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COUNTABLY COMPACTNESS AND BAIRE SPACE PROPERTY

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Dedicated to the memory of Professor Walter Roelcke (10.12.1928–24.12.2005)

Abstract. In the present article τ-bounded spaces are investigated. It is shown that for every infinite cardinal τ there exists a meager Hausdorff τ-bounded space.

1. Introduction. All spaces are assumed to be Hausdorff spaces. We shall use the terminology from [4, 10, 11] which contain a survey of results. Denote by $|X|$ the cardinality of a space X, by wX the weight of X, by βX the Stone-Čech compactification of a Tychonoff space X. A space is called a Baire space, if the intersection of every countable family of open dense subsets is a dense subset. A space is called a meager space or a first category space, if it is a union of a countable family of nowhere dense subsets.

Our main interest is the following question posed by W. Roelcke: Is there a Hausdorff ω-bounded space which is not Baire?

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In 1972, Z. Frolik [6] constructed an example of a meager countably compact space. Then in 1996, J. R. Porter [8] constructed an example of a countably compact, separable and meager space. We show that for every infinite cardinal τ, there exists a τ-bounded meager space. However, the following problems remain unsolved:

Question 1.1. *Is there a countably compact or an ω-bounded k-space which is not Baire?*

Question 1.2 (W. Roelcke). *Is there a sequentially compact space which is not Baire? Is there a sequentially compact ω-bounded space which is not Baire?*

A subset Z of a topological space X is called bounded in X if for any locally finite family γ of open subsets of X the set $\{U \in \gamma \colon Z \cap U \neq \emptyset\}$ is finite. A space X is called feebly compact if the set X is bounded in in the space X.

Any countably compact space is feebly compact. A completely regular space is feebly compact if and only if it is pseudocompact.

Theorem 1.3. *Let Z be a bounded G_δ-subset of a regular space X. Then:*
1. *The subspace Z is a Baire space.*
2. *If $Z \subset Y \subset \mathrm{cl}_X Z$, then Y is a Baire space.*

P r o o f. Obviously, the assertion 2 follows from the assertion 1. Assertion 1 follows from Theorem 5.1. □

2. τ-bounded sets. Fix an infinite cardinal τ.

Definition 2.1. *A subset Z of a topological space X is called:*

(*a*) *τ-bounded in X, if the closure $\mathrm{cl}_X L$ in X of every subset $L \subseteq Z$ of cardinality $|L| \leq \tau$ is compact;*

(*b*) *weakly τ-bounded in X, if for every subset A of Z there exists a subset L of A such that $|L| \geq min\{\tau, |A|\}$ and $\mathrm{cl}_X L$ is compact;*

(*c*) *σ_τ-compact, if Z is a union of τ compact subsets of X;*

(*d*) *countably compact in X if any infinity countable subset of Z has an accumulation point in X.*

Definition 2.2 (see [10, 11]). *A space X is called:*

(*a*) *τ-bounded, if X is τ-bounded in X;*

(*b*) *totally τ-compact, if X is weakly τ-bounded in X;*

(*c*) *initially τ-compact, if every open cover of X of cardinality $\leq \tau$ contains a finite subcover.*

For $\tau = \omega$, i.e. for countable τ, we say that X is ω-bounded, or totally countably compact, or countably compact, respectively.

Definition 2.3 (see [1]). *For any subset A of a space X the set $\tau\text{-cl}_X A = \bigcup\{\text{cl}\, B\colon B \subseteq A, |B| \leq \tau\}$ is called the τ-closure of A in X. The set A is called τ-closed, if $A = \tau\text{-cl}\, A$.*

The τ-closure of a set is τ-closed (see [1]).

Theorem 2.4. *Let Z be a subset of a space X. The following assertions are equivalent:*

1. *The subset Z is τ-bounded in X.*
2. *$\tau\text{-cl}_X Z$ is τ-bounded space.*

P r o o f. The implication $2 \to 1$ is obvious. Suppose that Z is τ-bounded in X and $Y = \tau\text{-cl}_X Z$. Let $L \subseteq Y$ and $|L| \leq \tau$. For every point $y \in L$, there exists a subset $Z(y)$ of Z such that $y \in \text{cl}_X Z(y)$ and $|Z(y)| \leq \tau$. Put $Z(L) = \cup\{Z(y)\colon y \in L\}$. By construction, $L \subseteq \text{cl}_X Z(L) \subseteq Y$, $|Z(L)| \leq \tau$ and $\text{cl}_X Z(L)$ is compact. Hence $\text{cl}_Y L = \text{cl}_X L$ is compact. □

