Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **44** (2018), 413–438

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

n-DIMENSIONAL COPULAS AND WEAK DERIVATIVES

Nikolay Chervenov, Iordan Iordanov, Boyan Kostadinov

Communicated by I. D. Iliev

Abstract. In the copula theory universe the number of multivariate copulas is very limited. This is caused by both of non-trivial tasks - to check the n-increasing property and to define the copula. We generalize the notion of n-increasing property in terms of weak derivatives which allows us to simplify the otherwise complex former method. Furthermore, we demonstrate the applicability of our approach to the class of n-dimensional Archimedean copulas. Finally, we present a method which allows us to obtain a class of copulas as a solution of a boundary value problem in appropriate Sobolev spaces.

1. Introduction. Copulas are a key instrument in many scientific areas (see [16], [12], [13], [4], [14]). In the present article, using weak derivatives (i.e. derivatives in the sense of the distribution theory) we consider two main problems. At first we give two generalisations of the notion of an n-increasing function. This allows us easy to obtain several fundamental statements. In particular,

2010 *Mathematics Subject Classification:* 35L05, 35L20, 46FXX, 46F10, 46F12, 60E05, 62F99.

Key words: copula, n-increasing function, weak solution, test function.

we consider the so called bivariate Archimedean copulas in order to apply our approach. Another problem studied in the current work is the construction of n-dimensional copulas by using values of their derivative $\dfrac{\partial^n}{\partial x_1 \dots \partial x_n}$. For this, we use the already generalized notion of an n-increasing function as well as results of a variant of Goursat problem in n-dimensional cube in $\mathbb{R}^n$.

The present paper is based on some results obtained in the articles [5] and [6].

For the sake of brevity, in the exposure we use definitions about Sobolev Spaces and all other related concepts as they were introduced in [1]. The following notation is used frequently: $I^n = [0,1] \times \cdots \times [0,1]$ (n-multipliers).

2. *n*-increasing functions. Examples. In this section we generalise the definition of n-increasing function so let us first recall the well-known form of this notion.

Let n be a positive integer and let denote $\overline{\mathbb{R}}^n = \overline{\mathbb{R}} \times \cdots \times \overline{\mathbb{R}}$. For any $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}$, $b = (b_1, b_2, \dots, b_n) \in \mathbb{R}$, such that $a \leq b$ (i.e. $a_k \leq b_k$, for all $k = 1 \dots n$), we denote with $[a, b]$ the *n-box*

$$B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

The vertices of the *n-box* B are the points $c = (c_1, c_2, \dots, c_n)$ where each c_k is equal to either a_k or b_k. Let $G \subset \overline{\mathbb{R}}^n$ be a region and $H : G \to \mathbb{R}$ be a real function with domain G.

Then if $B = [a, b]$ is an n-box all of whose vertices are in G, then the H-volume of B is given by (see [16], Definition 2.10.1.)

$$V_H(B) = \sum \operatorname{sgn}(c)\, H(c), \tag{2.1}$$

where the sum is taken over all vertices c of B, and $\operatorname{sgn}(c)$ is given by

$$\operatorname{sgn}(c) = \begin{cases} 1, & \text{if } \ c_k = a_k \ \text{ for an even number of } k\text{'s },\\ -1, & \text{if } \ c_k = a_k \ \text{ for an odd number of } k\text{'s }. \end{cases}$$

The subsequent generalisations we perform are based on the two representations of the formula (2.1) we obtain.

If the function H is a smooth function, then the formula (2.1) has the representation

$$V_H(B) = \int\limits_{a_1}^{b_1} \cdots \int\limits_{a_n}^{b_n} H_{x_1 \cdots x_n}(\xi_1, \dots, \xi_n) d\xi_1 \cdots d\xi_n. \tag{2.2}$$

For the second representation we introduce the first order difference by

$$\begin{aligned}\Delta_{a_k}^{b_k} H(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) \\ =& H(x_1, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_n) - H(x_1, \ldots, x_{k-1}, a_k, x_{k+1}, \ldots, x_n).\end{aligned} \tag{2.3}$$

Then the H-volume of B is the nth order difference of H on B

$$V_H(B) = \Delta_a^b H(x) = \Delta_{a_n}^{b_n} \Delta_{a_{n-1}}^{b_{n-1}} \ldots \Delta_{a_2}^{b_2} \Delta_{a_1}^{b_1} H(x). \tag{2.4}$$

Remark 2.1. It is obvious that from (2.3) and (2.4) if H is a constant or more general – if H is a function of at most $(n-1)$ variables then the corresponding H-volume vanishes.

For an arbitrary $x = (x_1, \ldots, x_n) \in G$, let us set in (2.3) and (2.4) $a_k = x_k$, $b_k = x_k + h_k$ and $\Delta_{a_k}^{b_k} \equiv \Delta_{h_k}^k$, for $k = 1, 2, \ldots n$, where $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$ is fixed. Then the H-volume of the n-box with fixed vertex x, $B_h = [x, x + h]$ (for h enough small such that all of B_h vertices are in G), is given by

$$V_H(B_h) = \Delta_{h_1}^1 \Delta_{h_2}^2 \ldots \Delta_{h_n}^n H(x). \tag{2.5}$$

Based on the above reasoning we will generalize the following main definition of an n-increasing function.

We continue by assuming the following

Definition 2.2 (See Definition 2.10.2 from [16])**.** *An n-place real function H, with domain $G \subset \overline{\mathbb{R}}^n$, is n-increasing if $V_H(B) \geq 0$ for all n-boxes B whose vertices lie in G.*

Further, in the case when H is a smooth function, by the mean value theorem applied to (2.2) we obtain that H is n-increasing if and only if

$$V_H(B) = H_{x_1 \cdots x_n}(\bar{x})(b_1 - a_1) \cdots (b_n - a_n) \geq 0$$

for an appropriate $\overline{x} \in \mathbb{R}^n$, where we denoted $H_{x_1, \ldots, x_n} = \dfrac{\partial^n}{\partial x_1 \ldots \partial x_n} H$.

Remark 2.3. a) If the derivative $H_{x_1 \cdots x_n}$ exists and is non-negative in G, then the non-negativity of the H-volume follows immediately from (2.2).

Conversely, assuming that

$$H_{x_1 \cdots x_n}(\tilde{x}) < 0,$$

for a given point $\tilde{x}$, then inequality would be valid in a neighbourhood $V \ni \tilde{x}$ as this derivative is continuous. This combined with the relation $B \subset V$ contradicts the fact that H is a n-increasing function.

b) Therefore when H is a smooth function, then the condition $H_{x_1 \cdots x_n} \geq 0$, for each $x \in G$ implies that H is a n-increasing function. For example (see Example 2.22 from [16]) if we consider

$$H(x,y,z) = \frac{(x+1)(e^y-1)\sin z}{x+2e^y-1},$$

defined in

$$\left\{(x,y,z) \middle| -1 \leq x \leq 1,\ 0 \leq y \leq +\infty,\ 0 \leq z \leq \frac{\pi}{2}\right\},$$

and by establishing that

$$\frac{\partial^3}{\partial x \partial y \partial z} H(x,y,z) \geq 0,$$

we conclude that H is a 3-increasing function in its domain of definition.

Referring to the remark above, similar to the case of bivariate copulas considered in [11], we obtain the following

Definition 2.4. *A distribution $H \in \mathscr{D}'(G)$, where $G \subset \overline{\mathbb{R}}^n$ is a domain, is called* weak n-increasing distribution *in G if for any test function $\varphi \geq 0$ in $\mathscr{D}(G)$*

$$(H_{x_1 \cdots x_n}, \varphi) \geq 0. \tag{2.6}$$

Remark 2.5. It is proved in [11] that for $H \in \mathscr{D}'(C) \cap C^0(G)$, the new notion coincides with the classical Definition 2.2.

