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LOG-CONVEXITY OF WEIGHTED AREA INTEGRAL MEANS OF H^p FUNCTIONS ON THE UPPER HALF-PLANE

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Abstract. In the present work weighted area integral means $M_{p,\varphi}(f;\operatorname{Im} z)$ are studied and it is proved that the function $y \to \log M_{p,\varphi}(f;y)$ is convex in the case when f belongs to a Hardy space on the upper half-plane.

1. Introduction. In the present paper we study three weighted area integral means of holomorphic on the upper half plane functions. They are defined as follows

$$M^{(1)}_{p,\varphi}(f;y) = \frac{\int_1^y \varphi'(t) \ \int_{-\infty}^{+\infty} |f(x+it)|^p dx \ dt}{\int_1^y \varphi'(t)dt},$$

$$M^{(0)}_{p,\varphi}(f;y) = \frac{\int_0^y \varphi'(t) \ \int_{-\infty}^{+\infty} |f(x+it)|^p dx \ dt}{\int_0^y \varphi'(t)dt},$$

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$$M_{p,\varphi}^{(\infty)}(f;y) = \frac{\int_y^{+\infty} \varphi'(t)\ \int_{-\infty}^{+\infty} |f(x+it)|^p dx\ dt}{\int_y^{+\infty} \varphi'(t) dt}$$

where $p > 0$, $y > 0$, the functions f and φ are such that the integrals exist and the fraction can be defined as a continuous function on $(0; +\infty)$.

The goal is to find specific conditions on the functions f and φ under which each one of these three weighted area integral means is log-convex on $(0; +\infty)$. This goal is partially achieved in Theorems 12, 13, 14, 15 where some sufficient conditions are presented. Our theorems show that in the case when f belongs to the Hardy space H^p, $2 \le p < +\infty$, these three weighted area integral means are similar to the classical integral means

$$M_p^p(f;y) = \int_{-\infty}^{+\infty} |f(x+iy)|^p dx, \quad y \in (0; +\infty)$$

in terms of its monotonic growth and convexity behavior. Moreover, there is a specific weight φ and a specific holomorphic function f such that f does not belong to any Hardy space and nevertheless such a similarity still exists.

In addition, note that Theorems 8 and 12 can be stated and proved with any positive number as the lower limit of the integrals instead of 1. Therefore, in this paper, the weighted area integral mean $M_{p,\varphi}^{(1)}$ represents without loss of generality the more general notion of weighted area integral means when the integrals involved in the definition of $M_{p,\varphi}^{(1)}$ have the lower limit 1 replaced by any positive number.

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During the period 2011–2016, there was a series of papers by Ch. Wang, J. Xiao and K. Zhu on weighted area integral means. In [8] volume integral means of holomorphic in the unit ball of $\mathbb{C}^n$ functions were studied. Among various results they stated a conjecture about convexity of $\log M_{p,\alpha}(f,r)$ in $\log r$. In [7] authors studied monotonic growth and logarithmic convexity of integral means which are important from a geometric point of view. In [5], [6], [8] authors proved theorems about convexity of log of a weighted area integral mean in $\log r$ in the case of holomorphic functions in the unit disk of $\mathbb{C}$. They considered the weight function φ with $\varphi'(|z|^2) = (1-|z|^2)^\alpha$. In [2], [3], [4] authors studied the case when f is an entire function on $\mathbb{C}$ and the weight function φ with $\varphi'(|z|^2) = e^{-\alpha|z|^2}$.

Note that the case of holomorphic functions on the upper half plane remainded unexplored.

Thus, the present paper contains theorems about weighted area integral means in a new case. We apply the method demonstrated in [5] and modify it with some details that are relevant to our goals.

A great deal of our computations are done and checked with a freeware open-source computer algebra system Maxima (wxMaxima) which is published at `http://maxima.sf.net`.

2. Definitions.

Definition 1. *Let I, $I \subset (-\infty; +\infty)$, be a non-empty open interval, and $\mathcal{D}^n(I)$ stand for the class of all real valued functions such that have a finite n-th derivative everywhere in I. If the functions $q : I \to (0; +\infty)$, $\varphi : I \to (-\infty; +\infty)$ and $M : I \to (0; +\infty)$ are such that $q \in \mathcal{D}^2(I)$, $\varphi \in \mathcal{D}^3(I)$, $M \in \mathcal{D}^2(I)$ then the functions A, B_0, C_0, B, C, E_1, E_2, F_1 and F_2 are defined as follows*

$$A = (q\varphi')'\varphi - q\varphi'^2, \quad B_0 = (q\varphi')'\varphi^2, \quad C_0 = q\varphi^2\varphi'^2,$$

$$B = (q\varphi' M)'\varphi^2, \quad C = q\varphi^2\varphi'^2 M^2, \quad E_1 = A^2\varphi + E_2,$$

$$E_2 = Aq\varphi\varphi'^2 - B_0' q\varphi\varphi' + (q\varphi')'\varphi q(\varphi^2\varphi')',$$

$$F_1 = \frac{B - \sqrt{B^2 - 4AC}}{2A}, \quad F_2 = \frac{B + \sqrt{B^2 - 4AC}}{2A},$$

where $'$ denotes differentiation and F_1, F_2 are defined on the subset of I defined by the conditions $A \neq 0$, $B^2 - 4AC \geq 0$.

Note that if $A \neq 0$, $B^2 - 4AC \geq 0$ then the functions F_1, F_2 are well defined real valued functions such that $AF_i^2 - BF_i + C = 0$, $i = 1, 2$.

Example 2. The following examples are used in the main theorems

(1) If $I = (0; +\infty)$, $q(x) = 1$, $\varphi(x) = \int_1^x t^{-a} dt$, where $x \in I$ and the constant $a > 0$ then

$$A(x) = -x^{-a-1}(\varphi(x) + 1), \quad E_1(x) = ax^{-2a-2}\varphi^2(x).$$

(2) If $I = (0; +\infty)$, $q(x) = 1$, $\varphi(x) = \int_1^x e^{-t} dt$, where $x \in I$ then

$$A(x) = -e^{-x-1}, \quad E_1(x) = e^{-2x-1}\varphi^2(x).$$

(3) If $I=(0;+\infty)$, $q(x)=1$, $\varphi(x)=\int_0^x t^{-a}dt$, where $x\in I$ and the constant $a<1$ then

$$A(x)=(a-1)^{-1}x^{-2a},\quad E_1(x)=0.$$

(4) If $I=(0;+\infty)$, $q(x)=1$, $\varphi(x)=\int_0^x e^{-t}dt$, where $x\in I$ then

$$A(x)=-e^{-x},\quad E_1(x)=e^{-2x}\varphi^2(x).$$

(5) If $I=(0;+\infty)$, $q(x)=1$, $\varphi(x)=-\int_x^{+\infty} t^{-a}dt$, where $x\in I$ and the constant $a>1$ then

$$A(x)=(a-1)^{-1}x^{-2a},\quad E_1(x)=0.$$

(6) If $I=(0;+\infty)$, $q(x)=1$, $\varphi(x)=-\int_x^{+\infty} e^{-t}dt$, where $x\in I$ then

$$A(x)=0,\quad E_1(x)=0.$$

(7) If $I=(0;+\infty)$, $q(x)=1$, $\varphi(x)=-\int_x^{+\infty} t^a e^{-t}dt$, where $x\in I$ and the constant $a<0$ then

$$A(x)>0,\quad E_1(x)>0.$$

The computations which are needed in (1)–(7) are simple and straight-forward and because of this they are omitted.

Auxiliary example: $I=(0;+\infty)$, $q(x)=1$, $\varphi(x)=-\int_x^{+\infty} e^{t-e^t}dt$, where $x\in I$,

$$A=-e^{x-2e^x}<0,\quad E_2=0,\quad E_1=A^2\varphi+E_2<0.$$

3. Auxiliary results.

Lemma 3. *Let I, $I\subset(-\infty;+\infty)$, be a non-empty open interval. If the functions $q:I\to(0;+\infty)$, $\varphi:I\to(-\infty;+\infty)$ are such that $q\in\mathcal{D}^2(I)$, $\varphi\in\mathcal{D}^3(I)$ and $\varphi'(x)\neq 0$ for all $x\in I$, then the following identities hold on I*

$$E_2=A^2\varphi-(AB_0-Aq(\varphi^2\varphi')'+A'q\varphi^2\varphi'),\tag{1}$$

$$E_1=-q^2\varphi^2\varphi'^3\left(\frac{(q\varphi')'\varphi}{q\varphi'^2}\right)'.\tag{2}$$

Remark 4. Note that it follows by this lemma and the definition of E_2 that

$$\text{(3)}\qquad \left|\begin{aligned} A^2\varphi + (-E_2) &= AB_0 - Aq(\varphi^2\varphi')' + A'q\varphi^2\varphi', \\ \varphi(-E_2) &= (-1)(AC_0 - B_0'q\varphi^2\varphi' + B_0q(\varphi^2\varphi')'). \end{aligned}\right.$$