Corollary 2.5. *A space X contains a subset which is dense τ-bounded in X, if and only if X contains a dense τ-bounded subspace.*

Corollary 2.6. *Let $L \subseteq H \subseteq \tau\text{-cl}_X L \subseteq X$. The set L is τ-bounded in X, if and only if H is τ-bounded in X.*

Corollary 2.7. *Every τ-closed subset of a τ-bounded space is a τ-bounded space.*

Example 2.8 (see [4, Exercise 3.6.I]). Fix a maximal uncountable family $\{N_\alpha\colon \alpha \in A\}$ of almost disjoint infinite subsets of the set $\omega = \{0, 1, 2, \dots\}$. Let $A \cap \omega = \emptyset$ and $X = A \cup \omega$. Points in ω are declared to be isolated. For each $a \in A$ and every finite subset F of ω a set $V(a, F) = \{a\} \cup (N_a \setminus F)$ is a basic neighborhood of a in X. Then:

1. The set ω is weakly ω-bounded in X.
2. X does not contain a dense countably compact subspace.

3. The examples of Frolik and Porter.

Because in our constructions we use some ideas from the Frolik's and Porter's construction, we present succinctly these examples.

Construction 3.1. *Let $\{X_i\colon i \in \omega\}$ be a sequence of pairwise disjoint subspaces of a space X. Consider the subspaces $Y_0 = X, \dots, Y_{n+1} = \cup\{X_i\colon i \geq n\}$, $\dots$. Denote by $Y = X_{(X_0, X_1, \dots)}$ the set X with the topology defined as follows:*

U is open in Y, if and only if $U = \cup\{U_i : i \in \omega\}$, where U_i is an open set in the subspace Y_i of the space X. The system $\{Y_i : i \in \omega\} \cup \{U \subseteq X : U$ is open in $X\}$ is a subbase of the space Y. It is easy to check that:

a°. The topologies of X and Y coincide on each X_i and on $X \setminus Y_1$.

b°. The set Y_i is dense in Y, if and only if it is dense in X.

c°. If X is a Hausdorff space, then Y is a Hausdorff space too.

d°. A point $x \in X_n$ is an accumulation (complete accumulation) point of a set $L \subseteq Y_{n+1}$ in the topology of the space X, if and only if x is an accumulation (complete accumulation) point of L in the topology of the space Y.

e°. If $X_0, X_1, \ldots$ are dense subspaces of the space X, then Y is a meager space.

If $X_0 = Z$ and $X_i = \emptyset$ for all $i \geq 1$, then we write $X_Z = X_{(X_0, X_1, \ldots)}$. In this case, Z is an open subspace of the space X_Z.

Proposition 3.2. *If Z is a dense subspace of a space X, then:*

1. *X_Z is a Baire space, if and only if Z is a Baire space.*
2. *X_Z is a meager space, if and only if Z is a meager space.*

P r o o f. Obvious. □

Example 3.3 (Z. Frolik [6]). Let $X = \omega^* = \beta\omega \setminus \omega$, where ω is the discrete space of natural numbers. By Theorem 2.7 in [5], there exists a disjoint sequence $\{X_n : n \in \omega\}$ of countably compact dense subsets of X such that $|X_n| \leq \exp(\omega) = 2^\omega$ for all n. Put $Y = X_{(X_0, X_1, \ldots)}$. Since X_n are dense subsets of the space ω^*, Y is a meager space. As the cardinality of each infinite closed subset of X is $\exp(\exp(\omega))$, each infinite subset of Y has an accumulation point in $K = Y \setminus (\bigcup\{X_n : n \in \omega\})$. Hence Y is a countably compact meager space.

For every infinite subset L of X_0 the set $F = \mathrm{cl}_Y L$ is not a compact subset of Y. Suppose that F is compact. Since X_0 is countably compact, then there exists an accumulation point $x \in X_0 \cap F$ of the set L. Take a neighbourhood U of x in F for which $\mathrm{cl}_F U \subseteq Y_1 \cap F$ (the set $Y_1 \cap F$ is open in F), Hence $F \cap \mathrm{cl}_F U$ is an infinite compact subset of Y_1 of cardinality $\leq \exp(\omega)$, contradiction. Therefore Y is not totally countably compact.

