathcal{G}(U_\nu/K)$ of rank $u_\nu \leq \tau(p)$. Moreover, it follows from (5.1), Galois theory and [27], Sect. 15.1, Corollary b, that U_ν has a K-isomorphic copy in $I(U'_\nu/K)$, for the Galois extension U'_ν of K in K_{ur} with $\mathcal{G}(U'_\nu/K) \cong \mathbb{Z}_p^{r'}$. Therefore, $u_\nu \leq r'$, so [17], Theorem 4.4, proves the following:

(5.2) $\mathrm{ind}(D_\nu) \mid p^{(n-\nu)m'}$ and D_ν contains as a maximal subfield a K-isomorphic copy of a totally ramified extension Φ_ν of K in $K(p)$.

Statement (5.2) shows that $[D_\nu] \in \mathrm{Br}(\Phi_\nu/K)$, $[\Phi_\nu \colon K] = \mathrm{ind}(D_\nu)$ and $\widehat{\Phi}_\nu = \widehat{K}$. Hence, $\exp(D \otimes_K \Phi_\nu) \mid p^\nu$ and $r_p(\widehat{\Phi}_\nu) = r_p(\widehat{K})$, so it follows from (2.2) and Theorem 4.1 that $\mathrm{ind}(D \otimes_K \Phi_\nu) \mid p^{\nu\mu(p)}$, where $\mu(p) = [(m_p + \tau(p))/2]$. As $\mu(p,n) = (n-\nu)m' + \nu\mu(p)$, it is now easy to see that $\mathrm{ind}(D) \mid p^{\mu(p,n)}$, as required. Suppose finally that $(k,n) \in \mathbb{N}^2$ and $n \leq k \leq \mu(p,n)$. Then [17], Example 4.3, [26], Theorem 1, the above-noted properties of U'_ν, and those of intermediate fields of any finite abelian tamely and totally ramied extension of K, imply the existence of $D_{k,n} \in d(K)$ with $\mathrm{ind}(D_{k,n}) = p^k$ and $\exp(D_{k,n}) = p^n$. Moreover, one can ensure that $D_{k,n} \cong N_{k,n} \otimes_K D'_{k,n}$, for some $N_{k,n}$, $D'_{k,n} \in d(K)$, such that $N_{k,n}$ is NSR and $D'_{k,n}$ is totally ramified over K. Lemma 5.1 is proved. □

Next we show that, in the setting of (1.2) (a), $C(\widehat{K}(p)/\widehat{K})$ possesses a divisible subgroup with infinitely many elements of order p.

Lemma 5.2. *Let (E,ω) be an* HDV*-field with* $\mathrm{char}(E)=0$, $\widehat{E}$ *quasifinite and* $\mathrm{char}(\widehat{E})=p>0$, *and let $D(E(p)/E)$ be the maximal divisible subgroup of $C(E(p)/E)$. Then:*

(a) $r_p(E)=\infty$, *provided that $\widehat{E}$ is infinite;*

(b) $\mu_p(E)$ *is finite and* $C(E(p)/E)\cong D(E(p)/E)\oplus\mathbb{Z}/p^{\nu}\mathbb{Z}$, *where p^{ν} is the order of $\mu_p(E)$; in particular, $C(E(p)/E)=D(E(p)/E)$ if and only if $p^{\nu}=1$.*