P r o o f o f L e m m a 3. Let the functions q and φ meet the conditions from the lemma. Identity (1) follows from the computation[1]

$$\begin{aligned}
&\left(A^2\varphi - \left(AB_0 - Aq(\varphi^2\varphi')' + A'q\varphi^2\varphi'\right) - E_2\right)\varphi \\
&= A^2\varphi^2 - \left(AB_0 - Aq(\varphi^2\varphi')' + A'q\varphi^2\varphi'\right)\varphi \\
&\qquad\qquad\qquad\qquad - \left(AC_0 - B_0'q\varphi^2\varphi' + B_0q(\varphi^2\varphi')'\right) \\
&= A(B_0\varphi - C_0) - AB_0\varphi + \underline{Aq\varphi(\varphi^2\varphi')'} - \underline{\underline{A'q\varphi^3\varphi'}} \\
&\qquad\qquad\qquad\qquad - AC_0 + \underline{\underline{B_0'q\varphi^2\varphi'}} - \underline{B_0q(\varphi^2\varphi')'} \\
&= -2AC_0 + q(\varphi^2\varphi')'\,(A\varphi - B_0) - q\varphi\varphi'\left(A'\varphi^2 - B_0'\varphi\right) \\
&= -\underline{2AC_0} - q(\varphi^2\varphi')'q\varphi\varphi'^2 - q\varphi\varphi'\left(-\underline{2A\varphi\varphi'} + B_0\varphi' - C_0'\right) \\
&\qquad\qquad\qquad = q\varphi\varphi'\left(-q\varphi'(\varphi^2\varphi')' - (B_0\varphi' - C_0')\right) = 0.
\end{aligned}$$

In order to prove identity (2) note that by identity (1) it follows that $E_1 = 2A^2\varphi - AB_0 + Aq(\varphi^2\varphi')' - A'q\varphi^2\varphi'$. So,

$$\begin{aligned}
E_1 &= A(2A\varphi - B_0 + q(\varphi^2\varphi')') - A'q\varphi^2\varphi' \\
&= A\left(2(\underline{(q\varphi')'\varphi} - \underline{\underline{q\varphi'^2}})\varphi - \underline{(q\varphi')'\varphi^2} + q(\underline{\underline{2\varphi\varphi'^2}} + \varphi^2\varphi'')\right) - A'q\varphi^2\varphi' \\
&= A\left((q\varphi')'\varphi^2 + q\varphi^2\varphi''\right) - A'q\varphi^2\varphi' = A\left((q\varphi')'\varphi' + q\varphi'\varphi''\right)\frac{\varphi^2}{\varphi'} - A'q\varphi^2\varphi' \\
&= -\frac{\varphi^2}{\varphi'}(A'q\varphi'^2 - A(q\varphi'^2)') = -\frac{\varphi^2}{\varphi'}(q\varphi'^2)^2\left(\frac{A}{q\varphi'^2}\right)' = -q^2\varphi^2\varphi'^3\left(\frac{A}{q\varphi'^2}\right)' \\
&\qquad = -q^2\varphi^2\varphi'^3\left(\frac{(q\varphi')'\varphi - q\varphi'^2}{q\varphi'^2}\right)' = -q^2\varphi^2\varphi'^3\left(\frac{\varphi(q\varphi')'}{q\varphi'^2}\right)'. \ \square
\end{aligned}$$

Lemma 5. *Let I, $I \subset (-\infty; +\infty)$, be a non-empty open interval. Assume the functions $q : I \to (0; +\infty)$, $\varphi : I \to (-\infty; +\infty)$ and $M : I \to (0; +\infty)$ are*

[1] $A'\varphi^2 - B_0'\varphi + C_0' = (-2A\varphi + B_0)\varphi'$ follows from $A\varphi^2 - B_0\varphi + C_0 = 0$ on I.

such that $q \in \mathcal{D}^2(I)$, $\varphi \in \mathcal{D}^3(I)$, $M \in \mathcal{D}^2(I)$. *Then, the following identities hold on* I

$$B' q\varphi^2\varphi' M = B(B - B_0 M) + B_0' q\varphi^2\varphi' M^2 + (\varphi^2\varphi')' qM(B - B_0 M) + C\varphi^2 \left(q\frac{M'}{M}\right)', \tag{4}$$

$$C' q\varphi^2\varphi' M = Cq(\varphi^2\varphi')' M + 2BC - B_0 CM. \tag{5}$$

Proof. The proof of identity (4) is as follows

$$\begin{aligned}
B' q\varphi^2\varphi' M &= \left(M\left(B_0 + q\varphi^2\varphi'\,\frac{M'}{M}\right)\right)' q\varphi^2\varphi' M \\
&= M'(B_0 + q\varphi^2\varphi'\frac{M'}{M}) q\varphi^2\varphi' M + M\left(B_0' + \left(\varphi^2\varphi'\, q\frac{M'}{M}\right)'\right) q\varphi^2\varphi' M \\
&= M' B q\varphi^2\varphi' + B_0' q\varphi^2\varphi' M^2 \\
&\qquad + (\varphi^2\varphi')'\, q\frac{M'}{M} q\varphi^2\varphi' M^2 + \varphi^2\varphi' \left(q\frac{M'}{M}\right)' q\varphi^2\varphi' M^2 \\
&= Bq\varphi^2\varphi' M' + B_0' q\varphi^2\varphi' M^2 \\
&\qquad + (\varphi^2\varphi')'\, q\varphi^2\varphi' qMM' + q\varphi^2\varphi'\varphi^2\varphi' M^2 \left(q\frac{M'}{M}\right)' \\
&= B(B - B_0 M) + B_0' q\varphi^2\varphi' M^2 \\
&\qquad + (\varphi^2\varphi')' qM(B - B_0 M) + C\varphi^2 \left(q\frac{M'}{M}\right)'.
\end{aligned}$$

The proof of identity (5) is as follows

$$\begin{aligned}
C' q\varphi^2\varphi' M &= (q\varphi'\,\varphi'\varphi^2\, M^2)' q\varphi^2\varphi' M \\
&= \left((q\varphi')'\,\varphi^2\varphi'\, M^2 + q\varphi'\,(\varphi^2\varphi')'\, M^2 + q\varphi'\,\varphi'\varphi^2\, 2MM'\right) q\varphi^2\varphi' M \\
&= B_0 CM + q(\varphi^2\varphi')'\, CM + 2q\varphi^2\varphi'^2 M^2\, q\varphi^2\varphi' M' \\
&= B_0 CM + q(\varphi^2\varphi')'\, CM + 2C(B - B_0 M) \\
&= Cq(\varphi^2\varphi')'\, M + 2BC - B_0 CM.
\end{aligned}$$

□

Lemma 6. *Let I, $I \subset (-\infty;+\infty)$, be a non-empty open interval. Assume the functions $q : I \to (0;+\infty)$, $\varphi : I \to (-\infty;+\infty)$, $M : I \to (0;+\infty)$ and $h : I \to (-\infty;+\infty)$ meet the conditions*

(*i*) $q \in \mathcal{D}^2(I)$, $\varphi \in \mathcal{D}^3(I)$, $M \in \mathcal{D}^2(I)$,

(*ii*) *there exists a non empty open subinterval J of I, $J \subseteq I$, such that $A(x) \neq 0$ for all $x \in J$ and $B^2 - 4AC \geq 0$ on J,*

(*iii*) $h' = \varphi' M$ *on I.*

Then, the following identities hold on J

$$(h - F_1)' A(F_2 - F_1) q\varphi^2\varphi' M = F_1 \left((F_1 - \varphi M)(A^2(F_1 - \varphi M) + E_1 M) + C\varphi^2 \left(q \frac{M'}{M} \right)' \right), \tag{6}$$

$$(h - F_2)' A(F_1 - F_2) q\varphi^2\varphi' M = F_2 \left((F_2 - \varphi M)(A^2(F_2 - \varphi M) + E_1 M) + C\varphi^2 \left(q \frac{M'}{M} \right)' \right). \tag{7}$$

P r o o f. Each of these identities is a result of direct simple and rather long computations.

The proof of identity (6) is as follows.

$$(h - F_1)' A(F_2 - F_1) q\varphi^2\varphi' M = (h' - F_1') A(F_2 - F_1) q\varphi^2\varphi' M$$

$$= \varphi' M A(F_2 - F_1) q\varphi^2\varphi' M - F_1' A(F_2 - F_1) q\varphi^2\varphi' M. \tag{8}$$

By the definition of the functions F_1, F_2 it follows that

$$A(F_2 - F_1) = \sqrt{B^2 - 4AC} = -2AF_1 + B,$$

$$AF_1^2 - BF_1 + C = 0 \implies A'F_1^2 - B'F_1 + C' = F_1'(-2AF_1 + B)$$

and hence $F_1' A(F_2 - F_1) = A'F_1^2 - B'F_1 + C'$.

So, from (8) it follows that

$$(h - F_1)' A(F_2 - F_1) q\varphi^2\varphi' M = A(F_2 - F_1)C - (A'F_1^2 - B'F_1 + C') q\varphi^2\varphi' M$$

$$= A(F_2 - F_1)(-AF_1^2 + BF_1) - A'F_1^2 q\varphi^2\varphi' M + B'F_1 q\varphi^2\varphi' M - C' q\varphi^2\varphi' M.$$