Example 3.4 (J. R. Porter [8]). Since ω^* contains every separable extremally disconnected space, ω^* contains a countable dense-in-itself subset S (see [7, Theorem 1.8.3], [9, Exercise 6Q2]). Consider the subspace $X = \mathrm{cl}_{\beta\omega} S$ of ω^*. The space $Y = X_S$ is a countably compact meager space. Since Y is separable and not compact, Y is not an ω-bounded space. We show that Y is totally countably compact. If D is any countable subset of X, there is a countable family of continuous functions $\{f_n\colon X \to [0,1]\colon n \in \omega\}$ which separates the points of D. The diagonal product $f = \triangle\{f_n\colon n \in \omega\}\colon X \to [0,1]^\omega$ is a continuous mapping which separates the points of D.

Let L be an infinity countable subset of Y. Since X is compact there is a continuous mapping $g\colon X \to Z$ onto a metrizable compact space Z which separates the points of $L \cup S$. The set $g(L)$ contains an infinite convergent sequence H with a limit $c \in Z$. Put $A = L \cap g^{-1}(H)$. Then A is a discrete subspace of the spaces X and Y, the set $B = \mathrm{cl}_X A \setminus A \subseteq g^{-1}(c)$ is uncountable and $|S \cap B| \le 1$. Therefore there exist a point $b \in B \setminus S$ and an open set U of X such that $b\, U$ and $\mathrm{cl}_X U \cap B \cap S = \emptyset$. The set $E = U \cap A$ is infinite and discrete in X and Y. Since $\mathrm{cl}_X E \subseteq \mathrm{cl}_X U$, we obtain that $(\mathrm{cl}_X E \setminus E) \cap S = \emptyset$, $\mathrm{cl}_X E$ is a compact subset of X and the topologies of the spaces X and Y coincide on $\mathrm{cl}_X E$. We have constructed an infinite subset E of L for which $\mathrm{cl}_Y E$ is compact. Hence Y is totally countably compact.

The following assertion is obvious.

Proposition 3.5. *If a subset Z is countably compact in a space X, then the set Z is bounded in X.*

Example 3.6. Let $\mathbb{Q}_0$ be the set of rational numbers of the segment $[0,1]$. Denote by $\mathcal{T}$ the usual Euclidean topology on $[0,1]$. By X we denote the set $[0,1]$ with the topology $\mathcal{T}_1$ generated by the open base $\mathcal{T} \cup \{U \cap \mathbb{Q}_0\colon U \in \mathcal{T}\}$. The space X has the following properties:

– $\mathbb{Q}_0$ is an open dense subspace of the space X;
– the set $D = [0,1] \setminus \mathbb{Q}_0$ is closed, discrete and nowhere dense in X;
– X is not a countably compact space;
– the set $\mathbb{Q}_0$ is countably compact in the space X;
– X is a Hausdorff feebly compact first countable space;
– X is a meager space.

Example 3.7. Let S be a Hausdorff dense-in-itself separable countably compact first countable space with the topology $\mathcal{T}$. Fix a dense countable set Y in S. By X we denote the set S with the topology $\mathcal{T}_1$ generated by the open base $\mathcal{T} \cup \{U \cap Y\colon U \in \mathcal{T}\}$. The space X has the following properties:

– Y is an open dense subspace of the space X;
– the set $D = S \setminus Y$ is closed, discrete and nowhere dense in X;
– X is not a countably compact space;
– X is a Hausdorff feebly compact first countable space;
– X is a meager space.

4. Construction of meager τ-bounded spaces. Let τ be an infinite cardinal and by τ^+ denote the smallest cardinal greater than τ.

For every Tychonoff space X denote $\tau\text{-}\beta X = \tau\text{-}\mathrm{cl}_{\beta X} X$ and $b(X,\tau) = \{\mathrm{cl}_{\beta X} L\colon L \subseteq X, |L| \leq \tau\}$. The space $\tau\text{-}\beta X$ is the largest τ-bounded extension of the space X. Every continuous mapping $f : X \to Y$ into a τ-bounded space Y admits a continuous extension on $\tau\text{-}\beta X$. Now put $\tau\text{-}\theta X = (\tau\text{-}\beta X)_X$ in the sense of Construction 3.1 and $X^* = \tau\text{-}\beta X \setminus X$. Let $\tau\text{-}\eta X$ be the set $\tau\text{-}\beta X$ with the topology defined as follows: the space X is an open subspace of the space $\tau\text{-}\eta X$; a set V is a neighborhood of a point $x \in X^*$, if and only if $x \in V$, $V \cap X$ is open in X and $V \cap F$ is open in F for all $F \in b(X,\tau)$.