Proof. (b): Let ε be a primitive p-th root of unity in E_{sep}. It is well-known that $[E(\varepsilon)\colon E]\mid p-1$ (cf. [23], Ch. VIII, Sect. 3). Note also that every $E'\in\mathrm{Fe}(E)$ is a quasilocal field with $\mathrm{Br}(E')\cong\mathbb{Q}/\mathbb{Z}$; hence, the scalar extension map $\mathrm{Br}(E)\to\mathrm{Br}(E')$ is surjective. These facts, combined with (1.1) (b) and [27], Sect. 15.1, Proposition b, imply that if L is a cyclic p-extension of E in E_{sep}, then $L(\varepsilon)^*=L^*N(L(\varepsilon)/E(\varepsilon))$. When $\varepsilon\notin E$, this shows that $\varepsilon\in N(L(\varepsilon)/E(\varepsilon))$, which enables one to deduce from [15], Theorem 3, that $C(E(p)/E)=D(E(p)/E)$. Suppose now that $\mu_p(E)\neq\{1\}$ and denote by Γ_p the extension of E generated by the elements of $\mu_p(E_{\mathrm{sep}})$. It is known that, for any $n\in\mathbb{N}$, $\mathbb{Z}[X]$ contains the p^n-th cyclotomic polynomial $\Phi_{p^n}(X)$ (of degree $p^{n-1}(p-1)$), and the polynomial $\Phi_{p^n}(X+1)$ is p-Eisensteinian over $\mathbb{Z}$. This implies $p^{n-1}(p-1)\omega_{\Gamma_p}(\varepsilon_n-1)=\omega(p)$, for each $n\in\mathbb{N}$, $\varepsilon_n\in\Gamma_p$ being a primitive p^n-th root of unity. As ω is discrete and $\omega(p)\neq 0$, the noted fact proves that $\mu_p(E)$ is finite. In view of [5], II, Lemma 2.3, and the isomorphism $\mathrm{Br}(E)_p\cong\mathbb{Z}(p^{\infty})$, the obtained result yields $C(E(p)/E)\cong D(E(p)/E)\oplus\mathbb{Z}/p^{\nu}\mathbb{Z}$, as claimed by Lemma 5.2 (b).

(a): Assume that $\widehat{E}$ is infinite, fix a uniformizer $\pi\in E$ and elements $a_n\in E$, $n\in\mathbb{N}$, so that $\omega(a_n)=0$ and the residue classes $\hat{a}_n$, $n\in\mathbb{N}$, be linearly independent over the prime subfield $\mathbb{F}_p$ of $\widehat{E}$. It is easily verified that the cosets $(1+a_n\pi)E^{*p}$, $n\in\mathbb{N}$, are linearly independent over $\mathbb{F}_p$. This means that E^*/E^{*p} is an infinite group. At the same time, by local class field theory, if $L_1,\dots,L_n$ are cyclic extensions of E in $E(p)$ of degree p, and $L=L_1\dots L_n$, then $E^{*p}\le N(L/E)\le E^*$ and the index of $N(L/E)$ in E^* is equal to $[L\colon E]$. Finally, the quasilocality of E shows that if $a\in E^*\setminus E^{*p}$, $D\in d(E)$ and $\mathrm{ind}(D)=p$, then there is a cyclic degree p extension Y of E in $E(p)$, such that $D\cong(Y/E,\tau,a)$, for some generator τ of $\mathcal{G}(Y/E)$ (cf. [27], Sect. 15.5, and [5], I, Corollary 8.5). Hence, by [27], Sect. 15.1, Proposition b, $a\notin N(Y/E)$, which means that E^{*p} equals the intersection of the norm groups of cyclic extensions of E of degree p. Now it is clear that $r_p(E)=\infty$, so Lemma 5.2 is proved. □

We are now in a position to prove (1.2) (a). The fulfillment of the conditions of Lemma 5.2 ensures that $D(E(p)/E)$ contains infinitely many elements of order p. Hence, by Galois theory and the divisibility of $D(E(p)/E)$, every finite abelian p-group G is isomorphic to a subgroup of $D(E(p)/E)$. Assuming now that E is isomorphic to $\widehat{K}$, for some Henselian field (K, v), and using [33], Theorem A.24, one obtains further that K possesses a Galois extension U_G in K_{ur} with $\mathcal{G}(U_G/K) \cong G$. When the rank of G is at most $\tau(p)$, one deduces from [26], Theorem 1 (or [17], Example 4.3), that there is an NSR-algebra $D_G \in d(K)$ possessing a maximal subfield K-isomorphic to U_G. Thus it becomes clear that there exist $D_{k,n} \in d(K)$: $k, n \in \mathbb{N}, n \leq k \leq \tau(p)n$, such that $D_{k,n}/K$ is NSR, $\mathrm{ind}(D_{k,n}) = p^k$ and $\exp(D_{k,n}) = p^n$. The obtained result proves (1.2) (a), since Theorem 4.1 and the equality $r_p(E) = r_p(\widehat{K}) = \infty$ yield $\mathrm{Brd}_p(K) = \tau(p)$.