A disadvantage of Definition 2.4 relates to the search of weak derivatives of non-smooth functions (see Example 2.7 from [11]). Apart from that this definition does not take into account the fact that H belongs to a suitable Sobolev space in the cases under consideration.

Based on formula (2.5) we give a new generalisation of Definition 2.2. For $f \in C_0^\infty(\mathbb{R}^n)$, we fix a vector $h \in \mathbb{R}^n$ and if we set

$$\bar{H}(x) = \begin{cases} H(x), & x \in G, \\ 0, & x \notin G, \end{cases}$$

we conclude that

$$\int\limits_{\mathbb{R}^n} \left[\bar{H}(x+h) - \bar{H}(x)\right] f(x)dx = \int\limits_{\mathbb{R}^n} \bar{H}(z)\left[f(z-h) - f(z)\right] dz.$$

Consequently under the notations introduced above we obtain

$$\int_{\mathbb{R}^n} \Delta^1_{h_1} \cdots \Delta^n_{h_n} \bar{H}(x) f(x) dx = \int_{\mathbb{R}^n} \bar{H}(x) \Delta^1_{-h_1} \cdots \Delta^n_{-h_n} f(z) dz. \tag{2.7}$$

Now dividing by $h_i > 0$, $i = 1, \ldots, n$, and letting $(h_1, \ldots, h_n) \longrightarrow (0, \ldots, 0)$, for $f \geq 0$ and due to non-negativity of the left side of formula (2.7) we obtain

$$(-1)^n \int_G H(z) f_{z_1 \cdots z_n}(z) dz \geq 0. \tag{2.8}$$

If we assume that $H \in W^{1,p}(G)$, $p > n$, property (2.8) is immediately generalised for non-negative functions $f \in W^{n-1,p}(G)$, where $1/p + 1/q = 1$ (according to Theorem 3.22 and 3.9 from [1] under minimum requirements of the boundary ∂G, which are obviously satisfied for $G = I^n$).

In order for us to properly state the result above we need to remind that the notation

$$(\mu, \nu)$$

is used as a generalisation of the common scalar product of functions $\mu \in L_p$, $\nu \in L_q$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ (see Paragraph 3.9 from [1]) in the case when $\mu \in W^{1,p}$ and $\nu \in W^{-1,q}$ (In fact the derivative $\dfrac{\partial^n f}{\partial x_1 \cdots \partial x_n}$ of the function $f \in W^{n-1,q}$ belongs to the latter space).

We say that a domain G satisfies the *segment condition* (see [1]) if every $x \in \partial G$ has a neighbourhood U_x and a non-zero vector y_x such that if $z \in \overline{G} \cap U_x$, then $z + t y_x \in G$ for $0 < t < 1$ (we denote with ∂G the boundary of G).

Definition 2.6. *We say that $H \in W^{1,p}(G)$, where G satisfies the segment condition, is a weakly n*-increasing function *in G, if*

$$(-1)^n (H, f_{x_1 \cdots x_n}) \geq 0 \tag{2.9}$$

for all $f \geq 0$ in $W^{n-1,q}(G)$.

Remark 2.7. a) If the function H from the definition above is continuous (for example when $p > n$ this is true), then this condition leads to the common Definition 2.2. Actually if we refer to (2.9) and assume that the expression on the left hand side is negative, then for $\varepsilon > 0$ sufficiently small, such that $0 < h_k < \varepsilon$, for all $k = 1, \ldots, n$, we have

$$\int_G \Delta^1_{h_1} \cdots \Delta^n_{h_n} H(x) f(x) dx < 0,$$

(see (2.7)). Since H is continuous and f – non-negative, there is a $x \in G$ such that

$$\Delta^1_{h_1} \cdots \Delta^n_{h_n} H(x) < 0.$$

This implies that H is not an n-increasing function in the sense of Definition 2.2.

b) If in Definition 2.6 we assume that (2.9) is valid for smooth functions f only, then this limited definition takes us to the initial one, referring to Theorem 3.22 from [1].

Immediately after letting n tends to infinity in inequalities of the type (2.9) we obtain the following propositions

Corollary 2.8. *Let $H_m \in W^{1,p}(G)$ be an n-increasing functions for $m = 1, 2, \ldots$ in the domain $G \subset \mathbb{R}^n$ and let $H \in W^{1,p}(G)$ be the limit of H_m in $W^{1,p}(G)$. Then H is weakly n-increasing funciton in G.*

Corollary 2.9. *Let G_m be a sub-domain of the domain $G \subset \mathbb{R}^n$ and let the measure* $\operatorname{mes}(G \setminus G_m)$ *converge to zero when $m \to \infty$. Let $H \in W^{1,p}(G)$ be a weakly n-increasing function in G_m. Then H is a weakly n-increasing function in G.*

Remark 2.10. If we aim to verify inequality (2.9) by transferring derivatives from f to H using the Gauss formula (which does not require computations of weak, but rather of classical derivatives) we need information of behaviour of H, f and some of their derivatives over $\partial G = \partial I^n$ (as well as the coordinates of the unit normal vector to the boundary).

It is quite simple in the case when H vanishes on the sides of I^n passing through the origin (e.g. as we proceed in the next section for the Archimedean copulas). Then a variant of *Definition 2.6* is valid, where f and its derivatives vanish on the sides of I^n passing through the vertex $(1, \ldots, 1)$.

Lemma 2.11. *The function $H \in W^{1,p}(I^n)$ is weakly n-increasing function in I^n if the inequality*

$$(-1)^n (H, f_{x_1 \cdots x_n}) \geq 0$$

is fulfilled for all $f \geq 0$ in $W^{n-1,q}(I^n)$ such that f and its derivatives vanish on the sides of I^n passing through the vertex $(1, \ldots, 1)$.

Proof. a) When $n = 2$ it is enough f to vanish on the sides of $I^2 \equiv [0,1] \times [0,1]$ passing through the vertex $(1,1)$. Indeed, for an arbitrary $f \geq 0$ belonging to $C_0^\infty(\mathbb{R}^2)$ lets consider

$$g(x,y) = f(x,y) - f(x,1) - f(1,y) + f(1,1).$$

Obviously g vanishes on $x = 1$ or $y = 1$, and belongs to $C_0^\infty(\mathbb{R}^2)$. Let K be a constant, such that $g + K \geq 0$ in I^2. Therefore applying formula (2.7) for $g + K$ we observe non-negativity of the expression

$$(-1)^2 \int\limits_{I^2} H(x,y)(g(x,y)+K)_{xy}(x,y)\,dx\,dy = (-1)^2 \int\limits_{I^2} H(x,y) f_{xy}(x,y)\,dx\,dy,$$

as $\partial_{xy}(K + \Sigma) = 0$.