Lemma 4.1. *If X is a Tychonoff space then:*

1. *X is an open dense subspace of the spaces $\tau\text{-}\theta X$ and $\tau\text{-}\eta X$.*
2. *If X is a meager space, then $\tau\text{-}\theta X$ and $\tau\text{-}\eta X$ are meager spaces, too.*
3. *The topologies of the spaces $\tau\text{-}\theta X$ and $\tau - \eta X$ coincide on each $F \in b(X,\tau)$.*
4. *The space $\tau\text{-}\theta X$ is τ-bounded, if and only if the space $\tau\text{-}\eta X$ is τ-bounded.*
5. *If every subset of X of cardinality $\leq \tau$ is closed in X, then $\tau\text{-}\theta X$ and $\tau\text{-}\eta X$ are τ-bounded spaces and $F \in b(X,\tau)$ are compact subsets.*

P r o o f. Obvious. □

Definition 4.2 (W. Roelcke)**.** *A space is called a σcc^*-space, if the closure of each σ-compact subset is compact.*

We say that a space is a $\sigma_\tau cc^*$-space, if the closure of every σ_τ-compact subset is compact. Every $\sigma_\tau cc^*$-space is τ-bounded.

Lemma 4.3. *Let X be a normal space and every subset of X of cardinality $\leq \tau$ be closed in X. Then:*

1. *$X \bigcup F$ is open in $\tau\text{-}\eta X$ for each $F \in b(X,\tau)$.*
2. *If K is a compact subset of $\tau\text{-}\eta X$ then $K \subseteq F$ for some $F \in b(X,\tau)$.*
3. *$\tau\text{-}\eta X$ is a $\sigma_\tau cc^*$-space.*

P r o o f. The subspaces of X of cardinality $\leq \tau$ are closed and discrete subspaces of X. Hence, if $F, E \in b(X,\tau)$ and $F \subseteq E$, then F is a closed and open subset of E. This proves assertion 1. Let K be a compact subset of $\tau\text{-}\eta X$ and $K^\circ = K \setminus X$. Suppose that $K^\circ \setminus F$ is non-empty for every $F \in b(X,\tau)$. Then there exist a disjoint family $\{L_\alpha\colon \alpha < \tau^+, \alpha \text{ is ordinal}\}$ of subsets of X of cardinality $\leq \tau$ and a set $B = \{b_\alpha \in K^\circ\colon \alpha < \tau^+\}$ such that $b_\alpha \in F_\alpha = \mathrm{cl}\, L_\alpha$ for all $\alpha < \tau^+$. For that fix a point $b_1 \in K^\circ$ and a set $L_1 \subseteq X$ of cardinality $\leq \tau$

such that $b_1 \in \mathrm{cl}\, L_1$. If $\alpha > 1$ and $\{b_\beta, L_\beta \colon \beta < \alpha\}$ we have constructed, then we put $\Phi_\alpha = \mathrm{cl}(\cup\{L_\beta \colon \beta < \alpha\})$, fix a point $b_\alpha \in K^\circ \setminus \Phi_\alpha$ and a set $L \subseteq X$ of cardinality $\leq \tau$ such that $b_\alpha \in \mathrm{cl}\, L$. Because Φ_α is open $\Phi_{\alpha+1} = \mathrm{cl}(L \cup \Phi_\alpha)$, then for $L_\alpha = L \setminus \Phi_\alpha$ we have $b_\alpha \in \mathrm{cl}\, L_\alpha$. The set $U_\beta = X \bigcup \mathrm{cl}(\cup\{L_\alpha \colon \alpha < \beta\})$ is open for each $\beta < \tau^+$. Hence $K_\beta = K^\circ \smallsetminus U_\beta$ are non-empty compact subsets. Fix a complete accumulation point x of the set B. Since $|B \cap U_\beta| \leq \tau$ for all $\beta < \tau^+$, we have $x \in C = \cap\{K_\beta \colon \beta < \tau^+\}$. By construction, $x \in F$ for some $F \in b(X, \tau)$. If $E = (F \cap X) \setminus \cup\{L_\beta \colon \beta < \tau^+\}$, then $x \in \mathrm{cl}\, E$. Hence x is not an accumulation point of the set B. That is a contradiction. Therefore $K^\circ \subseteq \mathrm{cl}\, H$ for some subset H of X of cardinality $\leq \tau$. In this case the set $K \setminus \mathrm{cl}\, H$ is a finite set. Assertion 2 is proved. Assertion 3 follows from assertion 2. □

Remark 4.4. Let X be a normal space in which all countable subsets are closed. Denote by T_0 the topology on the space τ-βX, by T_1 the topology on τ-θX and by T_2 the topology on τ-ηX. Then:

1. $T_0 \subseteq T_1 \subseteq T_2$.