Now, identities (4) and (5) from Lemma 5 allow us to obtain

$$
\begin{aligned}
&(h-F_1)'A(F_2-F_1)q\varphi^2\varphi' M\\
&= -A^2F_1^2F_2 + ABF_1F_2 + A^2F_1^3 - ABF_1^2 - A'F_1^2 q\varphi^2\varphi' M\\
&\qquad + BF_1(B-B_0M) + B_0' q\varphi^2\varphi' M^2F_1 + (\varphi^2\varphi')' qM(B-B_0M)F_1\\
&\qquad + C\varphi^2\left(q\frac{M'}{M}\right)' F_1 - Cq(\varphi^2\varphi')' M - 2BC + B_0CM\\
&= -ACF_1 + \underline{BC} + A^2F_1^3 - \underline{ABF_1^2} - A'F_1^2 q\varphi^2\varphi' M\\
&\qquad + \underline{(AF_1^2+C)(B-B_0M)} + B_0' q\varphi^2\varphi' M^2F_1 + \underline{(\varphi^2\varphi')' qMBF_1}\\
&\qquad - (\varphi^2\varphi')' qB_0M^2F_1 + C\varphi^2\left(q\frac{M'}{M}\right)' F_1 - (-AF_1^2 + \underline{BF_1})q(\varphi^2\varphi')' M\\
&\qquad - \underline{2BC} + \underline{B_0CM}
\end{aligned}
$$

where all the underlined parts cancel out. Thus,

$$
\begin{aligned}
&(h-F_1)'A(F_2-F_1)q\varphi^2\varphi' M\\
&= -ACF_1 + A^2F_1^3 - A'F_1^2 q\varphi^2\varphi' M + B_0' q\varphi^2\varphi' M^2F_1\\
&\qquad - (\varphi^2\varphi')' qB_0M^2F_1 + C\varphi^2\left(q\frac{M'}{M}\right)' F_1 + AF_1^2 q(\varphi^2\varphi')' M\\
&= F_1\Bigg(A^2F_1^2 - (AB_0 - Aq(\varphi^2\varphi')' + A'q\varphi^2\varphi')F_1M\\
&\qquad -(AC_0 - B_0' q\varphi^2\varphi' + (\varphi^2\varphi')' qB_0)M^2 + C\varphi^2\left(q\frac{M'}{M}\right)'\Bigg).
\end{aligned}
$$

Finally, by identities (3) we obtain

$$
\begin{aligned}
&(h-F_1)'A(F_2-F_1)q\varphi^2\varphi' M\\
&\qquad = F_1\left((F_1-\varphi M)(A^2F_1 + E_2M) + C\varphi^2\left(q\frac{M'}{M}\right)'\right)\\
&\qquad = F_1\left((F_1-\varphi M)(A^2(F_1-\varphi M) + E_1M) + C\varphi^2\left(q\frac{M'}{M}\right)'\right).
\end{aligned}
$$

The computations that prove identity (7) are omitted as they are similar to those that prove identity (6). □

Lemma 7. *Let I, $I \subset (-\infty; +\infty)$, be a non-empty open interval. Assume the functions $q : I \to (0; +\infty)$, $\varphi : I \to (-\infty; +\infty)$, $M : I \to (0; +\infty)$ and $h : I \to (-\infty; +\infty)$ meet the conditions*

(i) $q \in \mathcal{D}^2(I)$, $\varphi \in \mathcal{D}^3(I)$, $M \in \mathcal{D}^2(I)$,

(ii) there exists a non empty open subinterval J of I, $J \subseteq I$, such that $A = 0$ and $\varphi' > 0$ on J and $B(x) \neq 0$ for all $x \in J$,

(iii) $h' = \varphi' M$ *on* I.

Then, the following identity holds on J

$$\left(h - \frac{C}{B}\right)' B^2 q\varphi' M = C^2 \left(q\frac{M'}{M}\right)'.$$

Proof. *Claim.* $\left(\dfrac{(q\varphi')'}{\varphi'}\right)' = 0$ on J and $\varphi(x) \neq 0$ for all $x \in J$.

Indeed, note that $A = 0$ implies $(q\varphi')'\varphi = q\varphi'^2 > 0$. So, $\varphi(x) \neq 0$ for all $x \in J$. Therefore, from $A = 0$ follows that $\dfrac{(q\varphi')'}{\varphi'} = \dfrac{q\varphi'}{\varphi}$ and

$$\left(\frac{(q\varphi')'}{\varphi'}\right)' = \left(\frac{q\varphi'}{\varphi}\right)' = \frac{A}{\varphi^2} = 0.$$

Thus, the claim is proved.

Now, the lemma follows from the following computations

$$\left(h - \frac{C}{B}\right)' Bq\varphi^2\varphi' M = \left(h' - \frac{C'B - CB'}{B^2}\right) Bq\varphi^2\varphi' M$$

$$= \varphi' M Bq\varphi^2\varphi' M + B'\frac{C}{B}q\varphi^2\varphi' M - C' q\varphi^2\varphi' M.$$

B' and C' are substituted accordingly to identities (4) and (5) from Lemma 5

$$\left(h - \frac{C}{B}\right)' Bq\varphi^2\varphi' M$$

$$= \underline{BC} + \frac{C}{B}\Big(\underline{B(B - B_0 M)} + B_0' q\varphi^2\varphi' M^2 + (\varphi^2\varphi')' qM(\underline{B} - B_0 M)$$

$$+C\varphi^2\left(q\frac{M'}{M}\right)'\Bigg) - \underline{(Cq(\varphi^2\varphi')'M + 2BC - B_0CM)}$$

where all the underlined parts cancel out. Thus,

$$\begin{aligned}
&\left(h-\frac{C}{B}\right)' Bq\varphi^2\varphi' M \\
&= \frac{C}{B}\left(B_0'q\varphi^2\varphi' M^2 - (\varphi^2\varphi')'qM^2B_0 + C\varphi^2\left(q\frac{M'}{M}\right)'\right) \\
&= \frac{C}{B}\left(qM^2(\varphi^2\varphi')^2\left(\frac{B_0}{\varphi^2\varphi'}\right)' + C\varphi^2\left(q\frac{M'}{M}\right)'\right) \\
&= \frac{C}{B}\left(qM^2(\varphi^2\varphi')^2\left(\frac{(q\varphi')'}{\varphi'}\right)' + C\varphi^2\left(q\frac{M'}{M}\right)'\right).
\end{aligned}$$

Finally, accordingly to the claim, it follows that

$$\left(h-\frac{C}{B}\right)' Bq\varphi^2\varphi' M = \frac{C^2}{B}\varphi^2\left(q\frac{M'}{M}\right)'. \qquad \square$$

4. Main theorems. In this section Theorems 8, 9, 10 are stated and proved, and these theorems represent the main theorems of the paper. These theorems are about the case when $q(x) = 1$ for all $x \in (0;+\infty)$.

Note that Definition 1 and the results from the previous section, all they are used in the proofs of the main theorems in the specific case when $q(x) = 1$ for all $x \in (0;+\infty)$. In particular,

$$A = \varphi''\varphi - \varphi'^2, \quad B = (\varphi' M)'\varphi^2, \quad C = \varphi^2\varphi'^2M^2,$$

and, by (2), $E_1 = -\varphi^2\varphi'^3\left(\frac{\varphi''\varphi}{\varphi'^2}\right)' = -\varphi^2(\varphi'^2\varphi'' + \varphi\varphi'\varphi''' - 2\varphi\varphi''^2)$.

The following Theorem can be stated and proved with any positive number as the lower limit of the integrals instead of 1.

Theorem 8. *Assume the functions* $\varphi : (0;+\infty) \to (-\infty;+\infty)$, $M : (0;+\infty) \to (0;+\infty)$ *and* $h : (0;+\infty) \to (-\infty;+\infty)$ *are such that* $\varphi \in \mathcal{D}^3(0;+\infty)$, $M \in \mathcal{D}^2(0;+\infty)$ *and meet the conditions* (*i*) $M' < 0$, *and* $(\log M)'' \geq 0$ *on* $(0; +\infty)$,

(*ii*) $\varphi' > 0$ *on* $(0; +\infty)$, *and* $\varphi(x) = \int_1^x \varphi'(t)dt$ *for all* $x \in (0; +\infty)$,

(*iii*) $\varphi'^2\varphi'' + \varphi\varphi'\varphi''' - 2\varphi\varphi''^2 \leq 0$ *on* $(0; +\infty)$,

(*iv*) $h(x) = \int_1^x \varphi'(t)M(t)dt$ *for all* $x \in (0; +\infty)$.

Then, $\frac{h}{\varphi}$ *and* $\log \frac{h}{\varphi}$ *both belong to* $\mathcal{D}^2(0; +\infty)$ *and moreover,*

$$\left(\frac{h}{\varphi}\right)' < 0, \left(\log \frac{h}{\varphi}\right)'' > 0 \text{ on } (0; +\infty).$$

Proof. *Claim 1.* $\frac{h}{\varphi} \in \mathcal{D}^2(0; +\infty)$.

Indeed, accordingly to the assumptions, it is clear that the functions h and φ belong to $\mathcal{D}^3(0; +\infty)$. Moreover, $\varphi(x) = 0 \iff x = 1$. So, it is sufficient to prove that $\frac{h}{\varphi}$ has asymptotic expansion of the form

$$\frac{h(x)}{\varphi(x)} = \alpha_0 + \alpha_1(x-1) + \alpha_2(x-1)^2 + o(x-1)^2,$$

as $x \to 1$, where α_0, α_1, α_2 are real numbers that do not depend on x.

The expansion is obtained as follows. Note that $h(1) = \varphi(1) = 0$ and

$$h(x) = h'(1)(x-1) + \frac{1}{2}h''(1)(x-1)^2 + \frac{1}{6}h'''(1)(x-1)^3 + o(x-1)^3, \tag{9}$$

$$\varphi(x) = \varphi'(1)(x-1) + \frac{1}{2}\varphi''(1)(x-1)^2 + \frac{1}{6}\varphi'''(1)(x-1)^3 + o(x-1)^3, \tag{10}$$

as $x \to 1$ with $\varphi'(1) \neq 0$ (by the assumptions of the theorem). Therefore,

$$\frac{h(x)}{\varphi(x)} = \beta_0 + \beta_1(x-1) + \beta_2(x-1)^2 + o(x-1)^2,$$

where $\beta_0 = \frac{h'(1)}{\varphi'(1)}$, $\beta_1 = \frac{1}{2}(h''(1) - \varphi''(1)\beta_0)\frac{1}{\varphi'(1)}$, and

$$\beta_2 = \frac{1}{6}(h'''(1) - 3\beta_1\varphi''(1) - \beta_0\varphi'''(1))\frac{1}{\varphi'(1)}.$$

So,

$$\frac{h(x)}{\varphi(x)} = M(1) + \frac{1}{2}M'(1)(x-1) + \frac{1}{6}(M''(1) + \frac{\varphi''(1)}{2\varphi'(1)}M'(1))(x-1)^2 + o(x-1)^2,$$

as $x \to 1$ and the claim is proved.