b) In the case $n \geq 3$ if $f \in C_0^\infty(\mathbb{R}^n)$ and $f \geq 0$, let us consider the function

$$\begin{aligned}(2.10)\quad & f^*(x_1,\ldots,x_n) = \\ & f(x_1,\ldots,x_n) - f(1,x_2,\ldots,x_n) - \sum_{k=1}^{m} \frac{(x_1-1)^k}{k!} \partial_{x_1}^k f(1,x_2,\ldots,x_n)\end{aligned}$$

which vanishes together with its derivatives up to order m on the side $\{x_1 = 1\}$ of I^n. Considering the side $\{x_2 = 1\} \cap I^n$, we set

$$\begin{aligned}(2.11)\quad & f^{**}(x_1,\ldots,x_n) = \\ & f^*(x_1,\ldots,x_n) - f^*(x_1,1,x_3,\ldots,x_n) - \sum_{s=1}^{m} \frac{(x_2-1)^s}{s!} \partial_{x_2}^s f^*(x_1,1,x_3,\ldots,x_n).\end{aligned}$$

It is a matter of direct verification that the function $f^{**}(x_1,\ldots,x_n)$ vanishes on the side $\{x_1 = 1\} \cap I^n$ together with its derivatives of order up to m. To prove this we observe first that by definition $f^*(x_1,\ldots,x_n)$ vanishes on $\{x_1 = 1\}$ and then that all terms of the sum $\sum\limits_{s=1}^{m} \frac{(x_2-1)^s}{s!} \partial_{x_2}^s f^*(x_1,1,x_3,\ldots,x_n)$ vanish on the side $\{x_1 = 1\}$ as

$$\partial_{x_k}^\beta \partial_{x_2}^s f^*(1,1,x_3,\ldots,x_n) = \partial_{x_2}^s \partial_{x_k}^\beta f^*(1,1,x_3,\ldots,x_n)$$

vanish on $\{x_1 = 1\}$. The last statement follows from the corresponding difference quotients (for $s = 0,1,\ldots m$) with respect to the variable x_2 in the point $x_2 = 1$.

After that we define

$$
\begin{aligned}
(2.12)\quad & f^{***}(x_1,\dots,x_n) = f^{**}(x_1,\dots,x_n) - \\
& f^{**}(x_1,x_2,1,x_4,\dots,x_n) - \sum_{r=1}^{m} \frac{(x_3-1)^r}{r!} \partial_{x_3}^r f^{**}(x_1,x_2,1,x_4,\dots,x_n).
\end{aligned}
$$

Similarly, f^{***}, together with its derivatives of order up to m, vanish over $\{x_3 = 1\} \cap I^n$ as well as over $\{x_1 = 1\} \cap I^n$ and $\{x_2 = 1\} \cap I^n$.

We continue the iterative process with the rest of the sides of I^n passing through the vertex $(1,\dots,1)$ until we obtain the function

$$
g(x_1,\dots,x_n) = f(x_1,\dots,x_n) + \Sigma(x_1,\dots,x_n), \tag{2.13}
$$

which vanishes together with its derivatives up to order m on the sides of I^n passing through the vertex $(1,\dots,1)$, where with $\Sigma(x_1,\dots,x_n)$ we denote the mentioned functions above within the iterative process.

Unlike the case a) here the derivative

$$
\frac{\partial^n \Sigma}{\partial x_1 \cdots \partial x_n}
$$

might be different from zero. Let K_n be a constant, such that

$$
\begin{cases}
(-1)^n (K_n + \Sigma) \geq 0, \\
(-1)^n \left(-K_n + \dfrac{\partial^n}{\partial x_1 \cdots \partial x_n} \Sigma \right) \leq 0.
\end{cases} \tag{2.14}
$$

Therefore applying formula (2.7) for the non-negative function

$$
j(x_1,\dots,x_n) = K_n(2-x_1) \prod_{i=2}^{n} (1+x_i) + f(x_1,\dots,x_n) + \Sigma(x_1,\dots,x_n)
$$

and letting $(h_1,\dots,h_n) \longrightarrow (0,\dots,0)$ we observe

$$
\begin{aligned}
& (-1)^n \int\limits_{I^n} H(x_1,\dots,x_n) f_{x_1\dots x_n}(x_1,\dots,x_n)\, dx_1 \dots dx_n \\
\geq\ & (-1)^n \int\limits_{I^n} H(x_1,\dots,x_n) \big(K_n - \Sigma_{x_1\dots x_n}(x_1,\dots,x_n)\big)\, dx_1 \dots dx_n \geq 0.
\end{aligned}
$$

The last inequality is valid when $H \geq 0$ which requirement in the case of cumulative distribution functions is fulfilled as they are grounded and non-decreasing on each argument (see [16]).

In conclusion we note again that the argument from Remark 2.7a) is valid. Basically if we have a non-negative function $f \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$(-1)^n \int\limits_G H f_{x_1 \cdots x_n} dx_1 \cdots dx_n < 0,$$

then

$$(-1)^n \int H \left(f_{x_1 \cdots x_n} - K + \frac{\partial^n \Sigma}{\partial x_1 \cdots \partial x_n} \right) dx_1 \cdots dx_n < 0$$

in terms of (2.14). This is where the conclusion of the sign of $\Delta_{h_1}^1 \cdots \Delta_{h_n}^n H(x)$ stems from. □

Example 2.12. a) (see [11] and Paragraph 2.2 from [16]). Let $n = 2$, we consider the function

$$W(x_1, x_2) = \max(x_1 + x_2 - 1, 0)$$

and let $f \in C_0^\infty(\mathbb{R}^2)$ vanish on the sides $x = 1$ and $y = 1$ of I^2.

After applying the Gauss formula twice we immediately obtain

$$\iint\limits_{I^2} W(\xi_1, \xi_2) f_{x_1 x_2}(\xi_1, \xi_2) d\xi_1 d\xi_2 = \frac{1}{\sqrt{2}} \int\limits_S f(\xi_1, \xi_2)\, ds \geq 0,$$

where $S = \{(x, y) \in I^2 \mid x + y - 1 = 0\}$ and ds is the elementary arc length of S. Hence $W(x, y)$ is 2-increasing.

b) Let $n \geq 3$, then the function

$$W^n(x_1, \ldots, x_n) = \max(x_1 + x_2 + \cdots + x_n - n + 1, 0),$$

(see Formula 2.10.8 and Exercise 2.34 from [16]) is n-increasing. Indeed, the plane

$$x_1 + \cdots + x_n = n - 1,$$

passes through the vertices of I^n, where $n - 1$ coordinates are 1 and one coordinate is 0. The function W^n coincides with $x_1 + \cdots + x_n - n + 1 > 0$ in the half-space non-containing the origin and defined by this plane. To prove that

W^n is not n-increasing let set in (2.9) $f(x) = \prod_{k=1}^{n} x_k$ when n is odd number or $f(x) = (1 - x_1) \prod_{k=2}^{n} x_k$ when n is even number. Then we have

$$(-1)^n f_{x_1 \cdots x_n} < 0,$$

and by Definition (2.6) the statement follows.

3. Archimedean copulas. Archimedean copulas are an important class of copulas which is considered and studied in details in many works (see chapter 4 in [16], as well as [13], [7], [14], [9]). These copulas find a wide range of applications as they are easy to construct and assess many nice properties. They are obtained via a certain formula. Since boundary conditions are easily verified a key point is whether the obtained function is n-increasing.

Let us recall the definition of copula. An n-dimensional copula (see [16]) is a function $C : I^n \longrightarrow I$ such that

1) for every $x \in I^n$, $C(x) = 0$ if at least one coordinate of x is 0;
2) if all coordinates of x are 1 except x_k , then $C(x) = x_k$;
3) C is n-increasing.