Our objective now is to prove (1.2) (b), (c) and (d). Suppose that (K, v) is Henselian, such that $v(K) \neq pv(K)$, $\mathrm{Brd}_p(K) < \infty$, and $\widehat{K}$ has a Henselian discrete valuation ω whose residue field $\tilde{k}$ is quasifinite with $\mathrm{char}(\tilde{k}) \neq p$. Then $\widehat{K}$ is quasilocal and $\mathrm{Brd}_p(K)$ is determined by Theorem 4.1 (a). Also, the conditions on ω ensure that $\widehat{K}^*/\widehat{K}^{*p} \cong \tilde{k}^*/\tilde{k}^{*p} \times \omega(\widehat{K})/p\omega(\widehat{K})$. This allows to prove those of the following statements, for which we assume that $\mu_p(\widehat{K}) \neq \{1\}$:

(5.3)
(a) $r_p(\widehat{K}) \leq 2$ and $r_p(\widehat{K}) = 2 \leftrightarrow \mu_p(\widehat{K}) \neq \{1\}$ (cf. [16], Ch. 2, (3.5));
(b) If $\mu_p(\widehat{K}) = \{1\}$, then finite extensions of $\widehat{K}$ in $\widehat{K}(p)$ are inertial relative to ω, and $\mathcal{G}(\widehat{K}(p)/\widehat{K}) \cong \mathcal{G}(\tilde{k}(p)/\tilde{k}) \cong \mathbb{Z}_p$ (see [39], Theorem 2, and [4], Lemma 1.1);
(c) $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ is a Demushkin group when $\mu_p(\tilde{k}) \neq \{1\}$ (cf. [36], Lemma 7);
(d) $\mathcal{G}(\widehat{K}_{\mathrm{ab}}(p)/\widehat{K}) \cong \mathbb{Z}_p \oplus \mathbb{Z}/p^\nu\mathbb{Z}$, provided that $\mu_p(\tilde{k})$ is of finite order p^ν; $\mathcal{G}(\widehat{K}_{\mathrm{ab}}(p)/\widehat{K}) \cong \mathbb{Z}_p^2$, if $\mu_p(\tilde{k})$ is infinite (apply (5.1) (a) in the former case, and use Kummer theory in the latter one).

The inequality $p \neq \mathrm{char}(\tilde{k})$ and the quasilocality of $\widehat{K}$ show that $\mathrm{Brd}_p(K)$ can be determined by applying Theorem 4.1. In view of (5.3) (a), (b) and the divisibility of $\mathrm{Br}(\widehat{K})_p$, this proves (1.2) (b) and (c). The former part of (1.2) (d) follows from (5.3) (c), (d) and Lemma 5.1; in this case, $\mu(p, n)$ is equal to $n + \min\{n, \nu\}[\tau(p)/2]$, for each $n \in \mathbb{N}$. For the proof of the latter one, we use the concluding part of (5.3) (d), which implies every finite abelian p-group G of rank ≤ 2 is isomorphic to $\mathcal{G}(U_G/K)$, for some Galois extension U_G of K in K_{ur}. This gives us the possibility to complete the proof of (1.2) (d), arguing along the lines drawn at the end of the proof of (1.2) (a).