Let us define the value of $\frac{h}{\varphi}$ at $x = 1$ to be equal to

$$\lim_{x\to 1} \frac{h(x)}{\varphi(x)} = M(1)$$

and note that $M(1) > 0$. Moreover,

$$\left(\frac{h(x)}{\varphi(x)}\right)'\bigg|_{x=1} = \frac{1}{2}M'(1), \quad \left(\frac{h(x)}{\varphi(x)}\right)''\bigg|_{x=1} = \frac{1}{3}(M''(1) + \frac{\varphi''(1)}{2\varphi'(1)}M'(1)),$$

where $|_{x=1}$ stands for 'the value at $x = 1$'.

Now, it follows from the claim and from $\frac{h}{\varphi} > 0$ on $(0; +\infty)$ that $\log \frac{h}{\varphi}$ is well defined on $(0; +\infty)$ and belongs to $\mathcal{D}^2(0; +\infty)$.

Here, it is verified that the functions h, φ and M satisfy an important simple inequality.

Claim 2. $h - \varphi M > 0$ on $(0; 1) \cup (1; +\infty)$.

This inequality holds because of

$$h(x) - \varphi(x)M(x) = (-1)\int_1^x \varphi(t)M'(t)dt > 0$$

for all $x \in (0; 1) \cup (1; +\infty)$.

Hence, the derivative

$$\left(\frac{h}{\varphi}\right)' = (-1)\frac{\varphi'}{\varphi^2}(h - \varphi M) < 0, \text{ on } (0; +\infty).$$

The derivative $\left(\log \frac{h}{\varphi}\right)''$ is calculated as follows.

Accordingly to the assumptions of the theorem, $E_1 \geq 0$ on $(0; +\infty)$. Furthermore, by the definition of C the inequality $C > 0$ holds on $(0; 1) \cup (1; +\infty)$, and

$$A(1) = \varphi''(1)\varphi(1) - \varphi'^2(1) = -\varphi'^2(1) < 0,$$

$$\left(\frac{A}{\varphi'^2}\right)' = \frac{\varphi\varphi'\varphi''' + \varphi'^2\varphi'' - 2\varphi\varphi''^2}{\varphi'^3} \leq 0 \text{ on } (0;+\infty)$$

and $\lim_{x\to 1} B(x) = B(1) = 0$, $\lim_{x\to 1} C(x) = C(1) = 0$.

So, $\frac{A}{\varphi'^2} = \frac{\varphi''\varphi}{\varphi'^2} - 1$ decreases on $(0;+\infty)$; therefore,

- $\left.\frac{A}{\varphi'^2}\right|_{x=1} < 0 \implies$ two possible cases exist: $A < 0$ on $(0;+\infty)$ or there exists a number $x_A \in (0;1)$ such that $A > 0$ on $(0;x_A)$, $A(x_A) = 0$ and $A < 0$ on $(x_A;+\infty)$ (these cases are discussed bellow as *Case 1* and *Case 2*);
- $\frac{\varphi''\varphi}{\varphi'^2}$ decreases on $(0;+\infty)$, its value $\left.\frac{\varphi''\varphi}{\varphi'^2}\right|_{x=1} = 0$ and hence $\varphi'' < 0$, $B = (\varphi''M + \varphi'M')\varphi^2 < 0$ on $(0;1)\cup(1;+\infty)$.

Now, by $\varphi'' < 0$ and

$$B^2 - 4AC = \varphi^2(\varphi^2\varphi'^2M'^2 + 2\varphi^2\varphi'\varphi''MM' + (\varphi''\varphi - 2\varphi'^2)^2M^2)$$

it follows that $B^2 - 4AC \geq 0$ on $(0;1)\cup(1;+\infty)$.

Therefore, the functions F_1, F_2 are real valued well defined functions on the subset of $(0;1)\cup(1;+\infty)$ where A does not vanish and

$$\lim_{x\to 1} F_1(x) = \lim_{x\to 1} F_2(x) = 0,$$

by the definition F_1 and F_2.

Case 1. Let us suppose that the function φ is such that

$$A < 0 \text{ on } (0;+\infty).$$

In this case, $F_2 < 0 < \frac{B}{2A} < F_1$ on $(0;1)\cup(1;+\infty)$. Moreover,

$$\left.(AF^2 - BF + C)\right|_{F=\varphi M} = -\varphi^3\varphi'MM' \begin{cases} < 0, & x \in (0;1), \\ > 0, & x \in (1;+\infty). \end{cases}$$

Therefore,

$$\begin{cases} \varphi M < F_2 < 0 < F_1, & \text{on } (0;1), \\ F_2 < 0 < \varphi M < F_1, & \text{on } (1;+\infty). \end{cases}$$

In order to prove $Ah^2 - Bh + C < 0$ on $(0;1) \cup (1;+\infty)$ it is sufficient to prove that values of h are outside the interval of roots of the equation $AF^2 - BF + C = 0$.

First, by identity (7) from Lemma 6 with $q = 1$, it follows that

$$(h - F_2)' A(F_1 - F_2)\varphi^2\varphi' M$$
$$= F_2\left((F_2 - \varphi M)(A^2(F_2 - \varphi M) + E_1 M) + C\varphi^2 \left(\frac{M'}{M}\right)'\right) < 0,$$

and hence $(h - F_2)' > 0$ on $(0;1)$. By the definition of F_2, it follows that the left hand side limit

$$\lim_{x\to 1^-}(h(x) - F_2(x)) = 0.$$

Thus, $h - F_2 < 0$, and hence $h < F_2 < F_1$ on $(0;1)$. Therefore,

$$Ah^2 - Bh + C < 0 \text{ on } (0;1).$$

Second, by the identity (6) from Lemma 6 with $q = 1$, it follows that

$$(h - F_1)' A(F_2 - F_1)\varphi^2\varphi' M$$
$$= F_1\left((F_1 - \varphi M)(A^2(F_1 - \varphi M) + E_1 M) + C\varphi^2 \left(\frac{M'}{M}\right)'\right) > 0;$$

hence, $(h - F_1)' > 0$ on $(1;+\infty)$. By the definition of F_1, it follows that the right hand side limit

$$\lim_{x\to 1^+}(h(x) - F_1(x)) = 0.$$

Thus, $h - F_1 > 0$, and hence, $F_2 < F_1 < h$ on $(1;+\infty)$. Therefore,

$$Ah^2 - Bh + C < 0 \text{ on } (1;+\infty).$$

Hence, in *Case 1*, $Ah^2 - Bh + C < 0$ on $(0;1) \cup (1;+\infty)$.

Case 2. Let us suppose that the function φ is such that there exists a number $x_A \in (0;1)$ such that $A > 0$ on $(0;x_A)$, $A(x_A) = 0$ and $A < 0$ on $(x_A;+\infty)$.

Note, if $x \in (1;+\infty)$ then the proof of $Ah^2 - Bh + C < 0$ on $(1;+\infty)$ is identical with the one showed in Second part of Case 1, so it is omitted here.

Thus, it is sufficient to prove $Ah^2 - Bh + C < 0$ on $(0;1)$.

In this case,

$$(AF^2 - BF + C)\big|_{F=\varphi M} = -\varphi^3\varphi' M M' < 0 \text{ on } (0;1).$$

Therefore,

$$\begin{cases} F_1 < \varphi M < F_2 < 0, & \text{on } (0; x_A), \\ \varphi M < F_2 < 0 < F_1, & \text{on } (x_A; 1). \end{cases}$$

Note, $\varphi M < h$ from *Claim 2*, and $h < 0$ on $(0; 1)$.

By identity (7) from Lemma 6 with $q = 1$, it follows that

$$\begin{aligned}(h - F_2)' A(F_1 - F_2)\varphi^2\varphi' M \\ = F_2 \left((F_2 - \varphi M)(A^2(F_2 - \varphi M) + E_1 M) + C\varphi^2 \left(\frac{M'}{M} \right)' \right) < 0,\end{aligned}$$

and hence, $(h - F_2)' > 0$ on $(0; x_A) \cup (x_A; 1)$. So, $h - F_2$ increases on $(0; x_A) \cup (x_A; 1)$.

Moreover, $h - F_2$ increases on $(0; 1)$ because of continuity of h and continuity of

$$F_2 = \frac{2C}{B - \sqrt{B^2 - 4AC}} \text{ at } x_A, \quad F_2(x_A) = \frac{C(x_A)}{B(x_A)}.$$

Therefore, $h - F_2 < 0$ on $(0; 1)$, as $\lim_{x \to 1^-} (h - F_2) = 0$ and hence,

$$\begin{cases} F_1 < \varphi M < h < F_2 < 0, & \text{on } (0; x_A), \\ \varphi M < h < F_2 < 0 < F_1, & \text{on } (x_A; 1). \end{cases}$$

Hence $Ah^2 - Bh + C\big|_{x_A} = -B \left(h - \frac{C}{B} \right)\Big|_{x_A} = -B(h - F_2)\big|_{x_A} < 0$, and

$$Ah^2 - Bh + C < 0 \text{ on } (0; 1).$$

Hence, in *Case 2*, $Ah^2 - Bh + C < 0$ on $(0; 1) \cup (1; +\infty)$.

Therefore

$$Ah^2 - Bh + C < 0 \text{ on } (0; 1) \cup (1; +\infty) \tag{11}$$

in both cases, *Case 1* and *Case 2*.