Let φ be a continuous and strictly decreasing function from $[0, 1]$ to $[0, +\infty]$ such that $\varphi(1) = 0$ and let

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & \text{if } 0 \le t \le \varphi(0), \\ 0, & \text{if } \varphi(0) \le t \le +\infty, \end{cases}$$

be the pseudo-inverse of φ. In fact $\varphi^{[-1]}(t)$ is continuous and non-increasing function in $[0, +\infty]$ and strictly decreasing in $[0, \varphi(0)]$. If $\varphi(0) = +\infty$, then $\varphi^{[-1]}$ coincides with φ^{-1} (see Definition 4.1.1 and Figure 4.1 from [16]).

The main result in the case of bivariate copulas is given by Theorem 4.1.4 from [16], i.e. the function

$$C : [0, 1] \times [0, 1] \to [0, 1],$$

defined by the formula

$$C(x_1, x_2) = \varphi^{[-1]}(\varphi(x_1) + \varphi(x_2)), \tag{3.1}$$

is bivariate copula if and only if φ is convex.

Conditions of C over the boundary of I^2 are trivially fulfilled (Lemma 4.1.2 from [16]) and it only needs to establish that C is a 2-increasing function. We use Definition (2.6), though Definition (2.4) could be also applied.

To obtain the proof we consider some supporting statements referring to approximation of functions. As usual, the regularisation kernel is defined by

$$J(x) = \begin{cases} k \exp\left(\dfrac{1}{x^2-1}\right), & -1 \le x \le 1, \\ 0, & |x| \ge 1, \end{cases} \tag{3.2}$$

where the constant k is such that

$$\int_{-\infty}^{+\infty} J(x)dx = 1.$$

By denoting

$$J_\varepsilon(x) = \frac{1}{\varepsilon} J\left(\frac{x}{\varepsilon}\right),$$

we obtain the common definition of a regularization (see Paragraph 2.28 from [1]):

$$\varphi_\varepsilon(x) = (J_\varepsilon * \varphi)(x) = \int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{\varepsilon} J\left(\frac{x-y}{\varepsilon}\right)\varphi(y)dy, \tag{3.3}$$

where it is assumed that φ is integrable and equal to zero outside its domain of definition (a subject of consideration in the respective case).

After changing variables $y = x - \varepsilon z$ we derive a new formula for the regularization

$$\varphi_\varepsilon(x) = \int_{-1}^{1} J(z)\varphi(x - \varepsilon z)dz. \tag{3.4}$$

The properties of this approximation are well known (see Theorem 2.29 from [1]), from where we only note that $\varphi_\varepsilon \in C^\infty(\mathbb{R})$.

Lemma 3.1. *Let φ be a monotonically decreasing function, i.e. for all x_1 and x_2 of its domain of definition such that $x_1 < x_2$ follows $\varphi(x_1) > \varphi(x_2)$.*

Then φ_ε is monotonically decreasing.

P r o o f. Since we have

$$x_1 - \varepsilon z < x_2 - \varepsilon z, \quad \text{for all } z \in [-1,1], \quad \text{and for any } \varepsilon > 0,$$

the statement follows from formula (3.4). □

Let φ be a convex function defined in the interval $X \subset \mathbb{R}$, i.e. for every interval $[x_1, x_2] \subset X$ and each $\lambda \in [0,1]$ the requirement below is fulfilled

$$\varphi((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)\varphi(x_1) + \lambda\varphi(x_2). \tag{3.5}$$

We need to point out that there is either an equality in (3.5) on the segment $[x_1, x_2]$ (when the graph of φ is the same as the respective chord) or the inequality is strict in the interval (x_1, x_2) (see Proposition 6 from [8]).

Again from (3.4) it is concluded that:

Lemma 3.2. *Let φ be a convex function. Then φ_ε also is convex function.*

P r o o f. We have that for $\varepsilon > 0$ and $z \in [-1,1]$ it is true that $x_1 - \varepsilon z < x_2 - \varepsilon z$ and

$$\varphi((1-\lambda)(x_1 - \varepsilon z) + \lambda(x_2 - \varepsilon z)) \leq (1-\lambda)\varphi(x_1 - \varepsilon z) + \lambda\varphi(x_2 - \varepsilon z).$$

Finally, we multiply by $J(z)$ and integrate in respect of z in the interval $[-1,1]$. □

For convenience we formulate the following:

Lemma 3.3. *If $y = f(x)$ and $x = g(y)$ are one-to-one and mutually-inverse functions (in accordance with the respective intervals of definition) and if f is convex and decreasing, then g is convex and decreasing as well.*

P r o o f. (See Paragraph 142, Proposition 4 from [8]). □

Remark 3.4. The properties of decreasing monotonicity and convexity in the case of smooth function φ are expressed via the inequalities: $\varphi' < 0$ and $\varphi'' > 0$ in the interval under consideration. These properties are transferred to the approximations φ_ε by formula (3.4) (the properties could be also derived straight from (3.3), even in the case when φ is not differentiable).

Remark 3.5. In case of differentiable and strictly monotone functions f and g (under notations from Lemma (3.3) the following properties are immediately verified):

$$\frac{\partial g}{\partial y} = \frac{1}{\frac{\partial f}{\partial x}} < 0,$$

$$\frac{\partial^2 g}{\partial y^2} = -\frac{1}{\left(\frac{\partial f}{\partial x}\right)^3} \cdot \frac{\partial^2 f}{\partial x^2} > 0.$$

Remark 3.6. If φ is strictly monotone, then φ_ε is strictly monotone and hence is invertible. Indeed, let x belong to the domain of φ and let $y_\varepsilon = \varphi_\varepsilon(x)$, $\varepsilon > 0$. We could immediately verify that $y_\varepsilon \longrightarrow \varphi(x)$ when $\varepsilon \to 0$ (in the case when we consider a continuous function φ).

We will need a more complicated result which is formulated in the following lemma.

Lemma 3.7. *Let $\varepsilon > 0$, $\eta > 0$ and let φ be a strictly monotone continuous function. Let x and x_ε belong to the domains of definition of φ and its regularization φ_η, respectively. Let us set $y_{\varepsilon,\eta} = \varphi_\eta(x_\varepsilon)$. Then if $x_\varepsilon \longrightarrow x$ when $\varepsilon \to 0$, we have $y_{\varepsilon,\eta} \longrightarrow y = \varphi(x)$ when $\varepsilon, \eta \to 0$.*

Proof. The proof follows from formula (3.4) and the estimates

$$\begin{aligned} |y_{\varepsilon,\eta} - y| &\le \int_{-1}^{1} J(z)\big|\varphi(x_\varepsilon - \eta z) - \varphi(x)\big|\,dz \\ &\le \sup_{|z|\le 1} \big|\varphi(x_\varepsilon - \eta z) - \varphi(x - \eta z)\big| + \sup_{|z|\le 1} \big|\varphi(x - \eta z) - \varphi(x)\big|. \ \square \end{aligned}$$

Theorem 3.8. *The function C from* (3.1) *is a bivariate copula if and only if the function φ is convex.*

Proof. We perform the proof in two steps using Definition 2.6.