We prove Theorem 1.1. The field $\widehat{K}$ is quasilocal, and is complete relative to a discrete valuation ω with a finite residue field $\tilde{k}$. This implies ω is Henselian, $\mu_p(\widehat{K})$ is finite, $\mathrm{Br}(\widehat{K}) \cong \mathbb{Q}/\mathbb{Z}$, and in case $p \neq \mathrm{char}(\tilde{k})$, $\varepsilon_p \in \widehat{K}$ if and only if p divides the order of $\tilde{k}^*$. When $\varepsilon_p \notin \widehat{K}$, $C(\widehat{K}(p)/\widehat{K})$ is divisible, by the following results (which are contained in (5.3) (b) and [32], Theorem 3, respectively):

(5.4) (a) $\mathcal{G}(\widehat{K}(p)/\widehat{K}) \cong \mathbb{Z}_p$, provided that $p \neq \mathrm{char}(\tilde{k})$;
(b) If $\mathrm{char}(\widehat{K}) = 0$ and $\mathrm{char}(\tilde{k}) = p$, then $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ is a free pro-p-group, and $\mathcal{G}(\widehat{K}(p)_{\mathrm{ab}}/\widehat{K}) \cong \mathbb{Z}_p^r$, where $r = r_p(\widehat{K})$; in addition, $\widehat{K}$ is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers and $r = [\widehat{K}\colon \mathbb{Q}_p] + 1$.

Note also that, by Theorem 4.1, $\mathrm{Brd}_p(K) = m_p$, and by (5.4) and [26], Theorem 1, each pair of p-primary integers admissible by Theorem 1.1 is an index-exponent pair of a suitably chosen NSR-algebra over K.

Consider finally the case where $\varepsilon_p \in \widehat{K}$. Then Theorem 4.1 yields $\mathrm{Brd}_p(K) = \mu(p,1)$, and Lemma 5.1 implies $(1,1)$ and $(p^k, p^n)\colon k, n \in \mathbb{N}, n \leq k \leq \mu(p,n)$, are all index-exponent p-primary K-pairs. This completes our proof. □

Remark 5.3. Theorem 1.1 retains validity, if $\widehat{K} \in \mathrm{Fe}(\mathbb{Q}'_\pi)$, for some π-adically closed field $\mathbb{Q}'_\pi$ (in the sense of [29]). This is fulfilled, if $\mathrm{char}(\widehat{K}) = 0$ and $\widehat{K}$ has a Henselian discrete valuation ω with a finite residue field $\tilde{k}$ of characteristic π. Also, (5.4) hold, if $\mu_p(\widehat{K}) = \{1\}$ (in case (b), with $\mathbb{Q}'_p$ instead of $\mathbb{Q}_p$). When $\mu_p(\widehat{K}) \neq \{1\}$ and $r = r_p(\widehat{K})$, we have: $r = 2$, provided $p \neq \pi$; $r = [\widehat{K}\colon \mathbb{Q}'_p] + 2$, if $p = \pi$ (see (5.3), [36], Lemma 7, and [21], Sect. 5, for the case of $\mathbb{Q}'_p = \mathbb{Q}_p$).

Corollary 5.4. *Let (K,v) be a Henselian field, such that $\tau(p) < \infty$, for some $p \in \mathbb{P}$, $p \neq \mathrm{char}(\widehat{K})$. Also, let $\widehat{K}$ have a Henselian discrete valuation ω with a quasifinite residue field $\tilde{k}$. Then* $\mathrm{abrd}_p(K) = 1 + [\tau(p)/2]$, *if* $p \neq \mathrm{char}(\tilde{k})$; $\mathrm{abrd}_p(K) = \max\{1, \tau(p)\}$, *if* $\mathrm{char}(\widehat{K}) = 0$ *and* $\mathrm{char}(\tilde{k}) = p$.