Claim. There exists the limit

$$\lim_{x \to 1} \frac{(-1)(Ah^2 - Bh + C)}{\varphi^2 h^2} > 0$$

and the second derivative

$$\left(\log \frac{h}{\varphi} \right)'' = \frac{-1}{\varphi^2 h^2}(Ah^2 - Bh + C) \text{ on } (0; +\infty).$$

In particular, $\left(\log\dfrac{h}{\varphi}\right)''$ is a well defined continuous function on $(0;+\infty)$.

The proof about the limit of this *Claim* is sketched out only. The limit is computed by using expansions (9), (10),

$$M(x) = M(1) + M'(1)(x-1) + \frac{1}{2}M''(1)(x-1)^2 + o(x-1)^2, \quad \text{as} \quad x \to 1,$$

and the values of the derivatives of h are substituted as follows

$$h'(1) = \varphi'(1)M(1), \quad h''(1) = \varphi''(1)M(1) + \varphi'(1)M'(1),$$

$$h'''(1) = \varphi'''(1)M(1) + 2\varphi''(1)M'(1) + \varphi'(1)M''(1).$$

The elements of the numerator, Ah^2, Bh and C, are calculated with a precision of $o(x-1)^4$. So, the numerator

$$(-1)(A(x)h(x)^2 - B(x)h(x) + C(x))$$

$$= \frac{\varphi'^4(1)M^2(1)}{6}\left(\frac{M'(1)}{M(1)}\frac{\varphi''(1)}{\varphi'(1)} + \frac{4M(1)M''(1) - 3M'^2(1)}{2\,M^2(1)}\right)(x-1)^4 + o(x-1)^4,$$

as $x \to 1$.

The denominator, φ^2h^2, is calculated with a precision of $o(x-1)^4$. So, the denominator

$$\varphi^2h^2 = M^2(1)\varphi'^4(1)(x-1)^4 + o(x-1)^4,$$

as $x \to 1$.

Thus, there exists the limit

$$\lim_{x\to 1}\frac{(-1)(Ah^2 - Bh + C)}{\varphi^2h^2} = \frac{1}{6}\left(\frac{M'(1)}{M(1)}\frac{\varphi''(1)}{\varphi'(1)} + \frac{4M(1)M''(1) - 3M'^2(1)}{2\,M^2(1)}\right).$$

Note that the limit is a positive number because of

$$M'(1)\varphi''(1) \geq 0,$$

$$4M(1)M''(1) - 3M'^2(1) = M(1)M''(1) + 3(M(1)M''(1) - M'^2(1)) > 0.$$

Now, we calculate the second derivative

$$\left(\log\frac{h}{\varphi}\right)'' = \left(\left(\log\frac{h}{\varphi}\right)'\right)' = \left(\frac{h'\varphi}{h\varphi'}\right)' = \left(\frac{\varphi M}{h}\right)' = \frac{(-1)(Ah^2 - Bh + C)}{\varphi^2h^2}$$

on $(0;1) \cup (1;+\infty)$ and it is continuous on $(0;1)$ and $(1;+\infty)$. The existence of the finite limit of the second derivative as $x \to 1$ is already proved in this *Claim.* This result and the existence of the finite derivative $\left(\log \frac{h}{\varphi}\right)''$ at $x = 1$ (note that $\log \frac{h}{\varphi} \in \mathcal{D}^2(0;+\infty)$), they imply that the second derivative $\left(\log \frac{h}{\varphi}\right)''$ is a continuous function at $x = 1$. Therefore, the second derivative $\left(\log \frac{h}{\varphi}\right)''$ is a continuous function on $(0;+\infty)$.

So, the *Claim* is proved.

By this *Claim* and (11) it follows that the second derivative

$$\left(\log \frac{h}{\varphi}\right)'' > 0 \text{ on } (0;+\infty).$$

Thus, the theorem is proved. □

Theorem 9. *Assume the functions $\varphi : [0;+\infty) \to [0;+\infty)$, $M : [0;+\infty) \to (0;+\infty)$ and $h : [0;+\infty) \to [0;+\infty)$ are such that $\varphi \in \mathcal{D}^3(0;+\infty)$, $M \in \mathcal{D}^2(0;+\infty)$ and meet the conditions*

(*i*) *the right hand side limit* $\lim\limits_{x\to 0^+} \dfrac{\varphi''(x)\varphi(x)}{\varphi'^2(x)} < 1$,

(*ii*) *the functions M and M' are continuous from the right at $x = 0$ and $M(0) \neq +\infty$ and $M'(0) \neq -\infty$,*

(*iii*) $M' < 0$, $(\log M)'' \geq 0$ *on* $(0;+\infty)$,

(*iv*) $\varphi' > 0$ *on* $(0;+\infty)$ *and* $\varphi(x) = \int_0^x \varphi'(t)dt$ *for all* $x \in (0;+\infty)$,

(*v*) $\varphi'^2\varphi'' + \varphi\varphi'\varphi''' - 2\varphi\varphi''^2 \leq 0$ *on* $(0;+\infty)$,

(*vi*) $h(x) = \int_0^x \varphi'(t)M(t)dt$ *for all* $x \in (0;+\infty)$.

Then, $\frac{h}{\varphi}$ *and* $\log \frac{h}{\varphi}$ *both belong to* $\mathcal{D}^2(0;+\infty)$ *and moreover,*

$$\left(\frac{h}{\varphi}\right)' < 0, \left(\log \frac{h}{\varphi}\right)'' > 0 \text{ on } (0;+\infty).$$

P r o o f. By the assumptions, it is clear that $\frac{h}{\varphi}$ and $\log \frac{h}{\varphi}$ both belong to $\mathcal{D}^2(0;+\infty)$. Furthermore,

$$h(x) - \varphi(x)M(x) = \int_0^x (-1)\varphi(t)M'(t)dt > 0 \text{ for all } x \in (0;+\infty).$$

Hence,

$$\left(\frac{h}{\varphi}\right)' = \frac{-\varphi'}{\varphi^2}(h - \varphi M) < 0 \text{ on } (0;+\infty).$$

Accordingly to the assumptions of the theorem, $E_1 \geq 0$ on $(0;+\infty)$. Furthermore,

$$\left(\frac{A}{\varphi'^2}\right)' = \frac{\varphi\varphi'\varphi''' + \varphi'^2\varphi'' - 2\varphi\varphi''^2}{\varphi'^3} \leq 0 \text{ on } (0;+\infty).$$

So, $\frac{A}{\varphi'^2} = \frac{\varphi''\varphi}{\varphi'^2} - 1$ decreases on $(0;+\infty)$ and $\frac{A}{\varphi'^2} \leq \lim_{x\to 0^+} \frac{A}{\varphi'^2} < 0$.

The inequalities $A < 0$, $C > 0$ on $(0;+\infty)$ imply that F_1, F_2 are well defined real valued functions on $(0;+\infty)$ and together with the inequality

$$(AF^2 - BF + C)\big|_{F=\varphi M} = -\varphi^3\varphi' MM' > 0$$

it follows that

$$F_2 < 0 < \varphi M < F_1 \text{ on } (0;+\infty).$$

We apply the identity (6) from Lemma 6 with $q = 1$ to obtain

$$\begin{aligned}&(h - F_1)'A(F_2 - F_1)\varphi^2\varphi' M \\ &\qquad = F_1\left((F_1 - \varphi M)(A^2(F_1 - \varphi M) + E_1 M) + C\varphi^2\left(\frac{M'}{M}\right)'\right) > 0.\end{aligned}$$

So, $(h - F_1)' > 0$ on $(0;+\infty)$.

Moreover, $(h - F_1)|_{0^+} = 0$ because of $h(0^+) = 0$ and

$$\begin{aligned}&\lim_{x\to 0^+} F_1(x) = \lim_{x\to 0^+} \frac{B - \sqrt{B^2 - 4AC}}{2A} \\ &= \lim_{x\to 0^+} \frac{(\varphi'' M + \varphi' M')\varphi^2 - \sqrt{(\varphi'' M + \varphi' M')^2\varphi^4 - 4A\varphi^2\varphi'^2 M^2}}{2A}\end{aligned}$$

$$= \lim_{x\to 0^+} \varphi \frac{\left((\frac{A}{\varphi'^2}+1)M+\frac{\varphi}{\varphi'}M'\right)-\sqrt{\left((\frac{A}{\varphi'^2}+1)M+\frac{\varphi}{\varphi'}M'\right)^2-4\frac{A}{\varphi'^2}M^2}}{2\frac{A}{\varphi'^2}} = 0.$$

Hence, $h - F_1 > 0$ on $(0;+\infty)$. Therefore, $F_2 < 0 < \varphi M < F_1 < h$ and

$$Ah^2 - Bh + C < 0 \text{ on } (0;+\infty).$$

Thus, the second derivative

$$\left(\log\frac{h}{\varphi}\right)'' = \left(\left(\log\frac{h}{\varphi}\right)'\right)' = \left(\frac{h'\varphi}{h\varphi'}\right)' = \left(\frac{\varphi M}{h}\right)' = \frac{(-1)(Ah^2-Bh+C)}{\varphi^2h^2} > 0$$

on $(0;+\infty)$.

The proof of the theorem is completed. □

Theorem 10. *Assume the functions* $\varphi : (0;+\infty) \to (-\infty;0)$, $M : (0;+\infty) \to (0;+\infty)$ *and* $h : (0;+\infty) \to (-\infty;0)$ *are such that* $\varphi \in \mathcal{D}^3(0;+\infty)$, $M \in \mathcal{D}^2(0;+\infty)$ *and meet the conditions*

(*i*) $M' < 0$ *and* $(\log M)'' \geq 0$ *on* $(0;+\infty)$,

(*ii*) $\varphi' > 0$ *on* $(0;+\infty)$ *and* $\varphi(x) = -\int_x^{+\infty} \varphi'(t)dt$, $\forall x \in (0;+\infty)$,

(*iii*) $\varphi'^2\varphi'' + \varphi\varphi'\varphi''' - 2\varphi\varphi''^2 \leq 0$ *on* $(0;+\infty)$,

(*iv*) $h(x) = -\int_x^{+\infty} \varphi'(t)M(t)dt$ *for all* $x \in (0;+\infty)$.