<u>Step 1.</u> Let φ be a smooth function and a) $\varphi(0) < +\infty$ or b) $\varphi(0) = +\infty$. In the case a) the line

$$l = \left\{(x_1, x_2) \in I^2 \;\middle|\; \varphi(x_1) + \varphi(x_2) = \varphi(0)\right\},$$

is monotone decreasing and connects the points $(0,1)$ and $(1,0)$. In the area I^2 below the line (i.e. the part of I^2 that contains the origin $(0,0)$) we have that

$$\varphi(x_1) + \varphi(x_2) > \varphi(0),$$

and consequently $C(x_1, x_2) = 0$ in this area, as well as on l. Above l the function C is smooth. Let D be the part of I^2 above that line. Then for $f \in C_0^\infty(\mathbb{R}^2)$ which is zero when $x_1 = 1$ or $x_2 = 1$ and is non-negative, we have

$$\iint_D \varphi^{-1}\left(\varphi(\xi_1) + \varphi(\xi_2)\right) f_{x_1x_2}(\xi_1, \xi_2)\,d\xi_1 d\xi_2$$

$$= \int\limits_{\partial D} \varphi^{-1}\left(\varphi(\xi_1)+\varphi(\xi_2)\right) f_{x_1}(\xi_1,\xi_2)\cos(n,\xi_2)ds$$

$$- \int\limits_{D} \frac{\partial}{\partial \xi_2}\left[\varphi^{-1}\left(\varphi(\xi_1)+\varphi(\xi_2)\right) f_{x_1}(\xi_1,\xi_2)\right] d\xi_1 d\xi_2$$

$$= \int\limits_{1}^{0} \xi_1 f_{x_1}(\xi_1,1)d\xi_1 - \int\limits_{\partial D} \frac{\partial}{\partial \xi_2}\varphi^{-1}\left(\varphi(\xi_1)+\varphi(\xi_2)\right) f(\xi_1,\xi_2)\cos(n,\xi_1)ds$$

$$+ \int\limits_{D} \frac{\partial^2}{\partial \xi_1 \partial \xi_2}\left[\varphi^{-1}\left(\varphi(\xi_1)+\varphi(\xi_2)\right) f(\xi_1,\xi_2)\right] d\xi_1 d\xi_2,$$

where n is the unit outward pointing normal to ∂D and the equalities describing the behaviour of $C(x_1,x_2)$ on the boundary of I^2 are taken into consideration, i.e.

$$C(\xi_1,1)=\xi_1,\ C(1,\xi_2)=\xi_2.$$

Thanks to properties from Remark 3.5 we instantly compute that

$$\frac{\partial^2}{\partial \xi_1 \partial \xi_2}\left[\varphi^{-1}\left(\varphi(\xi_1)+\varphi(\xi_2)\right)\right] = \frac{d^2}{dt^2}(\varphi^{-1}(t))\varphi'(\xi_1)\varphi'(\xi_2) > 0.$$

The rest of the last sum is of the form

$$x_1 f(x_1,1)\Big|_1^0 - \int\limits_1^0 f(x_1,1)dx_1 - \int\limits_l \frac{\partial}{\partial \xi_2}\left[\varphi^{-1}\left(\varphi(\xi_1)+\varphi(\xi_2)\right)\right]\cos(n,\xi_1)ds > 0$$

keeping in mind that $f(x_1,1)=0$ and

$$\frac{\partial}{\partial \xi_2}\left[\varphi^{-1}\left(\varphi(\xi_1)+\varphi(\xi_2)\right)\right] = \frac{1}{\frac{d\varphi(t)}{dt}}\varphi'(\xi_2) > 0$$

and $\cos(n,\xi_1)<0$ over l.

In the case b) the proof is similar, but D coincides with I^2 and the line l is defined as $\{x_2=0\}\cap I^2$.

Step 2. Let the function φ be continuous, strictly monotone, decreasing and convex but not smooth.

Now we consider φ_ε instead of φ and $C_\varepsilon^2 = \varphi_\varepsilon^{[-1]}\big(\varphi_\varepsilon(x_1)+\varphi_\varepsilon(x_2)\big)$.

In Step 1 we verified that

$$\int\limits_{I^2} C_\varepsilon^2(x_1,x_2) f_{x_1x_2}(x_1,x_2)\, dx_1 dx_2 \geq 0 \tag{3.6}$$

for each $f \in C_0^\infty(\mathbb{R}^2)$, where it equals zero on the side $\{x_1 = 1\}$ and $\{x_2 = 1\}$ of I^2.

What we need to do now is to let ε tend to zero, i.e. $\varepsilon \to 0$ in (3.6), so that we conclude that $C(x_1, x_2)$ is a 2-increasing function.

Let $y_\varepsilon = \varphi_\varepsilon(x_1) + \varphi_\varepsilon(x_2)$. Then apparently $y_\varepsilon \to \varphi(x_1) + \varphi(x_2)$, whenever $\varepsilon \to 0$. Applying Lemma 3.7 to (3.6), we obtain the requirement from Definition 2.6. $\square$

Remark 3.9. The main result concerning Theorem 3.8 could be established by applying Definition 2.4.

At first let the function φ be twice differentiable and let $t = \varphi(x_1)+\varphi(x_2)$. For $0 \le t < 1$ we have that

$$\frac{\partial^2 C^2}{\partial x_1 \partial x_2} = \frac{\partial^2}{\partial x_1 \partial x_2}\left[\varphi^{-1}\big(\varphi(x_1) + \varphi(x_2)\big)\right] = \frac{d^2 \varphi^{-1}(t)}{dt^2}\frac{\partial \varphi}{\partial x_1}\frac{\partial \varphi}{\partial x_2}.$$

According to Remark 3.5 this expression is non-negative.

If $t > 1$, $C^2(x_1, x_2) = 0$, then its derivative $\dfrac{\partial^2 C^2}{\partial x_1 \partial x_2}$ is zero.

Now we need to check the non-negativity of this derivative in the case $t = 1$. In a neighbourhood of $t = 1$ the first derivative (in terms of the Theory of Distributions) is equal to

$$\frac{d}{dt}\varphi^{-1}(t) = \begin{cases} \dfrac{d\varphi^{-1}}{dt}, & \text{if } t < 1, \\ 0, & \text{if } t \ge 1. \end{cases}$$

According to

$$\frac{d^2\varphi^{-1}(t)}{dt^2}(t) = \begin{cases} \dfrac{d^2}{dt^2}\varphi^{-1}, & \text{if } t < 1 \\ \left[0 - \lim\limits_{\substack{t \to 1 \\ t<1}} \dfrac{d\varphi^{-1}}{dt}(t)\right]\delta(t-1), & \text{if } t = 1 \\ 0, & \text{if } t > 1, \end{cases}$$

where $\delta(t-1)$ is the Dirac function with a support at the point 1.

The last distribution is non-negative since φ^{-1} is monotone and decreasing, i.e. its derivative is positive.

However now we need to obtain the inequality (3.6), which requires a double application of the Gauss formula, so that we can transfer the derivative from C^2 to f.

The general case of n-dimensional Archimedean copulas refers to functions (formula 4.6.1 from [16]),

$$C^n(x_1, \dots, x_n) = \varphi^{[-1]}(\varphi(x_1) + \cdots + \varphi(x_n)). \tag{3.7}$$

Definition 3.10 (See Definition 4.6.1 from [16]). *A function $g(t)$ is called completely monotonic on an interval J, if it is continuous there and has derivatives of all orders that alternate in sign, i.e., if it satisfies*

$$(-1)^k \frac{d^k}{dt^k} g(t) \geq 0 \tag{3.8}$$

for all t in the interior of J and $k = 0, 1, \dots, m$.