2. All non-isolated points of X are points of non-regularity of the space τ-θX. Hence $T_0 = T_1$, if and only if X is a discrete space.

3. Every point of X^* is a point of regularity of τ-θX. If L is an infinite set of non-isolated points of the space $X, |L| = \omega$ and $x \in \mathrm{cl}\, L$, then the space $\tau - \eta X$ is not regular at the point x. Hence $T_1 = T_2$, if and only if the set of non-isolated points of the space X is finite. The system $T_1 \cup \{X \cup F \colon F \in b(X, \tau)\}$ is a subbase of the topology T_2 provided that the subsets of X of cardinality $\leq \tau$ are closed in X.

4. If $|X| = \tau^+$, X is a normal space and all subsets of X of cardinality $\leq \tau$ are closed in X, then there exists a perfect continuous mapping of the subspace X^* of the space τ-ηX onto the space of ordinals $W(\tau^+) = \{\beta \colon \beta < \tau^+\}$.

A space X is called a P_τ-space, if the intersection of all τ open sets is open. A P_ω-space is called a P-space.

From Lemma 4.3 follows

Corollary 4.5. *If X is a normal P_τ-space, then τ-θX is a τ-bounded space and τ-ηX is a $\sigma_\tau cc^*$-space.*

Proposition 4.6. *For every infinite cardinal τ there exists a meager hereditarily paracompact P_τ-space S_τ with the properties:*

1. *S_τ is a commutative topological group.*

2. *$\chi(S_\tau) = w(S_\tau) = |S_\tau| = \tau^+$.*

P r o o f. Let $D = \{0, 1\}$ be the discrete group with the neutral element

0. Denote by X the set D^{τ^+}. For every point $x = (x_\alpha : \alpha < \tau^+) \in X$ and all $\beta < \tau^+$ we put $V(x, \beta) = \{y = (y_\alpha) \in X : x_\alpha = y_\alpha$ for all $\alpha < \beta\}$. Consider on X the topology with the base $B = \{V(x, \beta) : x \in X, \beta < \tau^+\}$. The system B is of the rank 1, i.e. for all $U, V \in B$ we have $U \cap V = \emptyset$, or $U \subseteq V$, or $V \subseteq U$. Because a space with a base of rank 1 is paracompact (see [2, Corollary 1]), X is a hereditarily paracompact topological group. By construction, X is a P_τ-space and $\chi(X) = \tau^+$. For every $n \in \omega$ we put $X_n = \{x = (x_\alpha) : |\alpha : x_\alpha \neq 0| \leq n\}$ and $S_\tau = \cup\{X_n : n \in \omega\}$. Every set X_n is nowhere dense in S_τ and $|S_\tau| = |X_n| = \tau^+$. □

From Lemma 4.1, Corollary 4.5 and Proposition 4.6 follows.

Corollary 4.7. *$B_\tau = \tau$-θS_τ is a meager τ-bounded Hausdorff space.*

Corollary 4.8. *$H_\tau = \tau$-ηS_τ is a meager Hausdorff $\sigma_\tau cc^*$-space.*

5. W_δ-sets. Let X be a space, Z be asubspace of X, $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ be a sequence of families of open non-empty subsets of the space X, and let $\pi = \{\pi_n : A_{n+1} \to A_n : n \in \omega\}$ be a sequence of single-valued mappings. A sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ is called *a spectral sequence* if $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for every $n \in \mathbb{N}$.

Consider the following conditions:

(SC1) $Z \subset \cup\{U_\beta : \beta \in A_n\}$ for each $n \in \mathbb{N}$.

(SC2) For each spectral sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ the set $H(\gamma, \pi, \alpha) = \cap\{U_{\alpha_n} : n \in \mathbb{N}\}$ is a subset of the subspace Z.

(SC3) $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subset U_\alpha$ and $\cup\{U_\beta \cap Z : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha \cap Z$ for all $\alpha \in A_n$ and $n \in \mathbb{N}$.

(SC4) For all $\alpha \in A_n$ and $n \in \mathbb{N}$ we have $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subset U_\alpha$ and the set $\cup\{U_\beta \cap Z : \beta \in \pi_n^{-1}(\alpha)\}$ is a dense subset of the subspace $U_\alpha \cap Z$.