Then, $\frac{h}{\varphi}$ *and* $\log\frac{h}{\varphi}$ *both belong to* $\mathcal{D}^2(0;+\infty)$ *and moreover,*

$$\left(\frac{h}{\varphi}\right)' < 0, \left(\log\frac{h}{\varphi}\right)'' > 0 \textit{ on } (0;+\infty).$$

P r o o f. By the assumptions, it is clear that $\frac{h}{\varphi}$ and $\log\frac{h}{\varphi}$ both belong to $\mathcal{D}^2(0;+\infty)$.

Claim. $h - \varphi M > 0$ on $(0;+\infty)$. Indeed, by the assumptions of the theorem

- $(h - \varphi M)' = -\varphi M' < 0$ on $(0;+\infty)$;

- M decreases on $(0;+\infty)$ and $M>0$. So, the limit $\lim\limits_{x\to+\infty} M(x)$ exists and it is a non-negative number.

Therefore, $h-\varphi M$ decreases on $(0;+\infty)$ and

$$\lim_{x\to+\infty}(h(x)-\varphi(x)M(x))=0.$$

Hence, $h-\varphi M>0$ on $(0;+\infty)$.

It follows from this *Claim* that the derivative

$$\left(\frac{h}{\varphi}\right)'=\frac{-\varphi'}{\varphi^2}(h-\varphi M)<0 \text{ on } (0;+\infty).$$

Note that for every $x\in(0;+\infty)$

$$(AF^2-BF+C)\big|_{F=\varphi M}=-\varphi^3\varphi' MM'<0. \tag{12}$$

Accordingly to the assumptions of the theorem, $E_1\geq 0$ on $(0;+\infty)$. Furthermore,

$$\left(\frac{A}{\varphi'^2}\right)'=\frac{\varphi\varphi'\varphi'''+\varphi'^2\varphi''-2\varphi\varphi''^2}{\varphi'^3}\leq 0$$

and $\dfrac{A}{\varphi'^2}$ decreases on $(0;+\infty)$.

Now, there are three cases to consider (and it is not possible to prove that $\varphi''<0$).

Case 1. $A<0$ on $(0;+\infty)$. In this case $B^2-4AC>0$ and hence F_1, F_2 are well defined real valued functions such that

$$\varphi M<F_2<0<F_1.$$

Hence, $0=\lim\limits_{x\to+\infty}\varphi(x)M(x)\leq\lim\limits_{x\to+\infty}F_2(x)\leq 0$ and

$$\lim_{x\to+\infty}(h(x)-F_2(x))=0.$$

We apply the identity (7) from Lemma 6 with $q=1$ to obtain

$$(h-F_2)'A(F_1-F_2)\varphi^2\varphi' M$$

$$=F_2\left((F_2-\varphi M)(A^2(F_2-\varphi M)+E_1M)+C\varphi^2\left(\frac{M'}{M}\right)'\right)<0.$$

So, $(h-F_2)'>0$ on $(0;+\infty)$ and $h-F_2$ increases on $(0;+\infty)$.

Therefore, $h-F_2<0$ on $(0;+\infty)$ and by $\varphi M<h<F_2<0<F_1$ it follows that

$$Ah^2-Bh+C<0 \text{ on } (0;+\infty).$$

Case 2. There exists $x_A\in(0;+\infty)$ such that $A>0$ on $(0;x_A)$, $A(x_A)=0$, $A<0$ on $(x_A;+\infty)$.

In this case,

- if $x\in(0;x_A)$ then

  $$(AF^2-BF+C)\big|_{F=\varphi M}=-\varphi^3\varphi' MM'<0,$$

  $C>0$ and hence F_1, F_2 are well defined real valued functions such that

  $$F_1<\varphi M<F_2<0.$$

  We apply the identity (7) from Lemma 6 with $q=1$ to obtain

  $$(h-F_2)'A(F_1-F_2)\varphi^2\varphi' M$$
  $$=F_2\left((F_2-\varphi M)(A^2(F_2-\varphi M)+E_1M)+C\varphi^2\left(\frac{M'}{M}\right)'\right)<0.$$

  So, $(h-F_2)'>0$ on $(0;x_A)$.

- if $x\in(x_A;+\infty)$ then $B^2-4AC>0$ and hence F_1, F_2 are well defined real valued functions such that

  $$\varphi M<F_2<0<F_1.$$

  Hence, $0=\lim\limits_{x\to+\infty}\varphi(x)M(x)\le\lim\limits_{x\to+\infty}F_2(x)\le 0$ and

  $$\lim_{x\to+\infty}(h(x)-F_2(x))=0.$$

  We apply the identity (7) from Lemma 6 with $q=1$ to obtain

  $$(h-F_2)'A(F_1-F_2)\varphi^2\varphi' M$$
  $$=F_2\left((F_2-\varphi M)(A^2(F_2-\varphi M)+E_1M)+C\varphi^2\left(\frac{M'}{M}\right)'\right)<0.$$

  So, $(h-F_2)'>0$ on $(x_A;+\infty)$.

Thus, $h - F_2$ increases on $(0; x_A)$ and on $(x_A; +\infty)$.

Moreover, in this case, $\varphi''(x_A) < 0$. Hence, the inequality $B = (\varphi'' M + \varphi' M')\varphi^2 < 0$ holds in a neighborhood of x_A. So,

$$F_2 = \frac{2C}{B - \sqrt{B^2 - 4AC}}$$

is continuous.

Hence, $h - F_2$ increases on $(0; +\infty)$. Therefore, $h - F_2 < 0$ on $(0; +\infty)$.

Now, we prove that $Ah^2 - Bh + C < 0$ on $(0; +\infty)$. Indeed,

- $Ah^2 - Bh + C < 0$ on $(0; x_A)$ because of $F_1 < \varphi M < h < F_2 < 0$ and $A > 0$;
- $Ah^2 - Bh + C < 0$ on $(x_A; +\infty)$ because of $\varphi M < h < F_2 < 0 < F_1$ and $A < 0$.

Case 3. $A > 0$ on $(0; +\infty)$. In this case,

$$(AF^2 - BF + C)\big|_{F=\varphi M} = -\varphi^3 \varphi' M M' < 0,$$

$C > 0$ and hence F_1, F_2 are well defined real valued functions such that

$$F_1 < \varphi M < F_2 < 0.$$

Hence, $0 = \lim\limits_{x \to +\infty} \varphi(x) M(x) \le \lim\limits_{x \to +\infty} F_2(x) \le 0$ and

$$\lim_{x \to +\infty} (h(x) - F_2(x)) = 0.$$

We apply the identity (7) from Lemma 6 with $q = 1$ to obtain

$$(h - F_2)' A(F_1 - F_2)\varphi^2 \varphi' M$$

$$= F_2 \left((F_2 - \varphi M)(A^2(F_2 - \varphi M) + E_1 M) + C\varphi^2 \left(\frac{M'}{M} \right)' \right) < 0.$$

So, $(h - F_2)' > 0$ on $(0; +\infty)$. Hence, $h - F_2$ increases on $(0; +\infty)$. Therefore, $h - F_2 < 0$ and $F_1 < \varphi M < h < F_2 < 0$ on $(0; +\infty)$.

So,

$$Ah^2 - Bh + C < 0 \text{ on } (0; +\infty)$$

holds in *Case 3* and moreover, it holds in all three cases.

The second derivative

$$\left(\log \frac{h}{\varphi} \right)'' = \left(\left(\log \frac{h}{\varphi} \right)' \right)' = \left(\frac{h' \varphi}{h \varphi'} \right)' = \left(\frac{\varphi M}{h} \right)' = \frac{(-1)(Ah^2 - Bh + C)}{\varphi^2 h^2} > 0$$

on $(0; +\infty)$.

Thus the theorem is proved. □

5. Applications. Here we start with a note about integral means of holomorphic on the upper half-plane functions proved in a paper [1] by G. Hardy, A. Ingham, G. Pólya in 1927. In Subsection 4.3 they proved that if the holomorphic function in a strip of the complex plane meets some conditions such as growth at infinity $O(e^{e^{k|z|}})$ and convergence of the integrals on the boundaries of the strip then in the case when $2 \le p < +\infty$ the integral mean

$$M(y) = \int_{-\infty}^{+\infty} |f(x+iy)|^p dx$$

has first derivative

$$M'(y) = p \int_{-\infty}^{+\infty} |f(x+iy)|^{p-2}(uu'_y + vv'_y)dx$$

and second derivative

$$M''(y) = p^2 \int_{-\infty}^{+\infty} |f(x+iy)|^{p-2}(u'^2_y + v'^2_y)dx$$

where u is the real part of f and v is the imaginary part of f. Moreover, they proved that

$$M'^2 \le M''M.$$

So, $(\log M)'' \ge 0$. Note that, $M'' \ge 0$.

In the present paper we consider such an integral mean of holomorphic on the upper half-plane function under the conditions $2 \le p < +\infty$ and

$$\sup_{y>0} M(y) < +\infty$$

i.e. we consider function f that belongs to the Hardy space H^p of holomorphic on the upper half-plane functions. It is well known (see "Bounded analytic functions" by J. Garnett) that such a function meets the growth condition $|f(x+iy)| = O(y^{-1/p})$ (both, as $y \to 0^+$ and $y \to +\infty$), the integral mean M is a non-increasing function on $(0; +\infty)$, the right hand side limit at $y = 0$ is

$$M(0) = M(0^+) = \lim_{y \to 0^+} M(y) = \sup_{y>0} M(y) < +\infty,$$

where $M(0)$ is defined to be the L^p norm to the power of p of the boundary values of f.