Lets note that if the pseudo-inverse $\varphi^{[-1]}$ of an Archimedean generator φ is completely monotonic, then $\varphi^{[-1]} = \varphi^{-1}$. We establish the following (see Theorem 4.6.2 from [16])

Theorem 3.11. *Let φ be continuous and strictly decreasing function from $[0, 1]$ to $[0, \infty]$, such that $\varphi(0) = \infty$ and $\varphi(1) = 0$, and let φ^{-1} denote the inverse of φ. If $C^n(x_1, \dots, x_n)$ is a function from I^n to I given by by (3.7), then C^n is a n-copula for all $n \geq 2$ if and only if φ^{-1} is completely monotonic on $[0, +\infty)$.*

Proof. We extend the continuous function $C^n(x_1, \dots, x_n)$ as zero outside the sides of I^n that pass through the origin $(0, \dots, 0)$. The function we obtain is also continuous as C^n vanishes over these sides. In order to define the regularization C^n_ε of C^n over I^n, we extend C^n as a continuous function outside I^n in the first quadrant $\mathbb{R}^n_+$. We denote again with C^n the obtained extension.

We use a mollifier, whose support is translated along the vector $(-1, \dots, -1)$ (see Section 2 from [11] for $n = 2$). This assure the annihilation of C^n_ε as well as all its derivatives over the sides of I^n passing through the origin $(0, \dots, 0)$.

At first, just as in the proof of Theorem 3.8, we verify the validity of property

$$(-1)^n \int\limits_{I^n} C^n_\varepsilon \cdot f_{x_1 \cdots x_n} dx_1 \cdots dx_n \geq 0 \tag{3.9}$$

for each function $f \in C_0^\infty(\mathbb{R}^n)$, $f \geq 0$. We use the result from Remark 2.11b) which claims that we may presume that f vanishes, together with all its derivatives, over the sides of I^n passing through the vertex $(1, \dots, 1)$. After multiple applications of Gauss formula we represent the integral from the left hand side of (3.9) in the form

$$\int\limits_{I^n} \frac{\partial^n}{\partial x_1 \cdots \partial x_n}(C_\varepsilon^n) \cdot f(x_1, \dots, x_n) dx_1 \cdots dx_n.$$

Due to the fact that the derivative from the last integral is equal to

$$\left(\frac{\partial^n C^n}{\partial x_1 \cdots \partial x_n}\right)_\varepsilon, \quad \varepsilon > 0,$$

(see Theorem 1.6.1 from [10]). In the case of a smooth function φ (what we assumed) we have

$$\frac{\partial^n C^n}{\partial x_1 \cdots \partial x_n} = \frac{d^n \varphi^{-1}}{dt^n} \cdot \varphi'(x_1) \cdots \varphi'(x_n) \geq 0,$$

so the inequality (3.9) is established.

To complete the proof we let n tend to infinity in (3.9) for $\varepsilon \to 0$ just as this was done in Theorem (3.8). □

Remark 3.12. The necessity of the convexity condition for the function φ ($\varphi'' > 0$ in the smooth case) could be established easily by the following counter-example.

The function

$$\varphi(x) = \begin{cases} 1 - x^2, & x \leq 1, \\ 0, & x \geq 1. \end{cases}$$

is strictly decreasing and concave. We thus have

$$\varphi^{[-1]}(y) = \begin{cases} \sqrt{1-y}, & y \leq 1, \\ 0, & y \geq 1. \end{cases}$$

Let we define

$$\widetilde{C} = \varphi^{[-1]}\Big(\varphi(x_1) + \varphi(x_2)\Big).$$

Then, as we have

$$\varphi(x_1) + \varphi(x_2) = \begin{cases} 2 - x_1^2 - x_2^2, & 0 \leq x_1 \leq 1, & 0 \leq x_2 \leq 1 \\ 1 - x_1^2, & 0 \leq x_1 \leq 1, & x_2 \geq 1 \\ 1 - x_2^2, & x_1 \geq 1, & 0 \leq x_2 \leq 1 \\ 0, & x_1 \geq 1, & x_2 \geq 1, \end{cases}$$

for $(x_1, x_2) \in I^2 \setminus \{x_1^2 + x_2^2 > 1\}$, we obtain

$$\begin{aligned}\widetilde{C}(x_1, x_2) &= \varphi^{[-1]}\Big(\varphi(x_1) + \varphi(x_2)\Big) \\ &= \varphi^{[-1]}\Big(2 - x_1^2 - x_2^2\Big) = \sqrt{1 - \Big(2 - x_1^2 - x_2^2\Big)} \quad = \sqrt{x_1^2 + x_2^2 - 1}.\end{aligned}$$

Finally,

$$\frac{\partial^2 \widetilde{C}}{\partial x_1 \partial x_2}(x_1, x_2) = -x_1 x_2 \big(x_1^2 + x_2^2 - 1\big)^{-\frac{3}{2}} < 0,$$

for all $(x_1, x_2) \in I^2 \setminus \{x_1^2 + x_2^2 > 1\}$.

4. Construction of n-dimensional copulas. The present paragraph summarises expositions in Paragraphs 4 and 5 from [11] concerning bivariate copulas. A key result is the paper [6] referring to the Goursat problem over the unit cube.

The main goal is to construct the function $C(x_1, \ldots, x_n)$ based on predetermined values of the derivative

$$\frac{\partial^n C}{\partial x_1 \ldots \partial x_n} = f(x_1, \ldots, x_n), \tag{4.1}$$

and the properties of C on the boundary of I^n. For convenience we recall the definition of a copula C, namely

Definition 4.1. *The function $C : I^n \longrightarrow I$ is called a n-dimensional copula or simply a n-copula, if the following criteria are met*

1) $C(u_1, \ldots, u_n) = 0$, *if $u_k = 0$ for at least one index $k = 1, \ldots, n$;*

2) *if all coordinates of u are 1 except u_k, then*

$$C(1, \ldots, 1, u_k, 1, \ldots, 1) = u_k;$$

3) *C is n-increasing.*

The function f from 4.1 cannot be an arbitrary element of the corresponding proper space, it needs to fulfil certain conditions:

1) non-negativity in I^n (in the sense of the Theory of distributions – see 2.4), so that condition 3) from Definition 4.1 is fulfilled,

2) conditions on C that assure the necessary prerequisites in 2) from Definition 4.1.

Finally let us assume that the integrals below are defined and convergent. Since (4.1) implies that

$$C(x_1, \ldots, x_n) = \int\limits_{B_{x_1,\ldots,x_n}} f(\xi_1, \ldots, \xi_n)\, d\xi_1 \ldots \xi_n,$$

where with $B_{x_1,\ldots,x_n}$ we denoted the n-box

$$B_{x_1,\ldots,x_n} = [0, x_1] \times \cdots \times [0, x_n],\ x_i \in [0,1],\ i = 1, \ldots, n,$$

conditions about f (i.e. about C on the boundary of I^n, and more precisely condition 2) from Definition 4.1) would have the form

$$\int\limits_{B_i} f(\xi_1, \ldots, \xi_n)\, d\xi_1 \ldots d\xi_n = x_i, \tag{4.2}$$

where

$$B_i = [0,1] \times \ldots \times [0, x_i] \times \ldots \times [0,1],\ x_i \in [0,1],\ i = 1, \ldots, n.$$

We limit ourselves to the case when the required copula C, is continuous, which in the context of the embedding theorem (see Theorem 4.12 from [1]) means that $C \in W^{1,p}(I^n), p > n$. From here it immediately follows that in terms of (4.1), the right hand side f is an element of $W^{1-n,p}(I^n)$. This imposes the necessity to further consider the integrals in (4.2).