P r o o f. In view of (1.1) (b) and (1.2), one may consider only the case where $\mu_p(\widehat{K}) \neq \{1\}$, $\mathrm{char}(\widehat{K}) = 0$, $\tilde{k}$ is finite and $\mathrm{char}(\tilde{k}) = p$. Then our conclusion follows from Remark 5.3 and the fact that $[\widehat{K}(p)\colon \widehat{K}] = \infty$. □

Conclusion. Assume that (K,v) is Henselian with $\widehat{K}$ possessing a Henselian discrete valuation ω whose residue field is quasifinite. Summing-up (1.1), (2.3) (b) and Corollary 2.2, observing that $\mathrm{Br}(\widehat{K}) \cong \mathbb{Q}/\mathbb{Z}$ and $\mathrm{Brd}_p(\widehat{K}) = 1$, $p \in \mathbb{P}$, and using results of this paper, one describes index-exponent K-pairs prime-to $\mathrm{char}(\widehat{K})$. The non-divisibility restriction is superfluous, if $\mathrm{char}(K) > 0$, (K,v) is maximally complete and $\widehat{K}$ satisfies the conditions of Corollary 3.6.

REFERENCES

[1] A. A. Albert. Structure of Algebras. American Mathematical Society Colloquium Publications, vol. **24**. New York, American Mathematical Society, 1939.

[2] R. Aravire, B. Jacob. p-algebras over maximally complete fields. With an Appendix by J.-P. Tignol. Proc. Sympos. Pure Math., vol. **58**, Part 2, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), 27-49, Providence, RI, Amer. Math. Soc., 1995.

[3] A. Blaszczok, F.-V. Kuhlmann. Algebraic independence of elements in immediate extensions of valued fields. *J. Algebra* **425** (2015), 179–214.

[4] I. D. Chipchakov. On the Galois cohomological dimensions of stable fields with Henselian valuations. *Comm. Algebra* **30**, *4* (2002), 1549–1574.

[5] I. D. Chipchakov. On the residue fields of Henselian valued stable fields. I – *J. Algebra* **319**, *1* (2008), 16–49; II – C. R. Acad. Bulgare Sci. **60**, *5* (2007), 471–478.

[6] I. D. Chipchakov. On the behaviour of Brauer p-dimensions under finitely-generated field extensions. *J. Algebra* **428** (2015), 190–204.

[7] I. D. Chipchakov. On Brauer p-dimensions and index-exponent relations over finitely-generated field extensions. *Manuscripta Math.* **148**, *3–4* (2015), 485–500.

[8] I. D. Chipchakov. On Brauer p-dimensions and absolute Brauer p-dimensions of Henselian fields. *J. Pure Appl. Algebra* **223**, *1* (2019), 10–29.

[9] I. D. Chipchakov. On the Brauer groups of quasilocal fields and the norm groups of their finite Galois extensions. Preprint, arXiv:math/0707.4245v6 [math.RA].

[10] P. M. Cohn. On extending valuations in division algebras. *Studia Sci. Math. Hungar.* **16**, *1–2* (1981), 65–70.

[11] P. K. Draxl. Skew Fields. London Mathematical Society Lecture Note Series, vol. 81. Cambridge, Cambridge University Press, 1983.

[12] P. K. Draxl. Ostrowski's theorem for Henselian valued skew fields. *J. Reine Angew. Math.* **354** (1984), 213–218.

[13] I. Efrat. On fields with finite Brauer groups. *Pacific J. Math.* **177**, *1* (1997), 33–46.

[14] I. Efrat. Valuations, orderings, and Milnor K-theory. Mathematical Surveys and Monographs, vol. **124**. Providence, RI, American Mathematical Society, 2006.

[15] B. Fein, D. Saltman, M. Schacher. Heights of cyclic field extensions. *Bull. Soc. Math. Belg. Sér. A* **40**, *2* (1988), 213–223.

[16] I. B. Fesenko, S. V. Vostokov. Local fields and their extensions. With a foreword by I. R. Shafarevich, 2nd ed. Translations of Mathematical Monographs, vol. **121**. Providence, RI, American Mathematical Society, 2002.

[17] B. Jacob, A. Wadsworth. Division algebras over Henselian fields. *J. Algebra* **128**, *1* (1990), 126–179.

[18] V. G. Khalin. The number of central skew fields of fixed index over multidimensional local fields. *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* (1989) *1*, 116–118, 126 (in Russian); English translation in *Vestn. Leningr. Univ. Math.* **22**, *1* (1989), 81–84.