Now, note if f is not the zero function then $M' < 0$ on $(0; +\infty)$. Indeed, if there is a $y_0 \in (0; +\infty)$ such that $M'(y_0) = 0$ then

- on the one hand, $M' \geq 0$ on $(y_0; +\infty)$ because of $M'' \geq 0$,
- on the other hand, $M' \leq 0$ on $(0; +\infty)$ as M is a non-increasing function.

Hence, $M' = 0$ on $(y_0; +\infty)$. Therefore, $M'' = 0$ on $(y_0; +\infty)$. So, $|f'|^2 = u_y'^2 + v_y'^2 = 0$ and $f = 0$ because it is the only constant function that belongs to the Hardy space H^p, $2 \leq p < +\infty$.

Thus, we have proved the following lemma

Lemma 11. *Let p be such that $2 \leq p < +\infty$, $f \in H^p$ (H^p is the Hardy space of holomorphic functions on the upper half-plane). If f is not the zero function then the integral mean*

$$M(y) = \int_{-\infty}^{+\infty} |f(x+iy)|^p dx$$

is a bounded continuous function on $[0; +\infty)$ such that $M \in \mathcal{D}^2(0; +\infty)$ and

$$M > 0, \quad M' < 0, \quad M'' > 0, \quad (\log M)'' \geq 0 \text{ on } (0; +\infty).$$

From this point of our paper through the its end p is such that $2 \leq p < +\infty$ and H^p is the Hardy space of holomorphic functions on the upper half-plane, $f \neq 0$, i.e. f is not the zero function and the integral mean

$$M(y) = \int_{-\infty}^{+\infty} |f(x+iy)|^p dx$$

is defined for $y \in [0; +\infty)$.

Note, Definition 1 from Section 2 and the results from Section 3, all they are used in the proofs of the following theorems in the specific case when $q(y) = 1$ for all $y \in (0; +\infty)$. In particular,

$$A = \varphi''\varphi - \varphi'^2, \quad B = (\varphi' M)'\varphi^2, \quad C = \varphi^2\varphi'^2 M^2,$$

and, by (2), $E_1 = -\varphi^2\varphi'^3 \left(\dfrac{\varphi''\varphi}{\varphi'^2}\right)' = -\varphi^2(\varphi'^2\varphi'' + \varphi\varphi'\varphi''' - 2\varphi\varphi''^2)$.

The following Theorem can be stated and proved with any positive number as the lower limit of the integrals instead of 1.

Theorem 12. *Let $2 \leq p < +\infty$, $f \in H^p \setminus \{0\}$. Assume the functions $\varphi : (0; +\infty) \to (-\infty; +\infty)$ and $h : (0; +\infty) \to (-\infty; +\infty)$ meet the conditions*

(i) $\varphi \in \mathcal{D}^3(0; +\infty)$, $\varphi(y) = \int_1^y \varphi'(t)dt$ *for all* $y \in (0; +\infty)$,

(ii) $\varphi' > 0$ *and* $\varphi'^2\varphi'' + \varphi\varphi'\varphi''' - 2\varphi\varphi''^2 \le 0$ *on* $(0; +\infty)$,

(iii) $h(y) = \int_1^y \varphi'(t)M(t)dt$ *for all* $y \in (0; +\infty)$.

Then, $\frac{h}{\varphi}$ *and* $\log \frac{h}{\varphi}$ *both belong to* $\mathcal{D}^2(0; +\infty)$ *and moreover,*

$$\left(\frac{h}{\varphi}\right)' < 0, \left(\log \frac{h}{\varphi}\right)'' > 0 \text{ on } (0; +\infty).$$

This theorem is a simple corollary of Lemma 11 and Theorem 8 and we omit the details. Theorem 12 holds with each one of the following functions

- $\varphi(y) = \int_1^y t^{-a}dt,\ a > 0;$
- $\varphi(y) = \int_1^y e^{-t}dt.$

Theorem 13. *Let* $2 \le p < +\infty$, $f \in H^p \setminus \{0\}$. *Assume the functions* $\varphi : (0; +\infty) \to (-\infty; +\infty)$ *and* $h : (0; +\infty) \to (-\infty; +\infty)$ *meet the conditions*

(i) $\varphi(y) = \int_0^y \varphi'(t)dt,\ \varphi' > 0$ *for all* $y \in (0; +\infty)$,

(ii) $\lim_{y\to 0^+} \frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} < 1,\ \varphi'^2\varphi'' + \varphi\varphi'\varphi''' - 2\varphi\varphi''^2 \le 0$ *on* $(0; +\infty)$,

or as an alternative

(ii′) $\lim_{y\to 0^+} \frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = 1,\ \varphi'^2\varphi'' + \varphi\varphi'\varphi''' - 2\varphi\varphi''^2 < 0$ *on* $(0; +\infty)$,

(iii) $h(y) = \int_0^y \varphi'(t)M(t)dt$ *for all* $y \in (0; +\infty)$.

Then, $\frac{h}{\varphi}$ *and* $\log \frac{h}{\varphi}$ *both belong to* $\mathcal{D}^2(0; +\infty)$ *and moreover,*

$$\left(\frac{h}{\varphi}\right)' < 0, \left(\log \frac{h}{\varphi}\right)'' > 0 \text{ on } (0; +\infty).$$

P r o o f. By Lemma 11, $M > 0$ and $M' < 0$, $(\log M)'' \geq 0$ on $(0; +\infty)$.

As in the proof of Theorem 9, by the assumptions, it is clear that $\dfrac{h}{\varphi}$ and $\log \dfrac{h}{\varphi}$ both belong to $\mathcal{D}^2(0; +\infty)$. Furthermore,

$$h(y) - \varphi(y)M(y) = \int_0^y (-1)\varphi(t)M'(t)dt > 0 \text{ for all } y \in (0; +\infty).$$

Hence,

$$\left(\frac{h}{\varphi}\right)' = \frac{-\varphi'}{\varphi^2}(h - \varphi M) < 0 \text{ on } (0; +\infty).$$

Accordingly to the assumptions of the theorem, $E_1 \geq 0$ on $(0; +\infty)$. Furthermore,

- in the case of the assumption (ii),

$$\left(\frac{A}{\varphi'^2}\right)' = \frac{\varphi\varphi'\varphi''' + \varphi'^2\varphi'' - 2\varphi\varphi''^2}{\varphi'^3} \leq 0 \text{ on } (0; +\infty).$$

So, $\dfrac{A}{\varphi'^2} = \dfrac{\varphi''\varphi}{\varphi'^2} - 1$ decreases on $(0; +\infty)$ and

$$\frac{A}{\varphi'^2} \leq \lim_{y \to 0^+} \frac{A}{\varphi'^2} < 0.$$

- in the case of the assumption (ii'),

$$\left(\frac{A}{\varphi'^2}\right)' = \frac{\varphi\varphi'\varphi''' + \varphi'^2\varphi'' - 2\varphi\varphi''^2}{\varphi'^3} < 0 \text{ on } (0; +\infty).$$

So, $\dfrac{A}{\varphi'^2} = \dfrac{\varphi''\varphi}{\varphi'^2} - 1$ decreases on $(0; +\infty)$ and

$$\frac{A}{\varphi'^2} < \lim_{y \to 0^+} \frac{A}{\varphi'^2} = 0.$$

Thus, in both cases, $A < 0$ on $(0; +\infty)$.

The inequalities $A < 0$, $C > 0$ on $(0; +\infty)$ imply that the functions F_1, F_2 are real valued well defined functions on $(0; +\infty)$ and together with the inequality

$$(AF^2 - BF + C)\big|_{F = \varphi M} = -\varphi^3\varphi' M M' > 0$$

it follows that

$$F_2 < 0 < \varphi M < F_1 \text{ on } (0;+\infty). \tag{13}$$

We apply the identity (6) from Lemma 6 with $q = 1$ to obtain

$$(h - F_1)' A(F_2 - F_1)\varphi^2\varphi' M$$
$$= F_1 \left((F_1 - \varphi M)(A^2(F_1 - \varphi M) + E_1 M) + C\varphi^2 \left(\frac{M'}{M}\right)' \right) > 0.$$

So,

$$(h - F_1)' > 0 \text{ on } (0;+\infty). \tag{14}$$

Let $\varepsilon > 0$ and

$$M_\varepsilon(y) = \int_{-\infty}^{+\infty} |f(x + (y+\varepsilon)i)|^p dx, \quad \forall y \in (0;+\infty).$$

Thus,

$$M_\varepsilon(y) = M(y+\varepsilon),\ M_\varepsilon(0) = M(\varepsilon),\ M'_\varepsilon(0) = M'(\varepsilon),\ h_\varepsilon(y) = \int_0^y \varphi(t)M(t+\varepsilon)dt$$

and

$$B_\varepsilon(y) = (\varphi(y)M(y+\varepsilon) + \varphi'(y)M'(y+\varepsilon))\varphi^2(y)$$

$C_\varepsilon = \varphi^2\varphi'^2 M^2(y+\varepsilon)$,where $y > 0$. As it is above, functions

$$F_{1,\varepsilon} = \frac{B_\varepsilon - \sqrt{B_\varepsilon^2 - 4AC_\varepsilon}}{2A}, \quad F_{2,\varepsilon} = \frac{B_\varepsilon + \sqrt{B_\varepsilon^2 - 4AC_\varepsilon}}{2A}$$

are real valued well defined functions on $(0;+\infty)$ and together with the inequality

$$(AF^2 - B_\varepsilon F + C_\varepsilon)\big|_{F=\varphi M_\varepsilon} = -\varphi^3\varphi' M_\varepsilon M'_\varepsilon > 0$$

it follows that

$$F_{2,\varepsilon} < 0 < \varphi M_\varepsilon < F_{1,\varepsilon} \text{ on } (0;+\infty).$$

We apply the identity (6) from Lemma 6 wiht $q = 1$ to obtain

$$(h_\varepsilon - F_{1,\varepsilon})' A(F_{2,\varepsilon} - F_{1,\varepsilon})\varphi^2\varphi' M_\varepsilon$$
$$= F_{1,\varepsilon} \left((F_{1,\varepsilon} - \varphi M_\varepsilon)(A^2(F_{1,\varepsilon} - \varphi M_\varepsilon) + E_1 M_\varepsilon) + C\varphi^2 \left(\frac{M'_\varepsilon}{M_\varepsilon}\right)' \right) > 0.$$

So, $(h_\varepsilon - F_{1,\varepsilon})' > 0$ on $(0;+\infty)$.