At first let us assume that $f \in L_p(I^n)$. Then the main equation (4.1) is fulfilled in terms of almost everywhere (a.e.) as well as the condition 3) from Definition 4.1 if we assume $f(x_1, \ldots, x_n) \geq 0$. Under these conditions and from Fubini's theorem (see §1.54 in [1]) it follows that restrictions

$$g_i(x_i) = f(x_1, \ldots, x_i, \ldots, x_n),$$

over the sets

$$\big\{x_i \in [0,1] \,\big|\, (x_1, \ldots, x_i, \ldots, x_n) \in I^n\big\},$$

are from $L_1[0,1]$ for almost all values of the rest of the variables

$$(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n) \in I^n.$$

According to Paragraph 4, Theorem 2 from [15] means that, in terms of a.e., the derivative of

$$\int_0^{x_i} g(\xi_i)\, d\xi_i, \tag{4.3}$$

is equal to $g(x_i) = f(x_1, \ldots, x_n)$, i.e. (4.1) makes sense.

We generalise this situation under considerably weaker requirements of f. The approach here is different than the one applied in [11] for bivariate copulas, where additional local requirements over f have been imposed. In practice it turns out that the additional requirements are of the same kind – see Remark 4.3.

In order to properly impose the requirements of the type (4.2) for $f \in W^{1-n,p}(I^n)$, we consider the existing and uniquely defined family of functions $\{f_\alpha\}$, α – multi-index, $f_\alpha \in L_p(I^n)$, $|\alpha| \le n-1$, such that

$$(f, u) = \sum_{0 \le |\alpha| \le n-1} (\partial^\alpha u, f_\alpha)$$

for every $u \in W^{n-1,q}(I^n)$, where q is determined from the condition $\dfrac{1}{p} + \dfrac{1}{q} = 1$ (see Theorem 3.9 from [1] and pages 343 and 344 from [11]). By extending f_α as zero outside I^n, for the obtained $\widetilde{f} \in W^{1-n,p}(\mathbb{R}^n)$ we can define the regularization

$$\widetilde{f}_\varepsilon(x_1, \ldots, x_n) = \big(\widetilde{f} * J_\varepsilon\big)(x_1, \ldots, x_n), \tag{4.4}$$

where J_ε, $\varepsilon > 0$, is a mollifier (see Paragraph 2.28 from [1]). Using this regularization we give requirements for f of the type (4.2). Instead of (4.2) we assume that $f \in W^{n-1,p}(I^n)$ obeys the conditions below

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \widetilde{f}_\varepsilon(\xi_1, \ldots, \xi_n) \chi_{B_i}\, d\xi_1 \ldots d\xi_n = x_i \tag{4.5}$$

for all $i = 1, \ldots, n$ and for each n-box

$$B_i = [0,1] \times \ldots \times [0, x_i] \times \ldots \times [0,1],\ x_i \in [0,1],$$

where χ_{B_i} is the characteristic function of B_i.

The main result in this section is the following theorem.

Theorem 4.2. *Let the function $f \in W^{1-n,p}(I^n)$, $p > n$, be such that*

a) *f satisfies the conditions*

$$\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^n} \widetilde{f}_\varepsilon(\xi_1,\dots,\xi_n)\chi_{B_i}\, d\xi_1\dots d\xi_n = x_i$$

for all $i=1,\dots,n$ and for each n-box $B_i=[0,1]\times\dots\times[0,x_i]\times\dots\times[0,1]$, $x_i\in[0,1]$;

b) *f is non-negative in the sense of the theory of distributions, i.e.*

$$(f,\varphi)\ge 0,\ \textit{for all}\ \varphi\in W_0^{n-1,q}(I^n); \tag{4.6}$$

c) *f satisfies the regularity condition (R) (which we formulate below).*

Then there is in I^n a unique solution $C\in W^{1,p}(I^n)$ of the problem

$$(-1)^n\big(C,\varphi_{x_1\dots x_n}\big)=(f,\varphi),\ \textit{for each}\ \varphi\in W_0^{n-1,q}(I^n),\ \frac{1}{q}+\frac{1}{p}=1 \tag{4.7a}$$

$$C(1,\dots,1,u_k,1,\dots,1)=u_k. \tag{4.7b}$$

We do the proof in three steps:

I. There is a weak solution to the problem above, i.e. $C\in L_q(I^n)$, for which the boundary conditions 2) from Definition 4.1 may not be satisfied.

II. The Theorem regarding uniqueness of solutions from $W^{1,p}(I^n)$.

III. Theorem for regularity of C, i.e. mainly when the condition (R) is satisfied (see below), where it is established that the solution $C\in W^{1,p}(I^n)$.

I. The proof of this step is similar to the one exposed in [11] for the case $n=2$ (applying the Hahn-Banach Theorem).

II. Let $C\in W^{1,p}(I^n)$, $p>n$, be the difference of two weak solutions from the problem in Theorem 4.2, i.e.

$$\big(C_{x_1\dots x_n},\varphi\big)=0, \tag{4.8}$$

for every $\varphi\in C_0^\infty(I^n)$ and $C\in C^0(I^n)$ condition 1) of Definition 4.1.

We extend C as zero on the complement of the first quadrant of $\mathbb{R}^n$ and then – as a continuous function in the first quadrant (see Paragraph 261 from [8]). We denote again with C the obtained extension. (The last one could be avoided if we consider the restriction of C over the smaller cube $I^n_h = [0,h] \times \ldots \times [0,h]$, where $h \to 1-0$. If we establish $C = 0$ in I^n_h, then we would obtain $C = 0$ in I^n by continuity.)

The regularization $C_\varepsilon = C * \widetilde{J}_\varepsilon$ now is well defined where again we use mollifier $\widetilde{J}_\varepsilon$ translated in the direction of $(-1,\ldots,-1)$, so that the condition 1) of Definition 4.1 applied for C_ε is preserved.

From formula (4.8) for φ coinciding with the mollifier $\widetilde{J}_\varepsilon$ it follows $(C_{x_1\ldots x_n})_\varepsilon = 0$. According to Theorem 1.6.1 from [12], the last one means that in I^n

$$(C_\varepsilon)_{x_1\ldots x_n} = 0. \tag{4.9}$$

Immediately from here we obtain the following representation

$$C_\varepsilon(x_1,\ldots,x_n) = \sum_{i=1}^{n} f_i(\widehat{x}_i), \tag{4.10}$$

where $\widehat{x}_i = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$.

For the k-th side of I^n, $k = 1,\ldots,n$, it is fulfilled that

$$\sum_{i=1}^{n} f_i(\widehat{x}_i)\bigg|_{x_k=0} = 0. \tag{4.11}$$

Only $f_k(\widehat{x}_k)$ does not depend on x_k, i.e. it depends on the rest of the $n-1$ variables, while all other functions in the above sum depend on $n-2$ variables. After differentiating with respect to $(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)$, from this equation we obtain that $(f_k)_{x_1\ldots x_{k-1}x_{k+1}\ldots x_n} = 0$. Therefore f_k is equal to the sum of $(n-1)$ functions in $(n-2)$ variables. This is derived for each of the functions f_i in (4.10) when $k = 1,2,\ldots,n$, respectively. Then we consider the "edges" of I^n, where 2 or more variables are equal to 0. In the same way this leads to representation of f_i and thus of C_ε as sums of functions in less variables and finally – sum of functions in one variable. Last equalities of boundary conditions for $(n-1)$ variables equal to zero and arbitrary n-th variable show that sums of the derived functions in one variable are constant and consequently so is C_ε. Together with the boundary conditions $C_\varepsilon = 0$ over the sides passing through the origin, we find that $C_\varepsilon = 0$ in I^n. Hence $C = \lim\limits_{\varepsilon\to 0} C_\varepsilon = 0$.