The sequences γ and π are called a *sieve* of Z in X if they are Properties (SC1), (SC2) and (SC3).

The sequences γ and π are called a *dense sieve* of Z in X if they are Properties (SC1), (SC2) and (SC4).

The sequences γ and π are called a *pseudosieve* in X if for all $\alpha \in A_n$ and $n \in \mathbb{N}$ the set $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ is a subset of the subspace U_α.

Let $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ and $\pi = \{\pi_n : A_{n+1} \to A_n : n \in \omega\}$ be a pseudosieve in X. The set $\cup\{H(\gamma, \pi, \alpha) : \alpha$ *is a spectral sequence*$\}$ is called the limit of the pseudosive (γ, π) and is noted by $\lim_X(\gamma, \pi)$.

If (γ, π) is a sieve of Z in X, then $\lim_X(\gamma, \pi) = Z$. If (γ, π) is a dense sieve of Z in X, then $\lim_X(\gamma, \pi) \subset Z$.

A set Z is called a W_δ-subset of a space X if there exists a sieve (γ,π) of Z in X such that $\lim_X(\gamma,\pi) = Z$. Any G_δ-subset is a W_δ-subset.

A space X is called *sieve-complete* if X is an open continuous image of some Čech-complete space (see [3]). A regular space X is sieve-complete if and only if X is a W_δ-subset of the Wallman compactification ωX of the space X.

Theorem 5.1. *Z is a bounded W_δ-subset of a regular space X, then Z is a Baire space.*

Proof. Let $\gamma = \{\gamma_n = \{U_\alpha \colon \alpha \in A_n\} \colon n \in \mathbb{N}\}$ and $\pi = \{\pi_n \colon A_{n+1} \to A_n \colon n \in \omega\}$ be a sieve of Z in X. Suppose that Z is not a Baire space. Then there exists a sequence $\{V_n \colon n \in \mathbb{N}\}$ of open subsets of the space X such that:

– $V_{n+1} \subset V_n$ and the set $V_n \cap Z$ is dense in Z for each $n \in \mathbb{N}$

– the set $V = \cap\{V_n \cap Z \colon n \in \mathbb{N}\}$ is not dense in Z.

Obviously, $V \subset Z$. There exists an open subset W of X such that $W \cap Z \neq \emptyset$ and $W \cap V = \emptyset$. Then there exists a sequence $\{W_n \colon n \in \mathbb{N}\}$ of open subsets of the space X and a spectral sequence $\alpha = \{\alpha_n \colon n \in \mathbb{N}\}$ such that $\mathrm{cl}_X W_{n+1} \subset W_n \cap W \cap U_{\alpha_{n+1}} \cap V_n$ and the set $H_n = (W_n \cap Z) \setminus \mathrm{cl}_X W_{n+1}$ is non-empty for each $n \in \mathbb{N}$. Since $\cap\{W_n \colon n \in \mathbb{N}\}$ is the empty set, the family $\{H_n \colon n \in \mathbb{N}\}$ is locally finite in X and the set $H_n \cap Z$ is non-empty for each $n \in \mathbb{N}$, a contradiction. The proof is complete. □

Theorem 5.2. *If Z is a Baire subspace of a space X, then for any dense sieve (γ,π) of Z in X the set $\lim_X(\gamma,\pi)$ is dense in Z.*

Proof. Let $\gamma = \{\gamma_n = \{U_\alpha \colon \alpha \in A_n\} \colon n \in \mathbb{N}\}$ and $\pi = \{\pi_n \colon A_{n+1} \to A_n \colon n \in \omega\}$ be a dense sieve of Z in X. We consider that the sets A_n are well-ordered and for all $n \in \mathbb{N}$ and $\alpha,\beta \in A_{n+1}$ from $\alpha \leq \beta$ it follows that $\pi_n(\alpha) \leq \pi_n(\beta)$. For each $n \in \mathbb{N}$ and each $\alpha \in A_{n+1}$ we put $V_\alpha = (U_\alpha \cap Z) \setminus \mathrm{cl}_Z\{U_\beta \cap Z \colon \beta \in A_n, \beta < \alpha\}$ and $V_n = \cup\{V_\alpha \colon \alpha \in A_n\}$. The sets V_n are open and dense in Z. Hence $V = \cap\{V_n \cap Z \colon n \in \mathbb{N}\}$ is a dense subset of Z. By construction, $V \subset \lim_X(\gamma,\pi)$. The proof is complete. □

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