Moreover, $\lim\limits_{y\to 0^+}(h_\varepsilon - F_{1,\varepsilon}) = 0$ because of $\lim\limits_{y\to 0^+} h_\varepsilon = 0$ and

$$\lim_{y\to 0^+} F_{1,\varepsilon}(y) = \lim_{y\to 0^+} \frac{B_\varepsilon - \sqrt{B_\varepsilon^2 - 4AC_\varepsilon}}{2A}$$

$$= \lim_{y\to 0^+} \varphi \frac{\left(\left(\frac{A}{\varphi'^2}+1\right)M_\varepsilon + \frac{\varphi}{\varphi'}M'_\varepsilon\right) - \sqrt{\left(\left(\frac{A}{\varphi'^2}+1\right)M_\varepsilon + \frac{\varphi}{\varphi'}M'_\varepsilon\right)^2 - 4\frac{A}{\varphi'^2}M_\varepsilon^2}}{2\frac{A}{\varphi'^2}} = 0.$$

Hence, $h_\varepsilon - F_{1,\varepsilon} > 0$ on $(0;+\infty)$.

Fix $y > 0$. Hence, $h(y) - F_1(y) = \lim\limits_{\varepsilon\to 0^+}(h_\varepsilon - F_{1,\varepsilon}) \geq 0$.

So, $h - F_1 \geq 0$ on $(0;+\infty)$ and by (14) it follows that

$$h - F_1 > 0 \text{ on } (0;+\infty).$$

Therefore, by $A < 0$ and (13) it follows that $Ah^2 - Bh + C < 0$ on $(0;+\infty)$.

The second derivative

$$\left(\log\frac{h}{\varphi}\right)'' = \left(\left(\log\frac{h}{\varphi}\right)'\right)' = \left(\frac{h'\varphi}{h\varphi'}\right)' = \left(\frac{\varphi M}{h}\right)' = \frac{(-1)(Ah^2 - Bh + C)}{\varphi^2 h^2} > 0$$

on $(0;+\infty)$. □

Theorem 13 holds with each one of the following functions

- $\varphi(y) = \int_0^y t^{-a}dt$, $a < 1$;
- $\varphi(y) = \int_0^y e^{-t}dt$.

Theorem 14. *Let* $2 \leq p < +\infty$, $f \in H^p \setminus \{0\}$. *Assume the functions* $\varphi : (0;+\infty) \to (-\infty;+\infty)$ *and* $h : (0;+\infty) \to (-\infty;+\infty)$ *meet the conditions*

(*i*) $\varphi \in \mathcal{D}^3(0;+\infty)$, $\varphi(y) = -\int_y^{+\infty} \varphi'(t)dt$ *for all* $y \in (0;+\infty)$,

(*ii*) $\varphi' > 0$ *and* $\varphi'^2\varphi'' + \varphi\varphi'\varphi''' - 2\varphi\varphi''^2 \leq 0$ *on* $(0;+\infty)$,

(*iii*) $h(y) = -\int_y^{+\infty} \varphi'(t)M(t)dt$ *for all* $y \in (0;+\infty)$.

Then, $\frac{h}{\varphi}$ *and* $\log\frac{h}{\varphi}$ *both belong to* $\mathcal{D}^2(0;+\infty)$ *and moreover,*

$$\left(\frac{h}{\varphi}\right)' < 0, \left(\log\frac{h}{\varphi}\right)'' > 0 \text{ on } (0;+\infty).$$

This theorem is a simple corollary of Lemma 11 and Theorem 10 and we omit the details. Theorem 14 holds with each one of the following functions

- $\varphi(y) = -\int_y^{+\infty} t^{-a}dt,\ a > 1;$
- $\varphi(y) = -\int_y^{+\infty} t^{-a}e^{-t}dt,\ a < 0.$

Theorem 15. *Let* $2 \le p < +\infty$, $f \in H^p \setminus \{0\}$, $\varphi(y) = -\int_y^{+\infty} e^{-t}dt$, $h(y) = -\int_y^{+\infty} e^{-t}M(t)dt$ *for all* $y \in (0;+\infty)$. *Then,* $\frac{h}{\varphi}$ *and* $\log\frac{h}{\varphi}$ *both belong to* $\mathcal{D}^2(0;+\infty)$ *and moreover,*

$$\left(\frac{h}{\varphi}\right)' < 0, \left(\log\frac{h}{\varphi}\right)'' > 0 \text{ on } (0;+\infty).$$

P r o o f. By Lemma 11, $M \in \mathcal{D}^2(0;+\infty)$,

$$M > 0, \quad M' < 0, \quad M'' > 0, \quad (\log M)'' \ge 0 \text{ on } (0;+\infty).$$

By the assumptions, it is clear that $\frac{h}{\varphi}$ and $\log\frac{h}{\varphi}$ both belong to $\mathcal{D}^2(0;+\infty)$.

Claim. $h - \varphi M > 0$ on $(0;+\infty)$. Indeed, by the assumptions of the theorem

- $(h - \varphi M)' = -\varphi M' < 0$ on $(0;+\infty)$;
- M decreases on $(0;+\infty)$ and $M > 0$. So, the limit $\lim\limits_{y\to+\infty} M(y)$ exists and it is a non-negative number.

Therefore, $h - \varphi M$ decreases on $(0;+\infty)$ and

$$\lim_{y\to+\infty}(h(y) - \varphi(y)M(y)) = 0.$$

Hence, $h - \varphi M > 0$ on $(0; +\infty)$.

It follows from this Claim that the derivative

$$\left(\frac{h}{\varphi}\right)' = \frac{-\varphi'}{\varphi^2}(h - \varphi M) < 0 \text{ on } (0; +\infty).$$

Note,

$$A = \varphi''\varphi - \varphi'^2 = 0, \quad B = (\varphi'' M + \varphi' M')\varphi^2 < 0, \quad C = \varphi^2 \varphi'^2 M^2.$$

By Lemma 7 with $q = 1$,

$$\left(h - \frac{C}{B}\right)' B^2 \varphi' M = C^2 \left(\frac{M'}{M}\right)' \geq 0 \text{ on } (0; +\infty).$$

So, $\left(h - \frac{C}{B}\right)' \geq 0$ on $(0; +\infty)$ Therefore, $h - \frac{C}{B}$ increases on $(0; +\infty)$ and

$$\lim_{y \to +\infty} (h(y) - \frac{C(y)}{B(y)}) = \lim_{y \to +\infty} h(y) - \lim_{y \to +\infty} \frac{C(y)}{B(y)}$$
$$= \lim_{y \to +\infty} \frac{\varphi^2(y)\varphi'^2(y)M^2(y)}{(\varphi''(y)M(y) + \varphi'(y)M'(y))\varphi^2(y)} = \lim_{y \to +\infty} \frac{\varphi'(y)M(y)}{\frac{\varphi''(y)}{\varphi'(y)} + \frac{M'(y)}{M(y)}} = 0$$

because in the last equation the numerator tends to 0 and the denominator tends to sum of (-1) and a non-positive number.

Claim. $h - \frac{C}{B} < 0$ on $(0; +\infty)$.

Indeed, if there is $y_0 \in (0; +\infty)$ such that $h(y_0) - \frac{C(y_0)}{B(y_0)} = 0$ then $h - \frac{C}{B} = 0$ on $(y_0; +\infty)$. $\left(\frac{M'}{M}\right)' = 0$ on $(y_0; +\infty)$ which means that $\frac{M'}{M}$ is a non zero constant on $(y_0; +\infty)$ (because of f is not the zero function). The equation $h' = \left(\frac{C}{B}\right)'$ then gives us $\frac{M'}{M} = 1$ on $(y_0; +\infty)$. Therefore, $M(y) = e^y$.const and by $M > 0$, $M' \leq 0$ it follows that $M = 0$ on $(y_0; +\infty)$ which is impossible because of f is not the zero function.

Now, the second derivative

$$\left(\log \frac{h}{\varphi}\right)'' = \frac{B\left(h - \frac{C}{B}\right)}{h^2 \varphi^2} > 0 \text{ on } (0; +\infty). \qquad \square$$

Note that in some specific cases it seems reasonable to change parts of the proofs with an argument for $-Bh + C < 0$. In particular, such an approach will make us to use part of the proof of Lemma 7. However, we prefer not to do this.

Example 16 (An auxiliary example)**.** If

$$\varphi(y) = -\int_y^{+\infty} e^{t-e^t} dt, M(y) = e^{y^2}, h(y) = -\int_y^{+\infty} \varphi'(t)M(t)dt$$

then $\left(\frac{h}{\varphi}\right)' > 0$ and $\left(\log \frac{h}{\varphi}\right)'' > 0$ on $(0; +\infty)$.

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