III. Let $\widetilde{f^m} = \{f^m_\alpha\}$ be a smooth approximation of $f = \{f_\alpha\}$, $f_\alpha \in L_p(F^n)$ satisfying $\widetilde{f^m} \longrightarrow \widetilde{f}\ \ W^{n-1,p}(I^n)$. In fact we approximate in $L_p(I^n)$ the terms f_α of the family represeting f. We may even consider that $f^m_\alpha \in C^\infty_0(I^n)$ (see Theorem 2.1.9 from [1]).

Let C^m denote the solution of the problem in Theorem 4.2 with right hand side equal to f^m instead of f. Since it is immediately obtained that

$$\partial_{x_1 \ldots x_n} C^m = \sum_{|\alpha| \leq n-1} (-1)^{|\alpha|} \partial_\alpha f_\alpha^m,$$

upon integration (and with respect to the boundary conditions) we find that

$$C^m(x_1, \ldots, x_n) = \sum_{|\alpha| \leq m-1} (-1)^{|\alpha|} \mathscr{D}^{\alpha'} f_\alpha^m, \tag{4.12}$$

where for each multi-index $\alpha \in \mathbb{N}^n$, $\mathbb{N} = \{0, 1, 2, \ldots\}$ it is set that

$$\begin{cases} \alpha' = ({\alpha'}_1, \ldots, {\alpha'}_n) = (\alpha_1 - 1, \ldots, \alpha_n - 1) \\ \mathscr{D}^{\alpha'} = \mathscr{D}_1^{\alpha'} \ldots \mathscr{D}_n^{\alpha'}, \end{cases} \tag{4.13}$$

where for each $i = 1, \ldots, n$

$$\mathscr{D}_i^{\alpha'_i} \varphi = \begin{cases} \int\limits_0^{x_i} \varphi(x_1, \ldots, x_n) dx_i, & \text{whenever } \alpha'_i = -1, \\ \varphi(x_1, \ldots, x_n), & \text{whenever } \alpha'_i = 0, \\ \partial_i^{\alpha'_i} \varphi, & \text{whenever } \alpha'_i > 0. \end{cases} \tag{4.14}$$

In [6] this notation is illustrated when $n = 3$.

Now we are in position to formulate the regularity condition (R).

(R): *Let us assume that the functions f_α, representing f satisfy*

$$\partial_{x_i} \mathscr{D}^{\alpha'} f_\alpha \in L_p(I^n),\ i = 1, \ldots, n,\ |\alpha| \leq n - 1, \tag{4.15}$$

where

$$\partial_i \mathscr{D}^{\alpha'} \varphi = \begin{cases} \varphi, & \textit{whenever } \alpha'_i = -1 \\ \partial_{x_i} \mathscr{D}^{\alpha'_i} \varphi, & \textit{whenever } \alpha'_i \geq 0. \end{cases}$$

Immediately under these conditions from (4.12) it is obtained that the norms

$$\|(C^m)_{x_i}\|_{L_p(I^n)} \leq K, \tag{4.16}$$

are bounded by a constant K, which does not depend on m.

Now since the problem under consideration is linear and the boundary conditions – homogeneous, the corollary that $C \in W^{1,p}(I^n)$ is obtained by the Banack-Saks Theorem (see [2]), which claims that the average values of a suitable sequence of C^m converges strongly (i.e. in the norm of $W^{1,p}(I^n)$) to a limit C.

Remark 4.3. Let us compare the condition (R) with the local requirements (25 a-c) from [11]. Let $n = 2$ and let us also assume that the function

$$z(x, y) = \partial_x \int_0^y f(x, \eta) d\eta$$

belongs to $L_p(I^n)$. Let ξ and η be the dual variables of x and y and if we transfer the conditions of z and f to their Fourier transformations (see [3] or [11]) from the equality $z_y = \partial_x f(x, y)$ we establish (after some extensions of the functions under consideration in the entire space) the relation: $\widehat{z} = \dfrac{\xi}{\eta} \widehat{f}(\xi, \eta)$.

Therefore the local conditions considered in [11] are equivalent (in some sense) to the condition (R).

Acknowledgements. The authors are very grateful to the referee for the careful reading of the paper and the useful suggestions for improving of the exposition.

REFERENCES

[1] R. A. Adams, J. J. F. Fournier. Sobolev Spaces, 2nd edition. Pure and Applied Mathematics, vol. **140**. Amsterdam, Elsevier/Academic Press, 2003.

[2] S. Banach. Theory of linear operators. (Théorie des opérations linéaires.) Monografie Matematyczne, vol. **1**. Warszawa, Seminarium Matematyczne Uniwersytetu Warszawskiego, Instytut Matematyczny PAN, 1932 (in French).

[3] J. Bergh, J. Löfström. Interpolation Spaces, An Introduction. Berlin, Springer-Verlag, 1976.

[4] U. Cherubini, E. Luciano, W. Vecchiato. Copula Methods in Finance. Chichester, John Wiley & Sons Ltd., 2004.

[5] N. Chervenov, B. Kostadinov. Generalisation of the Notion of a n-increasing function. Archimedean Copulas. BAS Reports, Sofia, Bulgaria, 2018.

[6] N. Chervenov, I. Iordanov, B. Kostadinov. Goursat Problem over unit cube in first quadrant of $\mathbb{R}^n$ (with applications to existence of Copulas), AIP Conference Proceedings, vol. **2048**, 040022 (2018), doi: 10.1063/1.5082094.

[7] F. Durante, C. Sempi. Principles of copula theory. Boca Raton, FL, CRC Press, 2016.

[8] G. M. Fikhtengolts. Rate differential and integral calculus. Textbook. In 3 vols. T.1. 11th ed. (Kurs differentsialnogo i integralnogo ischisleniya. Uchebnik. V 3-kh tt. T. 1. 11-e izd.) 2017, (in Russian).

[9] C. Genest, J. MacKay. Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données (Archimedean copulas and families of two-dimensional distributions with fixed marginals). *Canad. J. Statist.* **14**, *2* (1986), 145–159 (in French).

[10] L. Hörmander. Linear Partial Differential Operators. Berlin, Springer-Verlag, 1976.

[11] I. Iordanov, N. Chervenov. Copulas on Sobolev Spaces, *Serdica Math J.* **42**, *3–4* (2016), 335–360.

[12] H. Joe. Multivariate models and dependence concepts. Monographs on Statistics and Applied Probability, vol. **73**. London, Chapman & Hall, 1997.

[13] H. Joe. Dependence Modeling with Copulas. Monographs on Statistics and Applied Probability, vol. **133**. Boca Raton, FL, CRC Press, 2014.

[14] A. J. McNeil, J. Nešlehová. Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions. *Ann. Statist.* **37**, *5B* (2009), 3059–3097.

[15] I. P. Natanson. Theory of functions of a real variable. Sofia, Nauka, 1971 (in Bulgarian).

[16] R. B. Nelsen. An introduction to copulas, 2nd edition. Springer Series in Statistics, New York, NY, Springer, 2006.

Nikolay Chervenov
Faculty of Mathematics and Informatics
Sofia University
5, James Bourcheir Blvd
1164 Sofia, Bulgaria
e-mail: `nikolay.tchervenov@gmail.com`

Iordan Iordanov
International Business School
7, Van Gogh Str.
1527 Sofia, Bulgaria
e-mail: iiordanov@ibsedu.bg

Boyan Kostadinov
Faculty of Applied Mathematics and Informatics
Technical University of Sofia
8, Kliment Ohridski Blvd
1000 Sofia, Bulgaria
e-mail: boyan.sv.kostadinov@gmail.com

Received January 13, 2019