[19] V. G. Khalin. p-algebras over a multidimensional local field. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (LOMI) **175** (1989), 121–127, 164–165 (in Russian); English translation in *J. Soviet. Math.* **57**, *6* (1991), 3516–3519.

[20] M. Kurihara. On two types of complete discrete valuation fields. *Compositio Math.* **63**, *2* (1987), 237–257.

[21] J. P. Labute. Classification of Demushkin groups. *Canad. J. Math.* **19** (1967), 106–132.

[22] T. Y. Lam. Orderings, valuations and quadratic forms. CBMS Regional Conference Series in Mathematics, vol. **52**. Providence, RI, American Mathematical Society, 1983.

[23] S. Lang. Algebra. Reading, Mass., Addison-Wesley Publ. Comp., Inc., 1965.

[24] A. S. Merkur'ev, A. A. Suslin. K-cohomology of Severi-Brauer varieties and norm residue homomomorphisms. *Izv. Akad. Nauk SSSR Ser. Mat.* **46**, *5* (1982), 1011–1046 (in Russian); English translation in Math. USSR Izv. **21**, *2* (1983), 307–340.

[25] J. S. Meyer. Division algebras with infinite genus. *Bull. Lond. Math. Soc.* **46**, *3* (2014), 463–468.

[26] P. Morandi. The Henselization of a valued division algebra. *J. Algebra* **122**, *1* (1989), 232–243.

[27] R. S. Pierce. Associative Algebras. Graduate Texts in Mathematics, vol. **88**. New York-Berlin, Springer-Verlag, 1982.

[28] V. P. Platonov, V. I. Yanchevskij. Dieudonné's conjecture on the structure of unitary groups over a division ring, and Hermitian K-theory. *Izv. Akad. Nauk SSSR Ser. Mat.* **48**, *6* (1984), 1266–1294 (in Russian); English translation in *Math. USSR-Izv.* **25**, *3* (1985), 573–599.

[29] A. Prestel, P. Roquette. Formally p-adic fields. Lecture Notes in Mathematics, vol. **1050**. Berlin, Springer-Verlag, 1984.

[30] O. F. G. Schilling. The Theory of Valuations. Mathematical Surveys, No. **4**. New York, N.Y., Amer. Math. Soc., 1950.

[31] J.-P. Serre. Local Fields. Transl. from the French original by M. J. Greenberg. Graduate Texts in Mathematics, vol. **67**. New York, Berlin, Springer-Verlag, 1979.

[32] J.-P. Serre. Galois Cohomology. Translated from the French by Patrick Ion and revised by the author. Berlin, Springer-Verlag, 1997.

[33] J.-P. Tignol, A. R. Wadsworth. Value Functions on Simple Algebras and Associated Graded Rings. Springer Monographs in Mathematics. Cham, Springer, 2015.

[34] S. V. Tikhonov. Division algebras of prime degree with infinite genus. Algebra, geometry, and number theory, Collected papers. Dedicated to Academician Vladimir Petrovich Platonov on the occasion of his 75th birthday, *Tr. Mat. Inst. Steklova*, **292**, (2016), 264-267 (in Russian); English translation in *Proc. Steklov Inst. Math.* **292** (2016), 256–259.

[35] M. Van den Bergh, A. Schofield. Division algebra coproducts of index n. Trans. Amer. Math. Soc. **341**, *2* (1994), 505–517.

[36] R. Ware. Galois groups of maximal p-extensions. Trans. Amer. Math. Soc. **333**, *2* (1992), 721-728.

[37] S. Warner. Topological Fields. North-Holland Math. Studies, vol. **157**. Notas de Matemática, vol. **126** Amsterdam, North-Holland Publ. Co., 1989.

[38] A. Weil. Basic Number Theory. Berlin, Springer-Verlag, 1967.

[39] G. Whaples. Algebraic extensions of arbitrary fields. *Duke Math. J.* **24** (1957), 201–204.

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